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TRACKING PERFORMANCE OF 'n' INTEGRAL-PLUS-TIME CONSTANT PLANTS WITH 'ONE' CONTROLLER

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In a unique and most easily comprehensible and applicable way, it will be shown how the outputs of n identical integral-plus-time constant plants: $G(s) = \frac{K}{s(s+\lambda)}$, with different output initial conditions, can be brought to track a reference, or command, input $r(t)$ through commissioning of only one controller $H(s)$. A three-part example, used in computer simulation, shall, most vividly, support the theoretical results.

1. INTRODUCTION

In many technical applications synchronizing control problems occur, see e. g. [1, 2]. Very often a technical process includes a number of identical units or plants which have to be controlled in parallel in order to track a common command signal. Problems of this kind are usually connected to multi-motor drives or positioning systems using either hydraulic or electric devices. These synchronizing control applications can be found in many production processes, as e. g. paper mills, textile industry, sheet rolling train, rolling mills, printing machines, cylinder presses for metal and fibre plastic parts forming, robotic manipulators, and many others.

The state of art for technically solving these problems usually consists in the application of n identical controllers $[H(s)]$ for controlling n such identical plants $[G(s)]$. This procedure will be defined in this contribution as the classical one-plant/one-controller philosophy. However, in this paper it will be shown how the outputs of n identical plants $[G(s)]$ can be made to track a common command input $r(t)$ with the use of just *one* controller $[H(s)]$. The dynamic behaviour of these plants to be synchronized can very often be described by a linear integral-plus-time constant one having the transfer function $G(s) = K/s(s + \lambda)$, which, for instance, corresponds to an armature-controlled dc-motor [3].

The paper is organized as follows: Sections 2 and 3 give a short overview of the classical solution of the above stated problem. Section 4 deals with the new technique of the multi-plant/one-controller philosophy, where this approach is introduced in single steps, starting from a simple 2-plant configuration. Section 5 presents the choice (optimization) of controller parameters. Examples for synchronization and tracking using computer simulation are considered in Section 6. The main results are finally summarized in Section 7.

2. CLASSICAL FEEDBACK SYSTEMS

Consider the classical feedback control system shown in Fig. 1. The system is comprised of an integral-plus-time constant plant: $G(s) = \frac{K}{s(s+\lambda)}$, controller $H(s)$ and a unity feedback. The transfer function of plant $G(s)$ is, say, that of an armature-controlled dc-motor with negligible armature inductance — with or without viscous friction damping [3]. As seen, we have been rather unorthodox in portraying also the output initial conditions: $y(0) = \alpha$, $\dot{y}(0) = \beta$. It is a matter of course that once stability of a linear closed-loop system is ascertained, then all the output initial conditions asymptotically tend to zero: the reason of being completely discarded. However, since throughout this work we shall show output initial conditions, then it would seem pertinent and plausible that right from the start they will be displayed. Referring to Fig. 1, we have

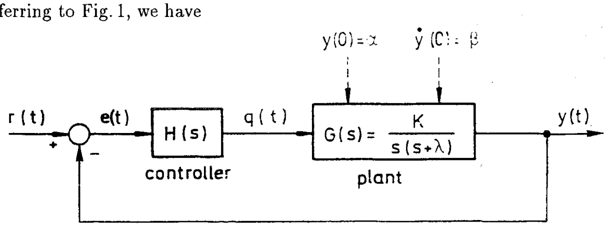

Fig. 1. Block diagram of classical feedback control system.

$$\ddot{y}(t) + \lambda\,\dot{y}(t) = K\,q(t) \tag{1}$$

Taking Laplace transform from both sides of Eq. (1), we get

$$s^2\,Y(s) + s\,y(0) - \dot{y}(0) + \lambda s Y(s) - \lambda y(0) = K\,Q(s) \tag{2}$$

where

$$y(0) = \alpha \tag{3.1}$$
$$\dot{y}(0) = \beta \tag{3.2}$$
$$Q(s) = \mathcal{L}[q(t)] = H(s)\,[R(s) - Y(s)]. \tag{3.3}$$

Upon substitution of Eqs. (3.1) to (3.3) in Eq. (2) and simplifying, we obtain

$$Y(s) = \frac{K\,H(s)}{s^2 + \lambda s + K\,H(s)}\,R(s) + \frac{s+\lambda}{s^2 + \lambda s + K\,H(s)}\,\alpha + \frac{1}{s^2 + \lambda s + K\,H(s)}\,\beta. \tag{4}$$

Let

$$e(t) = r(t) - y(t) \tag{5.1}$$

or

$$E(s) = R(s) - Y(s) \tag{5.2}$$

hence, from Eqs. (4) and (5.2), we get

$$E(s) = \frac{s[s+\lambda]}{s^2 + \lambda s + K\,H(s)}\,R(s) - \frac{[s+\lambda]\,\alpha + \beta}{s^2 + \lambda s + K\,H(s)}. \tag{6}$$

The task of the controller H(s) is to make the steady-state error equal to zero:

$$e(t)|_{t\to\infty} = 0 \Longrightarrow y(t)|_{t\to\infty} = r(t).$$

Two important points can be deduced from Eq. (6):

(i) System of Fig. 1 is stable *iff* all the roots of equation

$$s^2 + \lambda s + K\,H(s) = 0 \tag{7}$$

lie in the left-half of the s-plane.

(ii)

$$\begin{aligned}&[\text{No. of poles } R(s) \text{ at the origin of the } s\text{-plane}]-\\&[\text{No. of poles } H(s) \text{ at the origin of the } s\text{-plane}] = 1.\end{aligned} \tag{8}$$

Equation (8) implies that if $r(t)$ is the general polynomial

$$r(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_q t^q = \sum_{v=0}^{q} r_v t^v \tag{9.1}$$

or

$$R(s) = r_0\,\frac{0!}{s} + r_1\,\frac{1!}{s^2} + r_2\,\frac{2!}{s^3} + \cdots + r_q\,\frac{q!}{s^{q+1}} = \sum_{v=0}^{q} r_v\,\frac{v!}{s^{v+1}} \tag{9.2}$$

then $H(s)$ must be of the form

$$H(s) = A_0 + A_1\,\frac{1}{s} + A_2\,\frac{1}{s^2} + \cdots + A_q\,\frac{1}{s^q} = \sum_{h=0}^{q} A_h\,\frac{1}{s^h}. \tag{10}$$

Note: In Eq. (9.1), *some or all* the coefficients of $r(t)$ *except the last one* r_q may be zero; while in Eq. (10) *all* the parameters of $H(s)$, namely $A_0, A_1, A_2, \ldots, A_q$, must be *present* and all be *greater than zero*. Symbolically:

$$\begin{cases} r_q \neq 0 \\ A_0, A_1, A_2, \ldots, A_q > 0. \end{cases}$$

2.1. Step Response $[r(t) = r_0]$

Let

$$R(s) = r_0 \frac{1}{s} \quad \text{(step input, } q = 0) \tag{11}$$

$$H(s) = A_0 \quad (P\text{-controller}). \tag{12}$$

Substituting Eq. (12) in Eq. (7), we have

$$s^2 + \lambda s + K A_0 = 0 \tag{13}$$

both roots of Eq. (13) lie in the left-half of the s-plane $\forall A_0 > 0$. So, configuration of Fig. 1 *is stable* where the controller $H(s)$ is of P-type: $H(s) = A_0$, $A_0 > 0$. Upon substitution of Eqs. (11) and (12) in Eq. (6), we get

$$E(s) = \frac{[s+\lambda]\, r_0}{s^2 + \lambda s + K A_0} - \frac{[s+\lambda]\, \alpha + \beta}{s^2 + \lambda s + K A_0}$$

hence

$$e(t)|_{t\to\infty} = \lim_{s\to 0} s\, E(s) = 0 \Longrightarrow$$

$$y(t)|_{t\to\infty} = r(t) = r_0 .$$

2.2. Step-Plus-Ramp Response $[r(t) = r_0 + r_1 t]$

Let

$$R(s) = r_0 \frac{1}{s} + r_1 \frac{1}{s^2} \quad \text{(step-plus-ramp input, } q = 1),\ r_1 \neq 0 \tag{14}$$

$$H(s) = A_0 + A_1 \frac{1}{s} \quad (PI\text{-controller}). \tag{15}$$

Substituting Eq. (15) in Eq. (6) and simplifying, we get the characteristic equation

$$s^3 + \lambda s^2 + K A_0 s + K A_1 = 0. \tag{16}$$

The three roots of Eq. (16) would lie in the left-half of the s-plane, if $A_0,\ A_1 > 0$ and

$$\frac{A_1}{A_0} < \lambda \tag{17}$$

which is thoroughly studied in [3,4]. Again, if $A_0,\ A_1 > 0$ and inequality (17) is also satisfied, then the structure of Fig. 1 would be stable. Through substitution of Eqs. (14) and (15) in Eq. (6), one obtains

$$E(s) = \frac{[s+\lambda]\,[r_0 s + r_1]}{s^3 + \lambda s^2 + K A_0 s + K A_1} - \frac{[s+\lambda]\, \alpha + s\beta}{s^3 + \lambda s^2 + K A_0 s + K A_1}$$

hence

$$e(t)|_{t\to\infty} = \lim_{s\to 0} s\, E(s) = 0$$

$$y(t)|_{t\to\infty} = r(t) = r_0 + r_1 t$$

and so on and so forth. Inevitable and pertinent question: what should be the values of A_0 for the P-controller, and $A_0,\ A_1$ for the PI-controller? We shall deal with this important question after we have studied our proposed system in Section 4.

3. n-PLANT/n-CONTROLLER SYSTEM

Suppose that we have a fleet of n such identical integral-plus-time constant plants: $G(s) = \frac{K}{s(s+\lambda)}$, with, generally speaking, different output initial condition, but a *common command* input $r(t)$. The block structure of the system is shown in Fig. 2. To appreciate the practical application of Fig. 2: suppose one has n identical armature-controlled dc-motors with negligible armature inductances, without viscous friction damping or identical viscous friction damping. Furthermore, suppose the inertias on these identical dc-motors be also identical. So the transfer function of each one of these identical plants is given by [3]

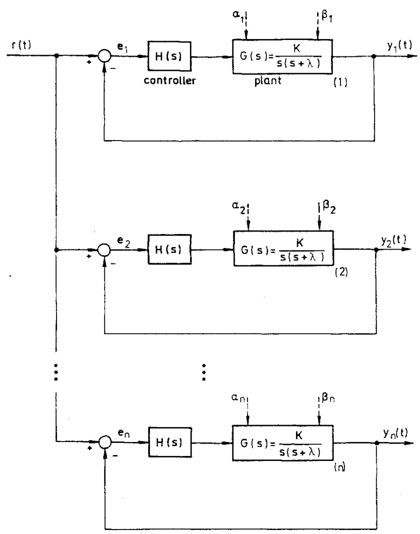

Fig. 2. Block diagram of n identical feedback system with a common input $r(t)$.

$$G(s) = \frac{Y_i(s)}{R(s)} = \frac{K}{s(s+\lambda)} \quad i = 1, 2, \ldots, n \tag{18}$$

where, for this particular example, $R(s)$ is the transform of common armature voltage and $Y_i(s)$ beeing the transform of angular displacement of motor shaft. Of course

$$\frac{Y_i(s)}{R(s)} = \frac{K}{s(s+\lambda)}$$

is meaningful if output initial conditions are zero: $\alpha_i = \beta_i = 0, \ i = 1, 2, ..., n$. If, however, both α_i and β_i are not zero, then $Y(s), \ i = 1, 2, ..., n$ must be represented as in Eq. (4). Consequently, as seen by Figs. 1 and 2, on the block diagrams inputs and outputs must be written as $r(t), \ y_i(t)$ and not as $R(s), \ Y_i(s), \ i = 1, 2, ..., n$.

From Fig. 2, it is observed that we have n isolated unity feed back systems, with a *common command* input $r(t)$. So, as in the case of one single unity feedback system discussed in Section 2, if $r(t)$ is a step one, then the n controllers $H(s)$ must be of P-type: $r(t) = r_0 \Longrightarrow H(s) = A_0, \ A_0 > 0$; while if $r(t)$ is a step-plus-ramp one, then the n controllers $H(s)$ must be of PI-type: $r(t) = r_0 + r_1 t, \ r_1 \neq 0 \Longrightarrow H(s) = A_0 + A_1 \frac{1}{s}, \ A_0, A_1 > 0$ and $\frac{A_1}{A_0} < \lambda$. It goes without saying that these P- or PI-controllers need not be identical. All that the configuration of Fig. 2 tells us is this that if $r(t) = r_0$ and $H(s) = A_0$, where for each one of the n feedback systems A_0 could have *different* positive values, then

$$y_i(t)|_{t\to\infty} = r(t) = r_0 \quad \forall \alpha_i, \beta_i \ i = 1, 2, \ldots, n.$$

Similarly, if $r(t) = r_0 + r_1 t$ and $H(s) = A_0 + A_1 \frac{1}{s}$, where, again, A_0 and A_1 could assume *different* positive values for each one of then controllers $H(s)$ as long as $\frac{A_1}{A_0} < \lambda$, then

$$y_i(t)|_{t\to\infty} = r(t) = r_0 + r_1 t \quad \forall \alpha_i, \beta_i \ i = 1, 2, \ldots, n.$$

As before, we do not want to involve ourselves at this stage as how one chooses values of A_0 for P-controllers: $H(s) = A_0$ and A_0, A_1 for PI-controllers: $H(s) = A_0 + A_1 \frac{1}{s}$.

So the structure of Fig. 2, which is, so to speak, the enlargement of Fig. 1, commands, in the most general sense, that tracking action of outputs of n plants can be achieved if n appropriate controllers $H(s)$ are commissioned. However, the question is justified whether we can have only *ONE* P-controller: $H(s) = A_0$ for the step input: $r(t) = r_0$ and *ONE* PI-controller: $H(s) = A_0 + A_1 \frac{1}{s}$ for the step-plus-ramp input: $r(t) = r_0 + r_1 t$. To generalize, if the command input is of polynomial type given by Eq. (9.1):

$$r(t) = \sum_{v=0}^{q} r_v \, t^v$$

then can we do with ONE controller given by Eq. (10):

$$H(s) = \sum_{h=0}^{q} A_h \frac{1}{s^h}?$$

The answer is *YES*. The attraction of the proposed scheme being that in *practical applications* this means tremendous saving in material cost.

To make the matter as plausible and comprehensible as possible, we shall first study the proposed technique for a '2-plant' system, then for a '3-plant' one. Armed with the technique and results obtained, the 'n-plant' case will be presented. A '3-part' example, which, most vividly, translates the theoretical results, will bring our study to a close.

4. THE NEW MULTI-PLANT/ONE-CONTROLLER PHILOSOPHY

4.1. 2-Plant/One-Controller System

Let us consider Fig. 3, which is comprised of 2 identical integral-plus-time constant plants: $G(s) = \frac{K}{s(s+\lambda)}$, with output initial conditions of (α_1, β_1), (α_2, β_2). The *common* input $x(t)$ is any function of time. The error signal formed between the two outputs: $\hat{e}_1(t) = y_1(t) - y_2(t)$ is *subtracted* from input $x(t)$ of plant (1) and *added* to input $x(t)$ of plant (2). If *all* the output initial conditions are zero: $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\hat{e}_1(t) = 0\ \forall t \geq 0$—see Eqs. (25.1) to (26.2). The formation of error signal $\hat{e}_1(t)$ is to *balance out* the possible *non-identical* output initial conditions and/or to eliminate the possible deterministic disturbances at the outputs $y_1(t)$ and $y_2(t)$. So, if $y_1(t) = y_2(t) \Longrightarrow e_1(t) = 0$: the feedback is *redundant* or *inactive*.

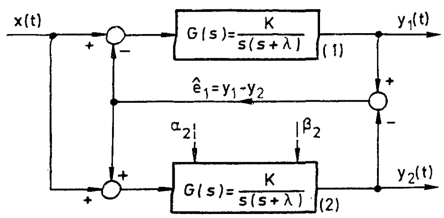

Fig. 3. Block diagram for interconnection of 2 identical plants: $G(s) = \frac{K}{s(s+\lambda)}$.

Now, from Fig. 3, we have

$$\ddot{y}_1(t) + \lambda\,\dot{y}_1(t) = K[x(t) - \hat{e}_1(t)] \tag{19.1}$$

$$\ddot{y}_2(t) + \lambda\,\dot{y}_2(t) = K[x(t) + \hat{e}_1(t)] \tag{19.2}$$

where

$$\hat{e}_1(t) = y_1(t) - y_2(t). \tag{20}$$

Substituting Eq. (20) in Eqs. (19.1) and (19.2), we get

$$\ddot{y}_1(t) + \lambda\,\dot{y}_1(t) + K\,y_1(t) - K\,y_2(t) = K\,x(t) \tag{21.1}$$

$$\ddot{y}_2(t) + \lambda\,\dot{y}_2(t) + K\,y_2(t) - K\,y_1(t) = K\,x(t). \tag{21.2}$$

Taking Laplace transform from both sides of Eqs. (21.1) and (21.2), and inserting the output initial conditions: $y_i(0) = \alpha_i$ and $\dot{y}_i(0) = \beta_i$, $i = 1, 2$, we obtain

$$[s^2 + \lambda s + K]\, Y_1(s) - K\, Y_2(s) = K\, X(s) + [s + \lambda]\, \alpha_1 + \beta_1 \tag{22.1}$$

$$[s^2 + \lambda s + K]\, Y_2(s) - K\, Y_1(s) = K\, X(s) + [s + \lambda]\, \alpha_2 + \beta_2. \tag{22.2}$$

Solving the simultaneous equations (22.1) and (22.2), we get

$$Y_1(s) = \frac{K}{s[s+\lambda)} X(s) + \frac{A_1(s)}{sD(s)} + \frac{B_1(s)}{s[s+\lambda]\, D(s)} \tag{23.1}$$

$$Y_2(s) = \frac{K}{s[s+\lambda)} X(s) + \frac{A_2(s)}{sD(s)} + \frac{B_2(s)}{s[s+\lambda]\, D(s)} \tag{23.2}$$

where

$$A_1(s) = \alpha_1 s^2 + \alpha_1 \lambda s + K \sum_{m=1}^{2} \alpha_m \tag{24.1}$$

$$A_2(s) = \alpha_2 s^2 + \alpha_2 \lambda s + K \sum_{m=1}^{2} \alpha_m \tag{24.2}$$

$$B_1(s) = \beta_1 s^2 + \beta_1 \lambda s + K \sum_{m=1}^{2} \beta_m \tag{24.3}$$

$$B_2(s) = \beta_2 s^2 + \beta_2 \lambda s + K \sum_{m=1}^{2} \beta_m \tag{24.4}$$

$$D(s) = s^2 + \lambda s + 2K. \tag{24.5}$$

It is obvious from Eqs. (24.1) to (24.5) that if: (i) $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, then

$$Y_1(s) = \frac{K}{s(s+\lambda)} X(s) \tag{25.1}$$

$$Y_2(s) = \frac{K}{s(s+\lambda)} X(s) \tag{25.2}$$

$$\Longrightarrow \quad y_1(t) = y_2(t) \quad \forall\, t \geq 0.$$

(ii) $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, then

$$Y_1(s) = \frac{K}{s(s+\lambda)} X(s) + \frac{1}{s}\alpha + \frac{1}{s(s+\lambda)}\beta \tag{26.1}$$

$$Y_2(s) = \frac{K}{s(s+\lambda)} X(s) + \frac{1}{s}\alpha + \frac{1}{s(s+\lambda)}\beta \tag{26.2}$$

$$\Longrightarrow \quad y_1(t) = y_2(t) \quad \forall\, t \geq 0.$$

Being either case, this implies that $\hat{e}_1(t) = y_1(t) - y_2 = 0 \;\forall\, t \geq 0$. That is, the feedback path formed for $\hat{e}_1(t)$ is, as was mentioned before, redundant or inactive.

In other words, either case (i) happening or case (ii), we now have two *isolated* plants: $G(s) = \frac{K}{s(s+\lambda}$ with a common input $x(t)$ — without any feedback action.

Now let us study Fig. 4, which is the same as Fig. 3 as far as common input $x(t)$ and outputs $y_1(t)$, $y_2(t)$ are concerned. The only difference between the two figures is that in Fig. 3, as was mentioned before, $x(t)$ is just any function of time; while in Fig. 4, $x(t)$ is the result of operation of controller $H(s)$ upon the *difference* between a *known* command input $r(t)$ and the output $y_1(t)$:

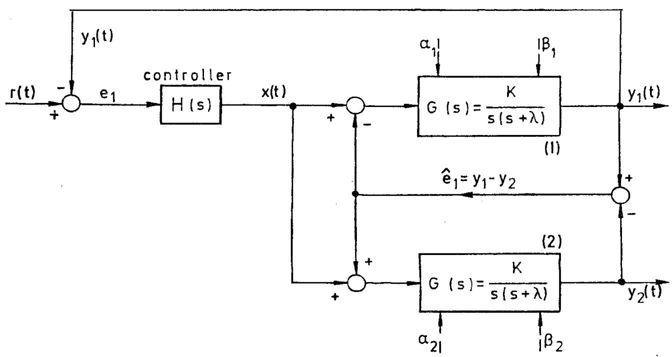

Fig. 4. Block diagram for tracking action of 2 identical plants: $G(s) = \frac{K}{s(s+\lambda)}$ with ONE controller $H(s)$.

$$e_1(t) = r(t) - y_1(t) \tag{27.1}$$

or

$$E_1(t) = R(s) - Y_1(s), \tag{27.2}$$

and

$$X(s) = E_1(s)\, H(S) \tag{28.1}$$

or

$$X(s) = [R(s) - Y_1(s)]\, H(s). \tag{28.2}$$

What we have done is simply this: the output $y_1(t)$ has been *fed back* to be *subtracted* from the *known* command input $r(t)$, forming an error signal $e_1(t) = r(t) - y_1(t)$. As seen, *common input* $x(t)$ is *not* just any function of time, as for Fig. 3, but the result of operation of controller $H(s)$ on this error signal $e_1(t)$ — as expressed by Eqs. (28.1) or (28.2). Of course, we could have as well taken the other output $y_2(t)$, but, as the final result would show, the end result would be the same. Throughout the rest of this paper, we assume that the *1st* output $y_1(t)$ is always taken to be subtracted from the common input $r(t)$. After all, there was *no* particular reason whatsoever as why we called one plant as plant (1) and the other as plant (2). So, always, that plant whose output is taken to be subtracted from r(t) shall be labelled as plant (1) and the remaining $n-1$ plants, in *any way* one wishes, as plant (2), plant (3),..., plant (n).

We shall now see how with this *one* controller $H(s)$ the steady-state outputs: $[y_1(t),\ y_2(t)]$ can be brought to *track* the known command input $r(t)$. Substituting Eq. (28.2) in Eqs. (23.1) and (23.2) and solving the arising simultaneous equations, we find $Y_1(s)$ and $Y_2(s)$ in relation to now the command input $R(s)$:

$$\begin{aligned} Y_1(s) &= \frac{K\,H(s)}{s^2+\lambda s+K\,H(s)}\,R(s)+\frac{[s+\lambda]\,A_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\ &\quad +\frac{B_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)}, \end{aligned} \tag{29.1}$$

$$\begin{aligned} Y_2(s) &= \frac{K\,H(s)}{s^2+\lambda s+K\,H(s)}\,R(s)+ \\ &\quad +\frac{[s+\lambda]\,A_2(s)+K\,H(s)\,[s+\lambda]\,[\alpha_2-\alpha_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\ &\quad +\frac{B_2(s)+K\,H(s)\,[\beta_2-\beta_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)}, \end{aligned} \tag{29.2}$$

where the expressions for $A_1(s)$, $A_2(s)$, $B_1(s)$, $B_2(s)$, $D(s)$ are those given by Eqs. (24.1) to (24.5), respectively.

Now, let

$$E_1(s) = R(s)-Y_1(s), \tag{30.1}$$

$$E_2(s) = R(s)-Y_2(s), \tag{30.2}$$

therefore, from Eqs. (29.1) and (29.2), we have

$$\begin{aligned} E_1(s) &= \frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s)-\frac{[s+\lambda]\,A_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\ &\quad -\frac{B_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)}, \end{aligned} \tag{31.1}$$

$$\begin{aligned} E_2(s) &= \frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s)-\frac{[s+\lambda]\,A_2(s)+K\,H(s)\,[s+\lambda]\,[\alpha_2-\alpha_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\ &\quad -\frac{B_2(s)+K\,H(s)\,[\beta_2-\beta_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)}. \end{aligned} \tag{31.2}$$

Note:

Comparing Eqs. (29.1) and (29.2) with Eq. (4), it is observed that the *first* term on the right-hand side of equality sign, namely $[KH(s)/s^2+\lambda s+KH(s)]\,R(s)$, is *exactly the same*. Consequently, comparison of Eqs. (31.1) and (31.2) with Eq. (6) shows that the *first* term on the right-hand of equality sign, that is $\{s[s+\lambda]/s^2+$

$\lambda s + KH(s)\} R(s)$, is *completely identical*. It has been seen that for configuration of Fig. 1 *iff* all the roots of Eq. (7): $s^2 + \lambda s + KH(s) = 0$ would lie in the left-hand of the s-plane, then the steady-state values of both terms on the right-hand of Eq. (6) would be zero. Hence, $e(t)|_{t\to\infty} = 0 \Longrightarrow y(t)|_{t\to\infty} = r(t)$. The denominators of *second* and *third* terms of Eqs. (31.1) and (31.2) on the right-hand side of equality sign have also the term $D(s)$, which is given by Eq. (24.5): $D(s) = s^2 + \lambda s + 2K$. Since $\lambda \,\&\, K > 0$, hence both roots of $D(s) = 0$ always lie in the left-hand of the s-plane: no stability concern with regard to $D(s)$. So we must concern ourselves *only* with the *first* term on the right-hand side of Eqs. (31.1) and (31.2) for *tracking* action, that is

$$\frac{s[s+\lambda]}{s^2 + \lambda s + K\, H(s)}\, R(s)$$

and the roots of its denominator for *stability*, namely

$$s^2 + \lambda s + K\, H(s) = 0. \tag{32}$$

Now, as was stated above, since the first term on the right-hand side of Eqs. (31.1) and (31.2) is exactly the same as that of Eq. (6), hence, as of structure of Fig. 1, if in Fig. 4 the command input $r(t)$ is of polynomial type given by Eq. (9.1):

$$r(t) = \sum_{v=0}^{q} r_v\, t^v$$

then the one controller $H(s)$ of Fig. 4 must have the transfer function given by Eq. (10):

$$H(s) = \sum_{h=0}^{q} A_h\, \frac{1}{s^h}.$$

So, as we saw before for Fig. 1, if command input $r(t)$ is a step one: $r(t) = r_0$, then controller $H(s)$ of Fig. 4 must be of P-type: $H(s) = A_0$. Upon substitution of $H(s) = A_0$ in Eq. (32), which is the same as Eq. (7), we get

$$s^2 + \lambda s + K\, A_0 = 0.$$

Both roots of the above quadratic equation lie in the left-hand of the s-plane, $\forall A_0 > 0$. Hence, from Eqs. (31.1) and (31.2), we have

$$\begin{aligned} e_1(t)|_{t\to\infty} &= e_2(t)|_{t\to\infty} = 0 \Longrightarrow \\ y_1(t)|_{t\to\infty} &= y_2(t)|_{t\to\infty} = r(t) = r_0. \end{aligned}$$

Likewise, if the command input $r(t)$ is step-plus-ramp one: $r(t) = r_0 + r_1\, t,\ r_1 \neq 0$, then the controller $H(s)$ of Fig. 4 should be of PI-type: $H(s) = A_0 + A_1\, \frac{1}{s}$. Substituting this new $H(s)$ in Eq. (32), we obtain the characteristic equation:

$$s^3 + \lambda\, s^2 + K\, A_0 s + K\, A_1 = 0.$$

All the three roots of the above equation lie in the left-half of the s-plane, if

$$A_0, A_1 > 0 \quad \text{and} \quad \frac{A_1}{A_0} < \lambda.$$

Being so, again from Eqs. (31.1) and (31.2), we get

$$\begin{aligned} e_1(t)|_{t\to\infty} &= e_2(t)|_{t\to\infty} = 0 \Longrightarrow \\ y_1(t)|_{t\to\infty} &= y_2(t)|_{t\to\infty} = r(t) = r_0 + r_1 t. \end{aligned}$$

These results resemble exactly that of Fig. 1. So, *one* P-controller: $H(s) = A_0$ for $r(t) = r_0$ or *one* PI-controller: $H(s) = A_0 + A_1 \frac{1}{s}$ for $r(t) = r_0 + r_1 t$ would bring the steady-state outputs $y_1(t)$ and $y_2(t)$ to *track* the command input $r(t)$.

4.2. 3-Plant/One-Controller System

We now combine Figs. 3 and 4 in a single Figure for a '3-plant' system, which is shown in Fig. 5.

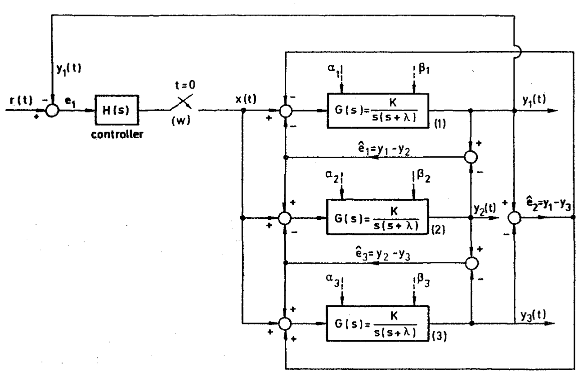

Fig. 5. Block diagram for tracking action of 3 identical plants: $G(s) = \frac{K}{s(s+\lambda)}$ with ONE controller $H(s)$.

If switch (w) is *open*, then the *common* input $x(t)$ is *any* function of time, as in Fig. 3; while if it is *closed*, then $x(t)$ is the result of operation of controller $H(s)$ upon the difference between the *known* command input $r(t)$ and the output $y_1(t)$, as in Fig. 4. Reader's special attention is invoked upon in observering how the 3 plants are *interconnected*, which is very important for both *stability* requirement and *tracking* behaviour: *Each one* of the outputs $y_1(t)$, $y_2(t)$, $y_3(t)$ is *compared* with the *other two* outputs to form the following *three possible* error signals:

$$\hat{e}_1(t) = y_1(t) - y_2(t) \tag{33.1}$$
$$\hat{e}_2(t) = y_1(t) - y_3(t) \tag{33.2}$$
$$\hat{e}_3(t) = y_2(t) - y_3(t). \tag{33.3}$$

These error signals are then *added* to or *subtracted* from inputs $x(t)$ of the three plants in the manner clearly shown in Fig. 5, which is of the same principle as for the simplest case shown in Fig. 3. Everything said in Section 4.1 regarding the presence or non-presence of the error signal $\hat{e}_1(t)$ is also true now for the three error signals $\hat{e}_1(t)$, $\hat{e}_2(t)$, $\hat{e}_3(t)$.

Now, suppose for the present that in Fig. 5 the switch (w) is *open* and the common input $x(t)$ is just *any* function of time. Similar to Eqs. (19.1) and (19.2), we get

$$\ddot{y}_1(t) + \lambda\dot{y}_1(t) = K[x(t) - \hat{e}_1(t) - \hat{e}_2(t)] \tag{34.1}$$
$$\ddot{y}_2(t) + \lambda\dot{y}_2(t) = K[x(t) + \hat{e}_1(t) - \hat{e}_3(t)] \tag{34.2}$$
$$\ddot{y}_3(t) + \lambda\dot{y}_3(t) = K[x(t) + \hat{e}_2(t) + \hat{e}_3(t)]. \tag{34.3}$$

Substituting Eqs. (33.1) to (33.3) in Eqs. (34.1) to (34.3), we get

$$\ddot{y}_1(t) + \lambda\dot{y}_1(t) + 2K\,y_1(t) - K\,y_2(t) - K\,y_3(t) = K\,x(t) \tag{35.1}$$
$$\ddot{y}_2(t) + \lambda\dot{y}_2(t) + 2K\,y_2(t) - K\,y_1(t) - K\,y_3(t) = K\,x(t) \tag{35.2}$$
$$\ddot{y}_3(t) + \lambda\dot{y}_3(t) + 2K\,y_3(t) - K\,y_1(t) - K\,y_2(t) = K\,x(t). \tag{35.3}$$

Taking Laplace transform from both sides of Eqs. (35.1) to (35.3), and inserting the output initial conditions: $y_i(0) = \alpha_i$ and $\dot{y}_i(0) = \beta$ $i = 1, 2, 3$, we have as in Eqs. (22.1) and (22.2):

$$[s^2 + \lambda s + 2K]\,Y_1(s) - K\,Y_2(s) - K\,Y_3(s) = K\,X(s) + [s + \lambda]\,\alpha_1 + \beta_1 \tag{36.1}$$
$$[s^2 + \lambda s + 2K]\,Y_2(s) - K\,Y_1(s) - K\,Y_3(s) = K\,X(s) + [s + \lambda]\,\alpha_2 + \beta_2 \tag{36.2}$$
$$[s^2 + \lambda s + 2K]\,Y_3(s) - K\,Y_1(s) - K\,Y_2(s) = K\,X(s) + [s + \lambda]\,\alpha_3 + \beta_3. \tag{36.3}$$

Solving the simultaneous equations (36.1) to (36.3), we get as in Eqs. (23.1) and (23.2):

$$Y_1(s) = \frac{K}{s[s+\lambda]}\,X(s) + \frac{A_1(s)}{s\,D(s)} + \frac{B_1(s)}{s[s+\lambda]\,D(s)} \tag{37.1}$$
$$Y_2(s) = \frac{K}{s[s+\lambda]}\,X(s) + \frac{A_2(s)}{s\,D(s)} + \frac{B_2(s)}{s[s+\lambda]\,D(s)} \tag{37.2}$$
$$Y_3(s) = \frac{K}{s[s+\lambda]}\,X(s) + \frac{A_3(s)}{s\,D(s)} + \frac{B_3(s)}{s[s+\lambda]\,D(s)} \tag{37.3}$$

where, similar to Eqs. (24.1) to (24.5):

$$A_i(s) = \alpha_i s^2 + \alpha_i \lambda s + K \sum_{m=1}^{3} \alpha_m \quad i = 1,2,3 \tag{38.1}$$

$$B_i(s) = \beta_i s^2 + \beta_i \lambda s + K \sum_{m=1}^{3} \beta_m \quad i = 1,2,3 \tag{38.2}$$

$$D(s) = s^2 + \lambda s + 3K. \tag{38.3}$$

Let us now return to Fig. 5 again. Suppose that at time $t = 0$: (i) a *known* command input $r(t)$ has been applied to the system, (ii) simultaneously, i.e. at time $t = 0$, the switch (w) has been *closed*. As seen by Fig. 5, similar to Fig. 4, $x(t)$ is *not* now just any function of time, but the result of operation of controller $H(s)$ upon the difference between the command input $r(t)$ and the output $y_1(t)$. So similar to Eqs. (28.1) and (28.2), we have

$$\text{or} \quad \begin{cases} X(x) = E_1(s)\, H(s) \\ X(s) = [R(s) - Y_1(s)]\, H(s). \end{cases}$$

Substituting the second of the above two equations in Eqs. (37.1) to (37.3), and solving the arising simultaneous equations, we get

$$\begin{aligned} Y_1(s) &= \frac{K\,H(s)}{s^2 + \lambda s + K\,H(s)} R(s) + \frac{[s+\lambda]\,A_1(s)}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \\ &+ \frac{B_1(s)}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \end{aligned} \tag{39.1}$$

$$\begin{aligned} Y_2(s) &= \frac{K\,H(s)}{s^2 + \lambda s + K\,H(s)} R(s) + \frac{[s+\lambda]\,A_2(s) + K\,H(s)\,[s+\lambda]\,[\alpha_2 - \alpha_1]}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \\ &+ \frac{B_2(s) + K\,H(s)\,[\beta_2 - \beta_1]}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \end{aligned} \tag{39.2}$$

$$\begin{aligned} Y_3(s) &= \frac{K\,H(s)}{s^2 + \lambda s + K\,H(s)} R(s) + \frac{[s+\lambda]\,A_3(s) + K\,H(s)\,[s+\lambda]\,[\alpha_3 - \alpha_1]}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \\ &+ \frac{B_3(s) + K\,H(s)\,[\beta_3 - \beta_1]}{[s^2 + \lambda s + K\,H(s)]\,D(s)} \end{aligned} \tag{39.3}$$

where the expressions for $A_1(s)$, $A_2(s)$, $A_3(s)$ are obtained from Eq. (38.1), for $B_1(s)$, $B_2(s)$, $B_3(s)$ from Eq. (38.2) and Eq. (38.3) provides us with $D(s)$.

As in the 2-plant case, let

$$E_1(s) = R(s) - Y_1(s) \tag{40.1}$$

$$E_2(s) = R(s) - Y_2(s) \tag{40.2}$$

$$E_3(s) = R(s) - Y_3(s). \tag{40.3}$$

Therefore, from Eqs. (39.1) to (39.3), we get

$$\begin{aligned}
E_1(s) &= \frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s) - \frac{[s+\lambda]\,A_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\
&- \frac{B_1(s)}{[s^2+\lambda s+K\,H(s)]\,D(s)} \qquad (41.1)\\
E_2(s) &= \frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s) - \frac{[s+\lambda]\,A_2(s)+K\,H(s)\,[s+\lambda]\,[\alpha_2-\alpha_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\
&- \frac{B_2(s)+K\,H(s)\,[\beta_2-\beta_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)} \qquad (41.2)\\
E_3(s) &= \frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s) - \frac{[s+\lambda]\,A_3(s)+K\,H(s)\,[s+\lambda]\,[\alpha_3-\alpha_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)} \\
&- \frac{B_3(s)+K\,H(s)\,[\beta_3-\beta_1]}{[s^2+\lambda s+K\,H(s)]\,D(s)}. \qquad (41.3)
\end{aligned}$$

Again, as in the 2-plant case, it is observed that the *first* term on the right-hand side of Eqs. (39.1) to (39.3), that is $[K\,H(s)/s^2+\lambda s+K\,H(s)]\,R(s)$, is the same as that of Eq. (4): similarly, the *first* term on the right-hand side of Eqs. (41.1) to (41.3), namely $\{s[s+\lambda]/s^2+\lambda s+K\,H(s)\}\,R(s)$, is identical with that of Eq. (6). The denominators of the second and third terms on the right-hand side of Eqs. (41.1) to (41.3) contain the term $D(s)$, which is given by Eq. (38.3): $D(s) = s^2+\lambda s+3K$. Since $\lambda\,\&\,K > 0$, hence, as before, both roots of $D(s) = 0$ always lie in the left-half of the s-plane. So, as in the 2-plant case, one must concern oneself only with the *first* term on the right-hand side of Eqs. (41.1) to (41.3) for *tracking* action, namely

$$\frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s)$$

and the roots of its denominator for stability, that is

$$s^2+\lambda s+K\,H(s) = 0.$$

As was stated for the '2-plant' case, if the common input $r(t)$ is of polynomial type given by Eq. (9.1):

$$r(t) = \sum_{v=0}^{q} r_v\,t^v.$$

then the *one* controller $H(s)$ of Fig. 5 must have the transfer function given by Eq. (10):

$$H(s) = \sum_{h=0}^{q} A_h\,\frac{1}{s^h}.$$

(i) $r(t) = r_0 \Longrightarrow H(s) = A_0$
$e_1(t)|_{t\to\infty} = e_2(t)|_{t\to\infty} = e_3(t)|_{t\to\infty} = 0 \Longrightarrow$
$y_1(t)|_{t\to\infty} = y_2(t)|_{t\to\infty} = y_3(t)|_{t\to\infty} = r(t) = r_0,$ if

$A_0 > 0.$

(ii) $r(t) = r_0 + r_1 t \Longrightarrow H(s) = A_0 + A_1 \dfrac{1}{s}$

$e_1(t)|_{t\to\infty} = e_2(t)|_{t\to\infty} = e_3(t)|_{t\to\infty} = 0 \Longrightarrow$

$y_1(t)|_{t\to\infty} = y_2(t)|_{t\to\infty} = y_3(t)|_{t\to\infty} = r(t) = r_0 + r_1 t,$ if

$A_0, A_1 > 0$ and $\dfrac{A_1}{A_0} < \lambda.$

Therefore, the arrangement of Fig. 5 for a '3-plant' system with *one* controller $H(s)$ will make the three steady-state outputs to *track* the command input $r(t)$, $H(s) = A_0$ for $r(t) = r_0$ and $H(s) = A_0 + A_1 \frac{1}{s}$ for $r(t) = r_0 + r_1 t$; while configuration of Fig. 2, with $n = 3$, would have required *three* such P- or PI-controllers. Now we shall use the results of '2- and 3-plant' system for the general 'n-plant' case.

4.3. n-Plant/*One*-Controller System

As in the case of '2- and 3-plant' systems, *tracking* action of an 'n-plant' system can be achieved in two stages:

(i) *Interconnection* of n plants, as in Fig. 3 to 5.

(ii) Taking *any one* of the outputs, say the 1*st* one, to be compared with the command input $r(t)$, as shown in Figs. 4 and 5.

The block diagram of the system is shown in Fig. 6. Let us elaborate on stage (i): For the interconnection of the n plants, as clearly seen in Figs. 3 to 5, *two* important rules must be strictly observed:

(a) Formation of *all* 'N' possible error signals $(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_N)$ from the 'n' plant outputs.

(b) Feeding back each one of these 'N' error signals for *addition* to or *subtraction from* the inputs $x(t)$ of the two plants which have produced this error signal.

The rules for addition to or subtraction from the inputs $x(t)$ of the two plants are of great significance, as far as the *stability* of the whole system is concerned, which is clearly seen in Figs. 3 to 5. For instance, in Fig. 5 since $\hat{e}_3 = y_2 - y_3$, then $\hat{e}_3$ must be *subtracted from* input $x(t)$ of plant (2) and *added to* input $x(t)$ of plant (3). Of course we could have made $\hat{e}_3 = y_3 - y_2$, thereupon $\hat{e}_3$ must now be *added to* input $x(t)$ of plant (2) and *subtracted from* input $x(t)$ of plant (3).

The relation between the total number of error signals 'N', which *can and must* be formed, and the 'n' outputs of the plants has been proved before [5] and is given by

$$N = \frac{1}{2}n(n-1).$$

For example, $n = 5 \Longrightarrow N = 10$. The '$N = 10$' error signals which can and must be formed are:

$$\hat{e}_1 = y_1 - y_2,\quad \hat{e}_2 = y_1 - y_3,\quad \hat{e}_3 = y_1 - y_4,\quad \hat{e}_4 = y_1 - y_5,\quad \hat{e}_5 = y_2 - y_3,$$

$$\hat{e}_6 = y_2 - y_4,\quad \hat{e}_7 = y_2 - y_5,\quad \hat{e}_8 = y_3 - y_4,\quad \hat{e}_9 = y_3 - y_5,\quad \hat{e}_{10} = y_4 - y_5.$$

So,

input to each plant is: $x(t)$ and ± 4 error signals.

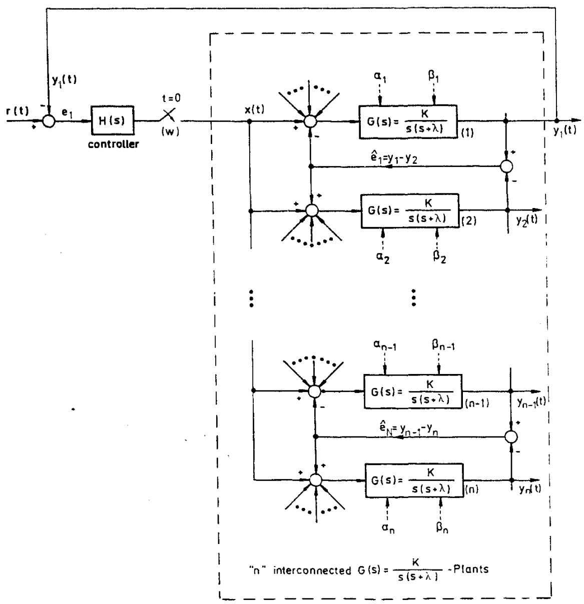

Fig. 6. Block diagram for tracking action of n plants: $G(s) = \frac{K}{s(s+\lambda)}$ with ONE controller $H(s)$.

Hence in Fig. 6, if, for instance, $n = 5 \Longrightarrow N = 10$, then

input to plant (1) is: $x(t) - \hat{e}_1(t) - \hat{e}_2(t) - \hat{e}_3(t) - \hat{e}_4(t)$
input to plant (2) is: $x(t) + \hat{e}_1(t) - \hat{e}_5(t) - \hat{e}_6(t) - \hat{e}_7(t)$
...
input to plant (5) is: $x(t) + \hat{e}_4(t) + \hat{e}_7(t) + \hat{e}_9(t) + \hat{e}_{10}(t)$

Still more insight to the interconnection of the plants can be cited in [5, 6, 7].

Let us return to Fig. 6, assuming for the present that the switch (w) is *open* and the common input $x(t)$ is *any* function of time. Hence, similar to Eqs. (19.1), (19.2) and (34.1) to (34.3), we can write

$$\ddot{y}_1(t) + \lambda\dot{y}_1(t) = K[x(t) - \hat{e}_1(t)\ldots] \tag{42.1}$$

$$\ddot{y}_2(t) + \lambda\dot{y}_2(t) = K[x(t) + \hat{e}_1(t)\ldots] \tag{42.2}$$

$$\vdots$$

$$\ddot{y}_{n-1}(t) + \lambda\dot{y}_{n-1}(t) = K[x(t)\cdots - \hat{e}_N(t)] \tag{42.n-1}$$

$$\ddot{y}_n(t) + \lambda\dot{y}_n(t) = K[x(t)\cdots + \hat{e}_N(t)] \tag{42.n}$$

where, as in the '2- and 3-plant' case

$$\hat{e}_1(t) = y_1(t) - y_2(t) \tag{43.1}$$

$$\vdots$$

$$\hat{e}_N(t) = y_{n-1}(t) - y_n(t) \tag{43.n}$$

Referring to Eqs. (19.1), (19.2) and (34.1) to (34.3), it is easily deduced that inside each one of the square brackets in Eqs. (42.1) to (42.n) there are: $x(t) \pm$ '$n-1$' error signals. So, similar to Eqs. (21.1), (21.2) and (35.1) to (35.3), upon substitution of Eqs. (43.1) to (43.n) in Eqs. (42.1) to (42.n), we have

$$\ddot{y}_1(t) + \lambda\dot{y}_1(t) + (n-1)\,Ky_1(t) - Ky_2(t) - Ky_3(t) - \cdots - Ky_n(t) = Kx(t) \tag{44.1}$$

$$\ddot{y}_2(t) + \lambda\dot{y}_2(t) + (n-1)\,Ky_2(t) - Ky_1(t) - Ky_3(t) - \cdots - Ky_n(t) = Kx(t) \tag{44.2}$$

$$\vdots$$

$$\ddot{y}_n(t) + \lambda\dot{y}_n(t) + (n-1)\,Ky_n(t) - Ky_1(t) - Ky_2(t) - \cdots - Ky_{n-1}(t) = Kx(t). \tag{44.n}$$

Taking Laplace transform from both sides of Eqs. (44.1) to (44.n), and inserting the output initial conditions:

$$\left.\begin{array}{l} y_i(0) = \alpha_i \\ \dot{y}_i(0) = \beta_i \end{array}\right. \quad i = 1, 2, \ldots, n$$

we get

$$[s^2+\lambda s(n-1)K]Y_1(s)-KY_2(s)-KY_3(s)-\cdots-KY_n(s)=KX(s)+[s+\lambda]\alpha_1+\beta_1 \tag{45.1}$$

$$[s^2+\lambda s(n-1)K]Y_2(s)-KY_1(s)-KY_3(s)-\cdots-KY_n(s)=KX(s)+[s+\lambda]\alpha_2+\beta_2 \tag{45.2}$$

$$\vdots$$

$$[s^2+\lambda s(n-1)K]Y_n(s)-KY_1(s)-KY_2(s)-\cdots-KY_{n-1}(s)=KX(s)+[s+\lambda]\alpha_n+\beta_n \tag{45.n}$$

Now, it is all-too-clear that Eqs. (22.1), (22.2) and (36.1) to (36.3) are *special cases* of Eqs. (45.1) to (45.n): $n = 2$ and $n = 3$, respectively.

It should be remembered, as in the previous two cases, that $X(s)$ in Eqs. (45.1) to (45.n) is the transform of *any* function of time $x(t)$.

From now on, matrix algebra can be commissioned, as in [5, 7], to finally get expressions relating outputs $Y_1(s)$, $Y_2(s), \ldots, Y_n(s)$ to command input $R(s)$—which, of course, is our final goal. However, scrutiny of results for the '2- and 3-plant' system would enable us to write down, stage-by-stage, all the results we require:
Referring to Eqs. (23.1), (23.2) and (37.1) to (37.3), we can write

$$Y_1(s) = \tfrac{K}{s[s+\lambda]}\,X(s) + \tfrac{A_1(s)}{s\,D(s)} + \tfrac{B_1(s)}{s[s+\lambda]\,D(s)} \tag{46.1}$$

$$Y_2(s) = \tfrac{K}{s[s+\lambda)}\,X(s) + \tfrac{A_2(s)}{s\,D(s)} + \tfrac{B_2(s)}{s[s+\lambda]\,D(s)} \tag{46.2}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$Y_n(s) = \tfrac{K}{s[s+\lambda)}\,X(s) + \tfrac{A_n(s)}{s\,D(s)} + \tfrac{B_n(s)}{s[s+\lambda]\,D(s)} \tag{46.n}$$

From Eqs. (24.1), (24.2) and (38.1):

$$A_i(s) = \alpha_i s^2 + \alpha_i \lambda s + K \sum_{m=1}^{n} \alpha_m \quad i = 1, 2, \ldots, n \tag{47.1}$$

and from Eqs. (24.3), (24.4) and (38.2):

$$B_i(s) = \beta_i s^2 + \beta_i \lambda s + K \sum_{m=1}^{n} \beta_m \quad i = 1, 2, \ldots, n \tag{47.2}$$

finally, from Eqs. (24.5) and (38.3):

$$D(s) = s^2 + \lambda s + n\,K. \tag{47.3}$$

Returning to Fig. 6 once more, assume, as for the '3-plant' case shown in Fig. 5, that at time $t = 0$ the command input $r(t)$ has been applied to the system and, simultaneously, the switch (w) has been *closed*. So, as in the previous two cases

$$\text{or} \begin{cases} X(s) = E_1(s)\,H(s) \\ X(s) = [R(s) - Y_1(s)]\,H(s). \end{cases}$$

Upon substitution of the second of the above two equations in Eqs. (46.1) to (46.n), and solving the arising n simultaneous equations, one obtains $Y_1(s)$, $Y_2(s), \ldots, Y_n(s)$ in terms of the command input $R(s)$:

$$Y_1(s) = \tfrac{KH(s)}{s^2+\lambda s+KH(s)} R(s) + \tfrac{[s+\lambda]A_1(s)}{[s^2+\lambda s+KH(s)]D(s)} + \tfrac{B_1(s)}{[s^2+\lambda s+KH(s)]D(s)} \tag{48.1}$$

$$Y_2(s) = \tfrac{KH(s)}{s^2+\lambda s+KH(s)} R(s) + \tfrac{[s+\lambda]A_2(s)+KH(s)[s+\lambda][\alpha_2-\alpha_1]}{[s^2+\lambda s+KH(s)]D(s)} + \tfrac{B_2(s)+KH(s)[\beta_2-\beta_1]}{[s^2+\lambda s+KH(s)]D(s)} \tag{48.2}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$Y_n(s) = \tfrac{KH(s)}{s^2+\lambda s+KH(s)} R(s) + \tfrac{[s+\lambda]A_n(s)+KH(s)[s+\lambda][\alpha_n-\alpha_1]}{[s^2+\lambda s+KH(s)]D(s)} + \tfrac{B_n(s)+KH(s)[\beta_n-\beta_1]}{[s^2+\lambda s+KH(s)]D(s)} \tag{48.n}$$

where the expressions for $A_1(s)$, $A_2(s), \dots, A_n(s)$ are obtained from Eq. (47.1), for $B_1(s)$, $B_2(s), \dots, B_n(s)$ from Eq. (47.2) and Eq. (47.3) furnishes us with $D(s)$.

As in the '2- and 3-plant' case, let

$$E_1(s) = R(s) - Y_1(s) \tag{49.1}$$

$$E_2(s) = R(s) - Y_2(s) \tag{49.2}$$

$$\vdots$$

$$E_n(s) = R(s) - Y_n(s) \tag{49.n}$$

Therefore, from Eqs. (48.1) to (48.n), we have

$$E_1(s) = \frac{s[s+\lambda]}{s^2+\lambda s+KH(s)} R(s) - \frac{[s+\lambda]A_1(s)}{[s^2+\lambda s+KH(s)]D(s)} - \frac{B_1(s)}{[s^2+\lambda s+KH(s)]D(s)} \tag{50.1}$$

$$E_2(s) = \frac{s[s+\lambda]}{s^2+\lambda s+KH(s)} R(s) - \frac{[s+\lambda]A_2(s)+KH(s)[s+\lambda][\alpha_2-\alpha_1]}{[s^2+\lambda s+KH(s)]D(s)} - \frac{B_2(s)+KH(s)[\beta_2-\beta_1]}{[s^2+\lambda s+KH(s)]D(s)} \tag{50.2}$$

$$\vdots$$

$$E_n(s) = \frac{s[s+\lambda]}{s^2+\lambda s+KH(s)} R(s) - \frac{[s+\lambda]A_n(s)+KH(s)[s+\lambda][\alpha_n-\alpha_1]}{[s^2+\lambda s+KH(s)]D(s)} - \frac{B_n(s)+KH(s)[\beta_n-\beta_1]}{[s^2+\lambda s+KH(s)]D(s)} \tag{50.n}$$

So, as in the previous two cases: $n = 2$ and $n = 3$, the *first* term on the right-hand side of Eqs. (48.1) to (48.n), namely $[K\,H(s)/s^2 + \lambda s + K\,H(s)]\,R(s)$, is the same as that of Eq. (4). Also, the *first* term on the right-hand side of Eqs. (50.1) to (50.n), that is $\{s[s+\lambda]/s^2 + \lambda s + K\,H(s)\}\,R(s)$, is identical with that of Eq. (6). The denominators of the second and third term on the right-hand side of Eqs. (50.1) to (50.n) contain the term $D(s)$, which is geiven by Eq. (47.3): $D(s) = s^2 + \lambda s + n\,K$. Since $\lambda\,\&\,K > 0$, hence, as in previous two cases, both roots of $D(s) = 0$ always lie in the left-half of the s-plane. Again, we must concern ourselves only with the *first* term on the right-hand side of Eqs. (50.1) to (50.n) for *tracking* action, namely

$$\frac{s[s+\lambda]}{s^2+\lambda s+K\,H(s)}\,R(s)$$

and the roots of its denominable for *stability*, that is

$$s^2 + \lambda s + K\,H(s) = 0.$$

So, as in the previous two cases: $n = 2$ und $n = 3$, if the command input $r(t)$ is of polynomial type given by Eq. (9.1):

$$r(t) = \sum_{v=0}^{q} r_v\, t^v$$

then the one controller $H(s)$ of Fig. 6 must have the transfer function given by Eq. (10):

$$H(s) = \sum_{h=0}^{q} A_h \frac{1}{s^h}.$$

(i) $r(t) = r_0 \Longrightarrow H(s) = A_0$

$e_1(t)|_{t\to\infty} = e_2(t)|_{t\to\infty} = \cdots = e_n(t)|_{t\to\infty} = 0 \Longrightarrow$

$y_1(t)|_{t\to\infty} = y_2(t)|_{t\to\infty} = \cdots = y_n(t)|_{t\to\infty} = r(t) = r_0$, if

$A_0 > 0$.

(ii) $r(t) = r_0 + r_1 t \Longrightarrow H(s) = A_0 + A_1 \frac{1}{s}$

$e_1(t)|_{t\to\infty} = e_2(t)|_{t\to\infty} = \cdots = e_n(t)|_{t\to\infty} = 0 \Longrightarrow$

$y_1(t)|_{t\to\infty} = y_2(t)|_{t\to\infty} = \cdots = y_n(t)|_{t\to\infty} = r(t) = r_0 + r_1 t$, if

$A_0, A_1 > 0$ and $\frac{A_1}{A_0} < \lambda$.

Therefore, the architecture of Fig. 6 for an 'n-plant' system with *one* controller $H(s)$ will make the n steady-steady outputs to *track* the command input $r(t)$, $H(s) = A_0$ for $r(t) = r_0$ and $H(s) = A_0 + A_1 \frac{1}{s}$ for $r(t) = r_0 + r_1 t$; while the classical, and generally accepted, arrangement of Fig. 2 would have required n such P- or PI-controllers ...

5. CHOICE OF CONTROLLER PARAMETERS $(A_0, A_1, \ldots)$

Stability and *optimality* are, in hierarchical order, the two most celebrated problems associated with control theory and design. Considering again the classical linear feedback control system shown in Fig. 1, once, as we have clearly seen, it has been decided what the transfer function of an *appropriate* controller $H(s)$ should be, in order that the steady-state output $y(t)$ would track the command input $r(t)$, then, naturally, this question arises: what values the parameters $(A_0, A_1, \ldots)$ of controller $H(s)$ should assume in order that a certain *cost function*, or *performance criterion*, J is minimized. Among many options [3], there has always been a particular attention and interest towards minimization of *integral square-error*:

$$J = \int_0^\infty e^2(t)\,\mathrm{d}t. \tag{51}$$

As was shown in Eq. (6), the transform $E(s)$ of the error signal $e(t)$ for the unity feed back system of Fig. 1 with $G(s) = \frac{K}{s(s+\lambda)}$, is given by

$$E(s) = \frac{s[s+\lambda]}{s^2 + \lambda s + K\,H(s)}\,R(s) - \frac{[s+\lambda]\,\alpha + \beta}{s^2 + \lambda s + K\,H(s)} \tag{6}$$

where $R(s)$ is the transform of command input $r(t)$, $\alpha = y(0)$ and $\beta = \dot{y}(0)$. Let us, for the sake of simplicity and explicity, assume that both output initial conditions be zero: $\alpha = \beta = 0$. Let us also assume that the command input be a step function: $r(t) = r_0$, or $R(s) = r_0 \frac{1}{s}$. Hence, according to Eq. (10), controller of Fig. 1 must be of P-type: $H(s) = A_0$. So, with $R(s) = r_0 \frac{1}{s}$, $H(s) = A_0$ and $\alpha = \beta = 0$, from Eq. (6) we get:

$$E(s) = \frac{c_1 s + c_0}{d_2 s^2 + d_1 s + d_0} \tag{52}$$

where

$$c_1 = r_0 \tag{53.1}$$
$$c_0 = r_0\,\lambda \tag{53.2}$$
$$d_2 = 1 \tag{53.3}$$
$$d_1 = \lambda \tag{53.4}$$
$$d_0 = K\,A_0. \tag{53.5}$$

Now by Parseval's theorem, Eq. (51) can also be written as

$$J = \int_0^\infty e^2(t)\,\mathrm{d}t + \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} E(s)\,E(-s)\,\mathrm{d}s. \tag{54}$$

Considering the general expression for $E(s)$, as given by Eq. (52), and using the well-known table of Parseval's integral [8], we get

$$J = \frac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2}. \tag{55}$$

Substituting values of $c_1, c_0, \ldots, d_0$ from Eqs. (53.1) to (53.5), we get

$$J = \int_0^\infty e^2(t)\,\mathrm{d}t = \frac{1}{2\lambda}\,r_0^2 + \frac{r_0^2\lambda}{2K\,A_0}. \tag{56}$$

As seen from Eq. (56), no absolute optimal value for A_0 exists which would minimize J: the *larger* the value of A_0, the *smaller* the value of integral square-error J.

So, with command input being a step function: $r(t) = r_0$, then in order to *minimize* the integral square-error J, the P-controller of Fig. 1 must be a *high-gain amplifier*, and

$$J_{(\min)} = J|_{A_0\to\infty} = \frac{1}{2\lambda}\,r_0^2. \tag{57}$$

Let us now consider the configuration of Fig. 2. Again, suppose that the common command input: $r(t) = r_0$ or $R(s) = r_0\,\frac{1}{s} \Longrightarrow$ all the n controllers must be of P-type: $H(s) = A_0$. Also, as before, assume, that all the output initial coditions are zero: $\alpha_i = \beta_i = 0,\ i = 1, 2, \ldots, n$. As just has been established, in order to minimize the integral square-error for *each one* of the n unity-feedback systems, then *each one* of the n controllers must be a *high-gain amplifier*: $H(s) = A_0,\ A_0 \to \infty$.

Let $J_i,\ i = 1, 2, \ldots, n$ be the integral square-error of the 'ith' unity-feedback system of Fig. 2, and assume that J_T be the total integral square-error of the *whole* of Fig. 2. From Eq. (56):

$$J_T = \sum_{i=1}^{n} J_i = n\left[\frac{1}{2\lambda}\,r_0^2 + \frac{r_0^2\,\lambda}{2K\,A_0}\right]. \tag{58}$$

As shown by Eq. (57):

$$J_{i(\min)} = J_i|_{A_0\to\infty} = \frac{1}{2\lambda}\,r_0^2 \quad i = 1, 2, \ldots, n \tag{59}$$

hence, the minimum of total integral square-error $J_{T(\min)}$ is given by

$$J_{T(\min)} = \sum_{i=1}^{n} J_{i(\min)} = \frac{n}{2\lambda} r_0^2 \tag{60}$$

Now let us consider the arrangement of Fig. 6, which employs only *one* controller $H(s)$. Again, we assume that the command input being step function: $r(t) = r_0$, or $R(s) = r_0 \frac{1}{s} \Longrightarrow H(s) = A_0$. As in Fig. 2, we assume that $\alpha_i = \beta_i = 0$, $i = 1, 2, \ldots, n$. This implies that the second and third terms on the right-hand of Eqs. (50.1) to (50.n) are zero:

$$E_i(s) = \frac{s[s+\lambda]}{s^2 + \lambda s + K\,H(s)}\,R(s) \quad i = 1, 2, \ldots, n. \tag{61}$$

Substituting $R(s) = r_0 \frac{1}{s}$ and $H(s) = A_0$ in Eq. (61), we get

$$E_i(s) = \frac{r_0 s + r_0 \lambda}{s^2 + \lambda s + K\,A_0} \quad i = 1, 2, \ldots, n. \tag{62}$$

It is noted that Eq. (62) is *exactly the same* as Eq. (52). Assuming that $\hat{J}_{T(\min)}$ be the total integral square-error for configuration of Fig. 6, then since Eq. (62) is the same as Eq. (52), hence the same line of approach as for Fig. 2 will bring us to exactly the same result as Eq. (60):

$$\hat{J}_{T(\min)} = \frac{n}{2\lambda} r_0^2. \tag{63}$$

So *one* high-gain amplifier for Fig. 6 gives the same minimum of total integral square-error

$$\hat{J}_{T(\min)} = \frac{n}{2\lambda} r_0^2$$

as n of such amplifiers for Fig. 2.

If, however, the output initial conditions were *not* all zero, again we would arrive at the *same* result that all the n controllers of Fig. 2 must be high-again amplifiers: $H(s) = A_0$, $A_0 \rightarrow \infty$; as well as the *one* controller of Fig. 6. The only difference being that the minimum total integral square-error for Fig. 2, that is $J_{T(\min)}$, would be *different* to that of Fig. 6, namely $\hat{J}_{T(\min)}$:

$$J_{T(\min)} \neq \hat{J}_{T(\min)}.$$

References [6, 7] deal with the minimization of total integral square-error with non-zero output initial conditions.

To be sure, in Fig. 2 if we were interested *only* in bringing the n steady-state outputs of the system to follow, or track, the command step input: $r(t) = r_0$, or $R(s) = r_0 \frac{1}{s}$,that is *no* minimization of a cost function, or performance criterion, then *all* the n controllers in that figure could be taken as *unity*: $H(s) = 1$. In other words, in the forward path of each one of the unity-feedback systems of Fig. 2 we would have only the plant: $G(s) = \frac{K}{s(s+\lambda)}$. Using Eq. (6) with $R(s) = r_0 \frac{1}{s}$ and $H(s) = 1$, we would have for the n error signals of Fig. 2:

$$E_i(s) = \frac{s+\lambda}{s^2+\lambda s+K} r_0 - \frac{[s+\lambda]\,\alpha_i + \beta_i}{s^2+\lambda s+K} \tag{64}$$

where

$$E_i(s) = R(s) - Y_i(s) \quad i = 1, 2, \ldots, n. \tag{64.1}$$

Hence

$$\begin{aligned} &e_i(t)|_{t\to\infty} = \lim_{s\to 0} s\, E_i(s) = 0 \Longrightarrow \\ &y_i(t)|_{t\to\infty} = r(t) = r_0 \quad \forall \text{ output initial conditions.} \\ &i = 1, 2, \ldots, n. \end{aligned}$$

Of course, $r(t) = r_0$ and $H(s) = 1$ is also valid for Fig. 6. All the aforesaid really means is this that if $r(t) = r_0$, then we could arrange a unity feedback around each plant and need not to resort to the plant interconnection as in Fig. 6, *if* no minimization of a certain performance criterion is in mind. On the contrary, if the command input is a step-plus-ramp one: $r(t) = r_0 + r_1 t$, $r_1 \neq 0$, then, as we have seen, we would definitely need n PI-controllers: $H(s) = A_0 + A_1 \frac{1}{s}$ for Fig. 2 as *one* for Fig. 6. The same goes with n $PII2$-controllers: $H(s) = A_0 + A_1 \frac{1}{s} + A_2 \frac{1}{s^2}$, when $r(t) = r_0 + r_1 t + r_2 t^2$, for Fig. 2 as *one* for Fig. 6, and so on. As was seen for the P-controller: $H(s) = A_0$, the parameters of PI-controller, i.e. A_0 and A_1, in Fig. 6 can be selected such that, again, a certain performance criterion is minimized; the same with the parameters of $PII2$-controller, namely A_0, A_1 and A_2.

6. SYNCHRONIZATION AND TRACKING

Referring to Fig. 6, since we are dealing with n plants with, generally speaking, *different* output initial conditions and we wish the n steady-state outputs to *track* the command input $r(t)$, then the presentation of the following two definitions are most imperative:

SYNCHRONIZATION: With command input $r(t)$, the *one* controller $H(s)$ and *different* output initial conditions, by *synchronization* it is meant that the n *steady-state* outputs be *identical*.

TRACKING: With command input $r(t)$, the *one* controller $H(s)$ and *different* output initial conditions, by *tracking* it is meant that the n *steady-state* outputs be *identical* and, moreover, these identical steady-state outputs also *track* the command input $r(t)$. As the ensuring example will show, it may well happen, due to *improper* choice of the *one* controller $H(s)$ in Fig. 6, that the n steady-state outputs are identical: *Synchronization*, but these identical steady-state outputs *lag* the command input $r(t)$: *No tracking*.

Example. In Fig. 5, let

$$K = 1, \quad \lambda = 1, \quad (\alpha_1 = 0,\ \beta_1 = 0), \quad (\alpha_2 = -1,\ \beta_2 = 0), \quad (\alpha_3 = 4,\ \beta_3 = 0).$$

That is, we have three identical integral-plus-time constant plants: $G(s) = \frac{1}{s(s+1)}$, with output initial conditions as shown above. As of Fig. 5, the output of plant (1) has been nominated to be subtracted from command input $r(t)$. As was mentioned before, *any one* of the other two outputs could have also been chosen for this task. Subsequently, we would have named that plant as plant (1), and the other two plants as plants (2) and (3). Figures 7 to 9 show, respectively, the result of computer simulation for

(i) $r(t) = 1$ ← step input, $H(s) = 1 \leftarrow$ (P-controller, $A_0 = 1$)

(ii) $r(t) = 0.42\,t$ ← ramp input, $H(s) = 1$

(iii) $r(t) = 0.42\,t$, $H(s) = 1 + 0.2\,\frac{1}{s} \leftarrow$ (PI-controller, $A_0 = 1,\ A_1 = 0.2$)

Discussion

(i) As seen in Fig. 7, the three *identical* steady-state outputs *follow* the command input $r(t) = 1$:
Synchronization and *Tracking*.

(ii) In this case, shown in Fig. 8, the three steady-state outputs are *identical* ramp function of slope $= 0.42$, as the input, but, however, these three identical steady-state outputs *lag* the command input $r(t) = 0.42\,t$ by a *fixed amount*:
Synchronization without *Tracking*.

Let Θ be this *fixed amount* of error between the three *synchronized* outputs and the command input $r(t)$. The value of Θ is obtained from the *first* term on the right-hand side of Eqs. (41.1) to (41.3):

$$\begin{aligned}
\Theta &= \lim e_i(t)|_{t\to\infty} = \lim_{s\to 0} s\,E_i(s) \quad i = 1,2,3 \\
&= \lim_{s\to 0} s\ \frac{s[s+\lambda]}{s^2+\lambda s + K\,H(s)}\,R(s)\Bigg|_{\substack{R(s)=0.42\,\frac{1}{s^2}\\ \lambda=K=1\\ H(s)=1}} \\
&= \lim_{s\to 0} \frac{0.42\,[s+1]}{s^2+s+1} = 0.42
\end{aligned}$$

This fixed amount of error: $\Theta = 0.42$ between the three steady-state outputs and the command input: $r(t) = 0.42t$ is clearly observed in Fig. 8.

(iii) Portrayed in Fig. 9, the three steady-state outputs are *identical* and *also* identical to the command input $r(t) = 0.42\,t$:
Synchronization and *Tracking*.

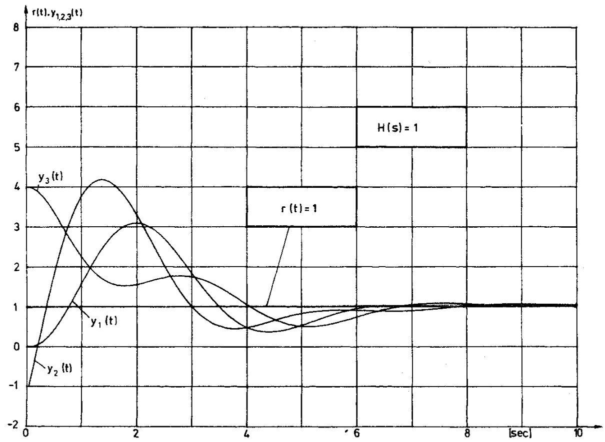

Fig. 7. Computer response of Fig. 5, with: $[r(t) = 1,\ H(s) = 1]$, $K = 1$, $\lambda = 1$, $(\alpha_1 = 0,\ \beta_1 = 0)$, $(\alpha_2 = -1,\ \beta_2 = 0)$, $(\alpha_3 = 4,\ \beta_3 = 0)$.

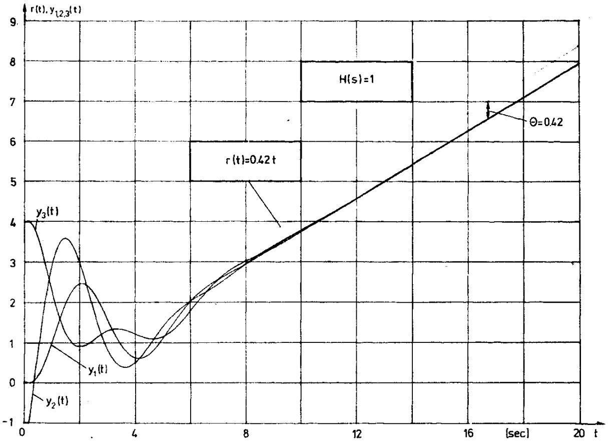

Fig. 8. Computer response of Fig. 5, with: $[r(t) = 0.42\,t,\ H(s) = 1]$, $K = 1$, $\lambda = 1$, $(\alpha_1 = 0,\ \beta_1 = 0)$, $(\alpha_2 = -1,\ \beta_2 = 0)$, $(\alpha_3 = 4,\ \beta_3 = 0)$.

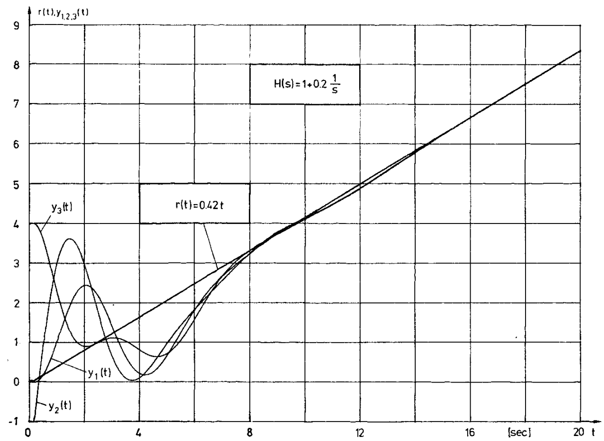

Fig. 9. Computer response of Fig. 5, with: $[r(t) = 1,\ H(s) = 1 + 0.2\,\frac{1}{s}]$, $K = 1$, $\lambda = 1$, $(\alpha_1 = 0,\ \beta_1 = 0)$, $(\alpha_2 = -1,\ \beta_2 = 0)$, $(\alpha_3 = 4,\ \beta_3 = 0)$.

It goes without saying that if optimization technique, as discussed for the P-controller: $H(s) = A_0$, were brought into this example, then, more likely than not, all round better results could have been obtained.

7. CONCLUSION

In this paper, the tracking action of n identical integral-plus-time constant plants: $G(s) = \frac{K}{s(s+\lambda)}$ through commissioning of only *one* controller $H(s)$ was studied. It was observed that if originally the plants are *directly* interconnected, in the manner fully discussed in the text, then upon operation of an appropriate controller $H(s)$ on the difference between the command input $r(t)$ and *any one* of the n outputs this goal can be achieved. A three-part example, being simulated on computer, most impressively illuminated the theoretical results. Obviously, the technique can easily be translated to discete-time systems: use of *one* digital controller. A most interesting observation was experieced where the parameters (K, λ) of the three identical plants, in the example given, were perturbed within the range of ± 30 % and still tracking behaviour, as portrayed in Figs. 7 and 9, was observed — with the same P-controller: $H(s) = 1$ and PI-controller: $H(s) = 1 + 0.2\,\frac{1}{s}$. However, the mathematical exposition of this observation, i.e. robustness of the system, is indeed beyond the scope of this work.

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REFERENCES

[1] G. Bleisteiner and W. v. Mangoldt (eds.): Handbuch der Regelungstechnik. Springer–Verlag, Berlin 1961.

[2] D. Schönert: Antriebs- und Regelprobleme kontinuierlicher Walzanlagen. Stahl und Eisen *99* (1979), 1361–1371.

[3] K. Ogata: Modern Control Engineering. Prentice-Hall, Englewood Cliffs, N. J. 1970.

[4] T. E. Fortmann and K. L. Hitz: An Introduction to Linear Control Theory. Marcel Dekker, New York 1977.

[5] H. Unbehauen and I. Vakilzadeh: Automatic error elimination between outputs of n identical simple-integral plants with different initial conditions. Internat. J. Control *47* (1988), 4, 1101–1116.

[6] H. Unbehauen and I. Vakilzadeh: Synchronization of n identical integral-plus-time constant plants. Control Theory Adv. Tech. *4* (1988), 1, 153–169.

[7] H. Unbehauen and I. Vakilzadeh: Optimal synchronization of n identical double-integral plants with identical inputs but different initial conditions. Optimal Control Appl. Methods *9* (1988), 4, 371–401.

[8] D. A. Pierre: Optimization Theory with Applications. J. Wiley, New York 1969.

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