KYBERNETIKA — VOLUME 29 (1993), NUMBER 5, PAGES 406-416

FEEDBACK REALIZATION OF OPEN LOOP DIAGONALIZERS

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In this work the feedback realization of open loop diagonalizers (*old*) of a linear, time-invariant, multivariable system is considered. In the first part of the paper, the properties of *old*'s which admit i) dynamic state feedback, ii) constant state feedback, iii) dynamic output feedback and iv) constant output feedback realization are investigated. Then, in the second part, dynamic (constant) output feedback decoupling problems are formulated as determining an open loop diagonalizer which admit the desired feedback realization. Finally, the solutions to these problems are obtained by determining the conditions of existence for such open loop diagonalizers.

1. INTRODUCTION

During the last decade a renewed interest has been witnessed in the decoupling problems. This interest is mainly focused on state feedback decoupling as in, for instance, Descusse, Lafay and Malabre [3], Dion and Commault [5] or block decoupling as in Descusse, Lafay and Malabre [4], Dion, Torres and Commault [7]. Dynamic output feedback decoupling, on the other hand, is considered by Hammer and Khargonekar [10] and by Eldem and Ozguler [8]. Also, Kučera [13] considered block decoupling by dynamic compensation with internal stability. Open loop block and scaler diagonalization are taken up in the works of Ozguler and Eldem [15].

This paper is basically a continuation of the works of Ozguler and Eldem [15]. It starts out with the investigation of the properties of *old*'s which admit a feedback realization. Four different types of feedback realizations are considered. These are namely, dynamic state feedback, constant state feedback, dynamic output feedback and constant output feedback. The objective is to formulate the feedback decoupling problems as determining a specific subset of the class of *old*'s which admit a particular feedback realization. This is done for dynamic and constant output feedback cases.

Throughout the paper linear, time-invariant, multivariable systems described by the following state space equations and input-output relations

$$\begin{aligned} x(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \\ y(s) &= Z(s)\,u(s), \quad Z(s) := C(sI - A)^{-1}B \end{aligned} \tag{1}$$

are considered. In the above equations $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ take values from n, m and p dimensional linear spaces over the field of real numbers R. A, B and C are constant matrices of appropriate dimensions. As usual $R(s)$ and $R[s]$ denote the field of rational functions and the ring of polynomials with coefficients from the field of real numbers. The Laurent series expansion of a matrix in $R^{p\times m}(s)$ is given by

$$B(s) := \sum_{i=-k}^{\infty} B_i\, s^{-i} \tag{2}$$

where B_i's are constant matrices. If $B_{-k} = \cdots = B_{-1} = 0$, then $B(s)$ is called *proper*. If B_0 is also zero then $B(s)$ is called *strictly proper*. $B(s)$ is called *right (left) biproper* if it is proper and B_0 has full column (row) rank. $B(s)$ is called simply *biproper* if B_0 is square and nonsingular. Note that a right (left) biproper $B(s)$ has a left (right) biproper left (right) inverse which will be denoted as ordinary inverse, i.e., $B(s)^{-1}$. Using the Laurent series expansion above, strictly proper ($B^-(s)$) and strictly polynomial ($B^+(s)$) parts of $B(s)$ are defined as

$$B^-(s) := \sum_{i=1}^{\infty} B_i\, s^{-i}, \quad B^+(s) := \sum_{i=-k}^{-1} B_i\, s^{-i}. \tag{3}$$

Static left (right) kernel of a rational matrix $B(s)$ is a linearly independent set of row (column) vectors $\{x_i\}$ such that $x_i B(s) = 0$ ($B(s)\, x_i = 0$). (In the rest of the paper kernel will be used instead of right kernel.) A basis for a static left (right) kernel can be obtained by picking out the *zero order rows (columns) of a minimal polynomial basis* (Forney [9]) for the kernel. It is well known in the literature that *the interactor*, first defined by Wolovich and Falb [17], plays a crucial role in decoupling problems. For a given strictly proper, $p \times m$ full row rank transfer matrix $Z(s)$, the interactor is defined as a lower left triangular polynomial matrix $X(s)$ such that $X(s)\,Z(s)$ is left biproper. The interactor can be expressed uniquely as $X(s) := H(s)\,D(s)$, where $D(s) := \mathrm{diag}\{s^{n_i}\}$ and $H(s)$ is a lower left triangular matrix with ones on the diagonal. The set of integers $\{n_i,\ i = 1, 2, \ldots, p\}$ are called the infinite zero orders of $Z(s)$ when the interactor is row reduced. The column degrees of $X(s)$, $(\partial(X(s))_{ci})$ are called essential orders and denoted as $n_{e,i}$ (Commault et al. [6]). In view of the fact that $X(s)\,Z(s)$ is left biproper, there exists a biproper $L(s) := [L_1(s)\ L_2(s)]$ such that

$$X(s)\,Z(s)\,[L_1(s) : L_2(s)] = [I : 0]. \tag{4}$$

Using this equation, the set of open loop diagonalizers of $Z(s)$ can be characterized easily as in Ozguler and Eldem [15]. For a given full row rank transfer matrix

$Z(s)$ and a given $p \times p$ strictly proper diagonal matrix $\Lambda(s)$, the set of open loop Λ-diagonalizers, $\mathrm{OLD}(Z, \Lambda)$ is defined as

$$\mathrm{OLD}(Z, \Lambda) := \{\text{proper } M(s) \mid Z(s)\, M(s) = \Lambda\}. \tag{5}$$

The following result, which we include for the sake of completeness, is a different version of Lemma 1 in Ozguler and Eldem [15].

Lemma 1. $\mathrm{OLD}(Z, \Lambda)$ is nonempty iff $X(s)\,\Lambda(s)$ is proper.

Proof. Since XZ is left biproper and M is proper the necessity is obvious by the definition of $\mathrm{OLD}(Z, \Lambda)$. For sufficiency, let $M := L_1 X \Lambda$. Then, in view of equation (4), M is an open loop diagonalizer. □

The above result basically implies that the infinite zero orders of the decoupled system is bounded below by $\partial(X(s))_{ci}$, which are the essential orders of the system to be decoupled.

In the feedback realization of the set of OLD's the following feedback control laws with constant precompensation is considered:

i) Dynamic state feedback: $u(s) = -F(s)\, x(s) + G\, v(s)$.

ii) Constant state feedback: $u(s) = -F\, x(s) + G\, v(s)$.

iii) Dynamic output feedback: $u(s) = -Z_c(s)\, y(s) + G\, v(s)$.

iv) Constant output feedback: $u(s) = -Z_c\, y(s) + G\, v(s)$.

In the above, $F(s)$ and $Z_c(s)$ are proper and F and Z_c are constant compensators in the feedback path. G is a full column rank constant precompensator.

Remark 1. The solvability conditions for general dynamic state feedback decoupling are well known in the literature. (See for instance, Hautus and Heynmann [11], Dion and Commault [5] and Ozguler and Eldem [15].) The constant state feedback case, also known as the Morgan's problem (Morgan [14]) remains still unsolved. The general versions of dynamic and constant output feedback decoupling has not been solved yet. The solution for restricted dynamic output feedback is given by Eldem and Ozguler [8]. The solution to dynamic output feedback decoupling with internal stability, where the initial transfer matrix is square and nonsingular, is due to Hammer and Khargonekar [10]. The solution for constant output feedback is given by Howze [12] and by Wolovich [16] for square transfer matrices.

2. PRELIMINARY RESULTS

For a given $p \times m$, full row rank, strictly proper transfer matrix $Z(s)$ and a strictly proper diagonal matrix Λ ($X(s)\,\Lambda$ proper), $\mathrm{OLD}(Z, \Lambda)$ can be characterized easily as

$$\mathrm{OLD}(Z, \Lambda) := \{M \mid M = L_1\, X(s)\, \Lambda + L_2 N\} \tag{6}$$

where L_1 and L_2 are as defined by equation (4) and N is an arbitrary proper matrix.

Lemma 2. Let M be in OLD(Z,Λ). Then M can be realized by dynamic state feedback iff it is right biproper.

Proof. Let $F(s)$ and G be a dynamic state feedback realization of $M(s)$. Then, $M = [I + F(s)(sI - A)^{-1}B]^{-1}G$. As $F(s)$ is proper, it follows that M is right biproper. Conversely, suppose that M is right biproper and define $F(s)$ and G as $F(s) := -s(M)^{-}(M)^{-1}((I - A/s)^{-1}B)^{-1}$; $G := (M)_0$. Clearly, $F(s)$ is proper and $F(s)(sI - A)^{-1}BM = -M^{-}$. Thus, $M = [I + F(s)(sI - A)^{-1}B]^{-1}G$ which implies that $(F(s), G)$ is a dynamic state feedback realization of $M(s)$. □

Remark 2. The above lemma implies that dynamic state feedback decoupling problem in equivalent to finding a right biproper element of OLD(Z,Λ). When $\Lambda = \mathrm{diag}\{s^{-n_{e,i}}\}$ this condition reduces to the solvability condition for dynamic state feedback decoupling given, for instance, in Dion and Commault [5]. It is also clear from the above result that only right biproper open loop diagonalizers admit feedback realizations. Therefore, *only* POLD(Z,Λ), *right biproper subset of* OLD(Z,Λ) *is considered in the rest of the paper.*

Lemma 3. Let M be in POLD(Z,Λ). Then, M is realizable by constant state feedback iff the first m columns of a basis of the static left kernel of

$$\begin{bmatrix} (F(s))^{-}(sI - A)^{-1}BM \\ K(sI - A)^{-1}BM \end{bmatrix} \tag{7}$$

are linearly independent. Here $F(s)$ is an arbitrary dynamic state feedback realization of M (which always exists as M is right biproper) and K is a constant matrix the rows of which span the left kernel of BM_0.

Proof. Let (F, G) be a constant state feedback realization of M. Then, $M = [I + F(sI - A)^{-1}B]^{-1}G$. This implies that $F(sI - A)^{-1}BM = -M^{-}$ as $G = M_0$. For any dynamic state feedback realization $F(s)$, it is also true that $F(s)(sI - A)^{-1}BM = -M^{-}$. Then, $[F - F(s)](sI - A)^{-1}BM = 0$. As $s(sI - A)^{-1}BM$ is right biproper, we have $[F - F(s)]_0[s(sI - A)^{-1}BM]_0 = [F - F(s)]_0[BM]_0 = 0$. Consequently, $F = [F(s)]_0 + LK$ for some constant matrix L. This implies that $[LK - F(s))^{-}](sI - A)^{-1}BM = 0$, i.e.

$$[-I : L]\begin{bmatrix} (F(s))^{-}(sI - A)^{-1}BM \\ K(sI - A)^{-1}BM \end{bmatrix} = 0. \tag{8}$$

For the converse, note that the above equation holds for some constant matrix L. Define F as $F := [F(s)]_0 + LK$. Then

$$\begin{aligned} F(sI - A)^{-1}BM &= \{[F(s)]_0 + LK\}(sI - A)^{-1}BM \\ &= \{F(s) - (F(s))^{-} + LK\}(sI - A)^{-1}BM \\ &= F(s)(sI - A)^{-1}BM = -M^{-}. \end{aligned} \tag{9}$$

Consequently, $M = [I + F(sI - A)^{-1}B]^{-1}G$. □

Lemma 4. Let M be in $\mathrm{POLD}(Z,\Lambda)$. Then M is realizable by dynamic (constant) output feedback iff $M^-\Lambda^{-1}$ is proper (constant).

Proof. If M can be realized by dynamic (constant) output feedback, then there exists a proper Z_c (constant Z_c) and a constant G such that $M = (I+Z_cZ)^{-1}G$, which implies that $Z_c ZM = -M^-$. Since $ZM = \Lambda$, it follows that $M^-\Lambda^{-1}$ is proper (constant). For sufficiency let $Z_c := -M^-\Lambda^{-1}$, then $Z_c\Lambda = Z_c ZM = -M^-$. Thus, $(I+Z_cZ)M = M_0 = G$, i.e., $M = (I+Z_cZ)^{-1}G$. □

Remark 3. Note that when Z is square and nonsingular we have $M = Z^{-1}\Lambda$. Thus, the properness of $M^-\Lambda^{-1}$ is equivalent to the properness of $Z^{-1}-(Z^{-1}\Lambda)_0\Lambda^{-1}$. If we let $(Z^{-1}\Lambda)_0 = G$ (the constant precompensator), then this condition further reduces to $(ZG)^{-1}-\Lambda^{-1}$ being proper (the off-diagonal terms of $(ZG)^{-1}$ are proper). This is exactly the same solvability condition given in Bayoumi and Duffield [1] and in Eldem and Ozguler [8] for dynamic output feedback decoupling of square transfer matrices (note that the restricted dynamic output feedback decoupling problem considered in Eldem and Ozguler [8] and the general version considered in this paper are equivalent problems for square transfer matrices). Moreover, for constant output feedback case, the condition given by the above lemma reduces to $(ZG)^{-1}-\Lambda^{-1}$ being constant (or equivalently the off-diagonal terms of $(ZG)^{-1}$ are constant) which is exactly the same solvability condition given for constant output feedback decoupling of square transfer matrices by Howze [12] and by Wolovich [16]. Thus, the above preliminary result shows the connection between our work and the previous results (on restricted cases) in the literature.

3. MAIN RESULTS

The problem of determining a desired feedback realization of a given M in $\mathrm{POLD}(Z,\Lambda)$ is easy as demonstrated in the previous section. In this section we present the necessary and sufficient conditions (in terms of Z and Λ) for the existence of an M in $\mathrm{POLD}(Z,\Lambda)$ which admit a desired feedback realization. This will be done for dynamic and constant output feedback cases.

In the rest of the paper the following characterization of the set of right biproper open loop Λ-diagonalizers of Z, $\mathrm{POLD}(Z,\Lambda)$ is going to be used:

$$\mathrm{POLD}(Z,\Lambda) = \{M(s)\,|\,M(s) = M_1(s) + M_2(s)\,N(s);\ N(s) \text{ is proper}\}. \tag{10}$$

Here, $M_1 := L_1X(s)\Lambda$ and $M_2 := L_2$, where L_1 and L_2 are as defined by equation (4). The problems to be treated in this section can now be formalized as follows:

Definition 1. *Dynamic output feedback Λ-decoupling* (Λ-DOOF): *GIVEN* a $p\times m$, full row rank, strictly proper transfer matrix Z and a diagonal, nonsingular, strictly proper Λ, *FIND* a proper Z_c and a full column rank constant G (if they exist) such that $Z(I+Z_cZ)^{-1}G = \Lambda$; or equivalently
FIND a proper N such that $M := M_1 + M_2N$ admits a dynamic output feedback realization, i.e., $(M_1+M_2N)^-\Lambda^{-1}$ is proper and $(M_1+M_2N)_0$ has full column rank.

Constant output feedback Λ-decoupling (Λ-*DCOF*) has a similar definition (proper Z_c is replaced by constant Z_c and $(M_1 + M_2 N)^{-}\Lambda^{-1}$ is constant).

In order to simplify the proof of the first main result the following Lemma is presented first.

Lemma 5. Given Z and Λ as above, there exists a dynamic output feedback control law (Z_c, G) such that $Z(I + Z_c Z)^{-1} G = \Lambda$ iff there exists a constant matrix G such that

$$X(s) - X(s)\, Z\, G\, \Lambda^{-1} \text{ is proper.} \tag{11}$$

Furthermore, if such G exists it has full column rank.

Proof. If (Z_c, G) is a solution then, $Z(I + Z_c Z)^{-1}\, G = (I + Z\, Z_c)^{-1}\, Z\, G = \Lambda$ which implies that $I - ZG\Lambda^{-1} = -ZZ_c$ or equivalently we have $X - XZG\Lambda^{-1} = -XZZ_c$. Since Z_c and XZ are proper, it follows that $X - XZG\Lambda^{-1}$ is proper.

Conversely, if $X - XZG\Lambda^{-1}$ is proper, then choose Z_c as $Z_c := -(XZ)^{-1}(X - XZG\Lambda^{-1})$ where $(XZ)^{-1}$ is the right biproper right inverse of XZ (which exists as XZ is left biproper). Clearly, Z_c is proper. Furthermore, as X^{-1} is strictly proper, it also follows that $I - ZG\Lambda^{-1}$ is strictly proper. Consequently, $ZG\Lambda^{-1}$ is biproper, i.e., G has full column rank. Since

$$\begin{aligned} Z(I + Z_c Z)^{-1} G &= (I + ZZ_c)^{-1} ZG \\ &= [I - Z(XZ)^{-1}(X - XZG\Lambda^{-1})]^{-1} ZG \\ &= [I - (I - ZG\Lambda^{-1})]^{-1} ZG = \Lambda \end{aligned} \tag{12}$$

(Z_c, G) is a solution, which concludes the proof. □

Remark 4. The above lemma points out the crucial role played by the constant precompensator in the design of decoupling feedback control. More specifically, it shows that the whole design is based on choosing a constant precompensator G such that $X - XZG\Lambda^{-1}$ is proper. This implies that one can decouple the system using certain linear combinations of the inputs only, i.e., by choosing a particular subspace of the input space. The choice for this subspace will be clear in the proof of Theorem 1. The importance of constant precompensators has also been emphasized by Eldem and Ozguler [8] in connection with diagonal decoupling via restricted state feedback (DDRSF) and via restricted dynamic output feedback (DDROF). Recall that the solvability condition for DDROF given in Eldem and Ozguler [8] is based on the notion of *diagonal causality degree dominance (dcdd)* of ZG. This condition is equivalent to $\Lambda^{-1} - \Lambda^{-1}ZG\Lambda^{-1}$ being proper. Multiplying this expression by $X\Lambda$ (which is proper) we obtain the condition given by equation (11). Thus, the properness of $X - XZG\Lambda^{-1}$ is necessary for *dcdd* as expected (because DDROF is a restricted version of the problem being considered in this paper). In this respect, the condition given in Lemma 5 can be interpreted as the generalization of the notion of dcdd for the problem considered in this paper.

Remark 5. Since G is the constant term of the Laurent series expansion of an open loop diagonalizer M, it follows that $G = M_{1,0}+M_{2,0}N_0$ for some constant matrix N_0. Hence, the question of existence of G such that $X - XZG\Lambda^{-1}$ is proper (Lemma 5) reduces to the question of existence of N_0 such that $X - XZ(M_{1,0} + M_{2,0}N_0)\,\Lambda^{-1}$ is proper. Through this observation we are led to Theorem 1. More precisely, Theorem 1 provides the necessary and sufficient conditions for the existence of such N_0 and the proof is based on the construction N_0.

Theorem 1. Λ-DDOF is solvable iff the first row of a basis of the static kernel of

$$[(X(s))_{ci} - ((\Lambda_i^{-1})^{+}X(s)\,ZM_{1,0})^{+}_{ci} : ((\Lambda_i^{-1})^{+}X(s)\,ZM_{2,0})^{+}] \tag{13}$$

is nonzero for each i. Here, Λ_i^{-1} denotes the ith diagonal element of Λ^{-1} and $(\cdot)_{ci}$ denotes the ith column.

Proof. If the hypothesis holds, there exists a constant vector N_i for each i $(= 1, 2, \ldots, p)$ such that

$$[(X(s))_{ci} - ((\Lambda_i^{-1})^{+}X(s)\,ZM_{1,0})_{ci} - ((\Lambda_i^{-1})^{+}X(s)\,ZM_{2,0})^{+}\,N_i] \tag{14}$$

is proper. Let $N_0 := [N_1, , \ldots, N_p]$. Then, in view of the above expression it follows that

$$[X(s) - X(s)\,ZM_{1,0}\Lambda^{-1} - X(s)\,ZM_{2,0}N_0\Lambda^{-1}] \tag{15}$$

is proper. Since $ZM_1 = \Lambda$ and $ZM_2 = 0$ we have

$$[X(s)\,Z\,(M_1 - M_{1,0} - M_{2,0}N_0)\,\Lambda^{-1}] = X(s)\,Z(M_1^{-} + M_2N^{-} + M_2^{-}N_0)\,\Lambda^{-1} \tag{16}$$

which is also proper for strictly proper part N^{-} of any proper N. Now let M_2^{-1} be a left biproper left inverse of M_2 such that $M_2^{-1}\,M_1 = 0$ (left inverse exists as M_2 is right biproper). Choose N^{-} as

$$N^{-} := -\left[(M_2^{-1}M_1^{-} + M_2^{-1}M_2^{-}N_0)\,\Lambda^{-1}\right]^{+}\Lambda. \tag{17}$$

Using equations (16) and (17) it follows that

$$\begin{bmatrix} X(s)\,Z \\ M_2^{-1} \end{bmatrix} \left(M_1^{-} + M_2^{-}N_0 + M_2N^{-}\right)\,\Lambda^{-1} \tag{18}$$

is proper. Since the first expression above is biproper, defining N as $N := N_0 + N^{-}$, implies that $(M_1 + M_2N)^{-}\Lambda^{-1}$ is proper. Equation (16) also implies that

$$X(s)\,Z\,(M_{1,0} + M_{2,0}N_0)\,\Lambda^{-1} = X(s) + Y(s) \tag{19}$$

where $Y(s)$ is proper. Since $X(s)$ is strictly polynomial and nonsingular, then the right hand side of the above equation is nonsingular. Therefore, $M_{1,0} + M_{2,0}N_0$ has full column rank. Thus, if Z_c and G are defined as $Z_c := (M_1 + M_2N)^{-}\Lambda^{-1}$ and $G := M_{1,0} + M_{2,0}N_0$, then (Z_c, G) is a solution of Λ-DDOF.

For necessity let (Z_c, G) be a solution of Λ-DDOF. Then $Z_c = (M_1 + M_2 N)^- \Lambda^{-1}$ for some proper matrix N. Thus, $(M_1^- + M_2 N^- + M_2^- N_0)^- \Lambda^{-1}$ is proper. Premultiplying this expression by $X(s)\, Z$ we obtain

$$X(s)\, Z \left(M_1^- + M_2 N^- + M_2^- N_0\right) \Lambda^{-1} = X(s) - X(s)\, Z M_{1,0} \Lambda^{-1} - X(s)\, Z M_{2,0} N_0 \Lambda^{-1} \tag{20}$$

which is also proper. Therefore,

$$(X(s))_{ci} - \left[(X(s)\, Z M_{1,0})_{ci} (\Lambda_i^{-1})^+\right]^+ - \left[X(s)\, Z M_{2,0} (N_0)_{ci} (\Lambda_i^{-1})^+\right]^+ = 0 \tag{21}$$

for each i, which concludes the proof. □

Lemma 4 implies that for a given open loop diagonalizer M there is a unique dynamic output feedback realization. Thus, the richness of the set of solutions of Λ-DDOF is only due to the richness of the subset of POLD(Z, Λ) which admit dynamic output feedback realization. On the other hand, the set of solutions of Λ-DCOF is a subset of the set of all solutions of Λ-DDOF. Therefore, the characterization of the set of solutions for Λ-DDOF, which is given by Theorem 2 below, is an important step towards the solution of Λ-DCOF. For this end, let (Z_c^*, G^*) be a solution of Λ-DDOF. Then, for some proper N^* we have $(I + Z Z_c^*)^{-1} G^* = M_1 + M_2 N^*$.

Theorem 2. (Z_c, G) is a solution of Λ-DDOF iff there exist a strictly proper matrix N_c and a constant matrix N_r such that

$$\begin{aligned} &\text{i)} \quad Z_c = Z_c^* - (M_2 N_c + M_2^- N_r)\, \Lambda^{-1}, \quad G = G^* + M_{2,0} N_r \\ &\text{ii)} \quad \mathrm{Im}(N_r) \in \text{ static Ker } \left[(\Lambda_i^{-1})^+ X(s)\, Z M_{2,0}\right]^+ \text{ and} \\ &\text{iii)} \quad N_c \Lambda^{-1} + (M_2^{-1} M_2^- N_r \Lambda^{-1})^+ \text{ is proper.} \end{aligned} \tag{22}$$

P r o o f. Let (Z_c, G) be a solution of Λ-DDOF. Then, for some proper N we have $Z_c = -(M_1 + M_2 N)^- \Lambda^{-1}$. Using a similar representation for Z_c^* we have

$$Z_c - Z_c^* = -\left[M_1 (N^- - (N^*)^-) + M_2^- (N_0 - N_0^*)\right] \Lambda^{-1}. \tag{23}$$

Since both Z_c and Z_c^* are proper, then

$$X(s)\, Z (Z_c - Z_c^*) = -X(s)\, Z M_2^- (N_0 - N_0^*)\, \Lambda^{-1} \tag{24}$$

is proper. This implies that N_r, defined as $N_r := N_0 - N_0^*$, is in the static kernel of $[X(s)\, Z M_{2,0} (\Lambda_i^{-1})^+]^+$. Also note that $G = G^* + M_{2,0} N_r$. Premultiplication of equation (21) by M_2^{-1} yields

$$M_2^{-1} (Z_c - Z_c^*) = -\left[N^- - (N^*)^- + M_2^{-1} M_2^- (N_0 - N_0^*)\right] \Lambda^{-1} \tag{25}$$

which is proper. Therefore, if N_c is defined as $N_c := N^- - (N^*)^-$, then it is easy to see that $N_c \Lambda^{-1} + (M_2^{-1} M_2^- N_r \Lambda^{-1})^+$ is proper.

For the converse, it has to be shown that Z_c is proper as defined by the hypothesis and (Z_c, G) solves Λ-DDOF. For the first claim, note that

$$\begin{bmatrix} X(s)\,Z \\ M_2^{-1} \end{bmatrix} Z_c = \begin{bmatrix} X(s)\,Z \\ M_2^{-1} \end{bmatrix} Z_c^* - \begin{bmatrix} 0 \\ N_c \end{bmatrix} \Lambda^{-1} - \begin{bmatrix} -X(s)\,ZM_{2,0}N_r \\ M_2^{-1}M_2^- N_r \end{bmatrix} \Lambda^{-1}. \quad (26)$$

By the choice of N_r and N_c, it follows that $X(s)\,ZM_{2,0}N_r\Lambda^{-1}$ and $(N_c+M_2^{-1}M_2^- N_r)$ Λ^{-1} are proper. Thus, the above expression is proper. Since Z_c is premultiplied by a biproper matrix in the above equation, it follows that Z_c is proper. For the second claim note that as $M_{2,0}N_r = G - G^*$

$$\begin{aligned} Z(I+Z_cZ)^{-1}G &= (I+ZZ_c)^{-1}ZG = (I+ZZ_c^* - ZM_2^- N_r\Lambda^{-1})^{-1}ZG = \\ &= (I+ZZ_c^* + ZM_{2,0}N_r\Lambda^{-1})^{-1}ZG \qquad (27)\\ &= (I{+}ZZ_c^*)^{-1}\left[I{-}ZG^*\Lambda^{-1}(I{+}ZZ_c^*)^{-1}{+}ZG\Lambda^{-1}(I{+}ZZ_c^*)^{-1}\right]^{-1}ZG \\ &= (I+ZZ_c^*)^{-1}(ZG\Lambda^{-1}(I+ZZ_c^*)^{-1})^{-1}ZG \\ &= \Lambda. \end{aligned}$$

thus, (Z_c, G) solves Λ-DDOF.

Theorem 3. Let (Z_c^*, G^*) be a solution of Λ-DDOF. Then, Λ-DCOF is solvable iff the first row of the static kernel of

$$\begin{bmatrix} 0 & : & (X(s)\,ZM_{2,0}(\Lambda_i^{-1})^+)^+ & : & 0 \\ Z(Z_c^*)_{ci}^- & : & Z\left[-M_2^-(\Lambda_i^{-1})^+ + M_2(M_2^{-1}M_2^-(\Lambda_i^{-1})^+)^+\right]^- & : & -ZM_2^- \end{bmatrix} \quad (28)$$

is nonzero for each i.

Proof. Let $[1\ N_i^T$ and $Y_i^T]^T$ be in the above static kernel. Define N_r as $N_r := [N_1, \ldots, N_p]$ and Z_c as

$$Z_c = Z_c^* - \left[M_2(W - (M_2^{-1}M_2^- N_r\Lambda^{-1})^+\Lambda)\,\Lambda^{-1} + M_2^- N_r\Lambda^{-1}\right] \quad (29)$$

where W is a strictly proper matrix such that

$$\begin{aligned} W_{ci} &= \Lambda_i\left[Y_i{+}M_2^{-1}(Z_c^*)_{ci}^-{+}M_2^{-1}[M_2(M_2^{-1}M_2^-(\Lambda_i^{-1})^+)^+{-}\right. \qquad (30)\\ &\quad \left.-M_2^-(\Lambda_i^{-1})^+]^- N_i{-}\ M_2^{-1}M_2^- Y_i\right]. \end{aligned}$$

Clearly, $W\Lambda^{-1}$ is proper. If G is defined as $G = G^* + M_{2,0}N_r$, then it is clear that N_r satisfies Theorem 2 ii). Defining N_c as $N_c := W - (M_2^{-1}M_2^- N_r\Lambda^{-1})^+\Lambda$, it also follows that N_c satisfies Theorem 2 iii). Thus (Z_c, G) is a solution of Λ-DDOF. Now note that in view of the hypothesis and the definition of W, we have

$$\begin{aligned} \begin{bmatrix} Z \\ M_2^{-1} \end{bmatrix} (Z_c)_{ci}^- &= \begin{bmatrix} Z \\ M_2^{-1} \end{bmatrix} (Z_c^*)_{ci}^- + \begin{bmatrix} Z \\ M_2^{-1} \end{bmatrix} \times \\ &\quad \times \left[M_2(M_2^{-1}M_2^-(\Lambda_i^{-1})^+)^+ - M_2^-(\Lambda_i^{-1})^+\right]^- N_i \\ &- \begin{bmatrix} Z \\ M_2^{-1} \end{bmatrix} \left[M_2(W_{ci}\Lambda_i^{-1})^- + M_2^-(W_{ci}\Lambda_i^{-1})_0\right] \qquad (31)\\ &= 0 \end{aligned}$$

which implies that $Z_c^- = 0$, i.e., Z_c is constant. Therefore (Z_c, G) is a solution of Λ-DCOF.

Conversely let (Z_c, G) be a solution of Λ-DCOF. Then, for some proper N we have

$$Z_c = Z_c^* - \left[M_2(N^- - (N^*)^-) + M_2^-(N_0 - N_0^*)\right]\Lambda^{-1}. \tag{32}$$

Multiplying the above equation by M_2^{-1} it can be easily shown that

$$(N - N^*)^- = -\left(M_2^{-1}M_2^-(N_0 - N_0^*)\Lambda^{-1}\right)^+\Lambda + W \tag{33}$$

where W is a strictly proper matrix such that $W\Lambda^{-1}$ is proper. Multiplying by $X(s)\,Z$ we also have

$$\left[X(s)\,ZM_{2,0}(\Lambda_i^{-1})^+\right]^+(N_0 - N_0^*)_{ci} = 0. \tag{34}$$

Column by column evaluation of $Z(Z_c)^-$ now yields

$$\begin{aligned} Z(Z_c)_{ci}^- = {} & Z(Z_c^*)_{ci}^- + Z\left[M_2(M_2^{-1}M_2^-(\Lambda_i^{-1})^+)^+ - M_2^-(\Lambda_i^{-1})^+\right]^-(N_0 - N_0^*)_{ci} \\ & -Z\left[M_2(W_{ci}\Lambda_i^{-1})\right]^- = 0 \end{aligned} \tag{35}$$

which implies that

$$\begin{aligned} & Z(Z_c^*)_{ci}^- + Z\left[M_2(M_2^{-1}M_2^-(\Lambda_i^{-1})^+)^+ - M_2^-(\Lambda_i^{-1})^+\right]^-(N_0 - N_0^*)_{ci} - \\ & Z\left[M_2^-(Y_{ci}\Lambda_i^{-1})_0\right] = 0. \end{aligned} \tag{36}$$

Thus, in view of equations (35) and (36) the hypothesis holds. □

4. CONCLUSIONS

In this work Λ-decoupling problems, where the closed loop transfer matrix Λ is specified, are considered. The problems are formulated by using the right biproper set $\mathrm{POLD}(Z, \Lambda)$ of open loop Λ-diagonalizers of a given strictly proper transfer matrix Z. The solutions are obtained via the feedback realizations of these open loop diagonalizers.

(Received February 18, 1993.)

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