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SHAPE OPTIMIZATION OF A NONLINEAR ELLIPTIC SYSTEM

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Shape optimization problem for a nonlinear elastic plate governed by von Kármán equation is considered. Using material derivative method, sensitivity analysis of the solution to the von Kármán system with respect to the variation of the domain is performed and a necessary optimality condition is derived.

1. INTRODUCTION

The paper deals with a shape optimization problem for a nonlinear elastic plate. The equilibrium state of this plate is described by a system of nonlinear, coupled elliptic equations of the fourth order. These equations, called von Kármán equations, were investigated in [3, 6, 11]. The optimization problem considered in this paper consists in minimizing the cost functional approximating the plate stiffness with respect to the domain occupied by the plate.

In literature [2, 4, 14, 15, 22, 23, 24] shape design sensitivity analysis for linear elliptic systems was developed by many authors. In recent years appeared papers investigating shape optimization problems for nonlinear elliptic systems, i.e. for shallow shells [9], buckled arches [10], plasma equation [13], contact problems in solid mechanics [15, 20, 24, 25]. The optimization problem for von Kármán system was considered in [5] only where the right hand side of von Kármán equations depends on a functional variable subject to optimization. The shape optimization problem for von Kármán system where the domain occupied by the system is the variable subject to optimization was not investigated in the literature except [21] where the author considered this problem in different formulation than in this paper and with the nonsmooth cost functional.

The aim of this paper is to determine the directional derivative of the cost functional of this shape optimization problem with respect to the variation of the domain occupied by the plate. In order to do it , we shall employ developed by Zolesio [14, 24] the material derivative method. We shall investigate the sensitivity of the solution to the nonlinear state system with respect to the variation of the domain. We shall formulate first order necessary optimality condition for this problem. The present work can be considered as a natural extension of results in [22] to nonlinear elliptic systems.

Throughout the paper we shall use the notation: $H^m(\Omega)$, $H_0^m(\Omega)$, $m = 0, 1, 2, 3, 4$ will denote the Sobolev spaces of order m with norm $\|\cdot\|_{H^m(\Omega)}$ [1], $\mathbf{n}=(n_1, n_2)$ is the unit outward versor to the boundary Γ, s$=(-n_2, n_1)$ is the unit tangent versor to the boundary Γ, $\frac{\partial v}{\partial n}$ is the outward normal derivative of a function v on the boundary Γ of domain Ω, $v_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j}$, $v_i = \frac{\partial v}{\partial x_i}$ for $i, j = 1, 2$, $\Delta v = v_{11} + v_{22}$, ∇v is a gradient of a function v with respect to a variable x.

2. NONLINEAR PLATE MODEL

Consider an elastic nonlinear plate occupying in the plane Ox_1x_2 domain $\Omega \subset R^2$. The domain Ω is bounded and simply connected. The boundary Γ of the domain Ω is Lipschitz continuous. We denote by $w = w(x)$, $x \in \Omega$ the displacement of the plate and by $f = f(x)$ the Airy's stress function [3, 11]. Let g be a perpendicular force bending the plate. In an equlibrium state the functions w and f satisfy the following system of von Kármán equations [3, 5, 6, 11, 21]:

$$\Delta^2 w = [f, w] + \lambda \Delta w + g \quad \text{in } \Omega \tag{1}$$

$$\Delta^2 f = -[w, w] \quad \text{in } \Omega \tag{2}$$

with the following boundary conditions:

$$w = f = 0, \ \frac{\partial w}{\partial n} = \frac{\partial f}{\partial n} = 0 \ \text{ on } \Gamma \tag{3}$$

where

$$[f, w] = f_{11}w_{22} + f_{22}w_{11} - 2f_{12}w_{12}\,, \qquad \Delta^2 w = (w_{11})_{11} + (w_{22})_{22} + 2(w_{11})_{22}$$

The condition (3) implies that the plate is clamped along the boundary Γ. We shall consider von Kármán equations (1)–(2) assuming that $\lambda \in R$ is a suitable small parameter such that $\lambda < \lambda_1$, and $\lambda_1 > 0$ is a constant satisfying [11]:

$$0 < \int_\Omega \nabla\phi\nabla\phi \,\mathrm{d}x \leq \frac{1}{\lambda_1}\|\phi\|^2_{H_0^2(\Omega)} \quad \forall\,\phi \in H_0^2(\Omega) \tag{4}$$

The condition (4) implies we shall consider small deflections of the nonlinear plate. The parameter λ indicates the intensity of compressive or tensile forces acting on the body. For suitable small $\lambda \in R$ the system (1)–(3) has a unique solution [3, 11].

We shall consider the weak solutions of problem (1)–(3). We denote by $a(\cdot,\cdot) : H^2(\Omega) \times H^2(\Omega) \to R$, $\tilde{b}(\cdot,\cdot) : H^2(\Omega) \times H^2(\Omega) \to R$, $b(\cdot,\cdot,\cdot) : H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \to R$ as well as by $l(\cdot) : H^2(\Omega) \to R$ the forms defined by:

$$a(u, z) = \int_\Omega (u_{11}z_{11} + u_{22}z_{22} + 2u_{12}z_{12})\,\mathrm{d}x \tag{5}$$

$$\tilde{b}(u, z) = \int_\Omega (u_1 z_1 + u_2 z_2)\,\mathrm{d}x \tag{6}$$

$$b(u,z,r) = \int_\Omega [(u_{12}z_2 - u_{22}z_1)r_1 + (u_{12}z_1 - u_{11}z_2)r_2]\,\mathrm{d}x \tag{7}$$

$$l(z) = \int_\Omega gz\,\mathrm{d}x \tag{8}$$

The bilinear forms (5) and (6) are symmetric, continuous as well as, respectively, $H^2(\Omega)$ and $H^1(\Omega)$ elliptic [18]. The trilinear form (7) is symmetric and bounded [3, 11]. $g \in L^2(R^2)$ is a given element.

In variational formulation problem (1)–(3) takes the form [3, 6, 11]: *for given element $g \in L^2(R^2)$ find functions $w \in H_0^2(\Omega)$ and $f \in H_0^2(\Omega)$ satisfying*:

$$a(w,\phi) = b(f,w,\phi) + \lambda\tilde{b}(w,\phi) + l(\phi) \quad \forall \phi \in H_0^2(\Omega) \tag{9}$$

$$a(f,\eta) = -b(w,w,\eta) \quad \forall \eta \in H_0^2(\Omega) \tag{10}$$

If λ satisfies condition (31) for $t = 0$ then the system (9)–(10) has a unique solution [3, 6, 11].

3. SHAPE OPTIMIZATION PROBLEM FORMULATION

Before we formulate the shape optimization problem we shall introduce a family of domains Ω_t depending on a parameter t. The domain Ω_t will be considered as an image of a mapping T_t of the reference domain Ω. We shall employ the speed method [14, 24] to describe the mapping T_t. We shall formulate the shape optimization problem for the variational system (9)–(10) in the perturbed domain Ω_t.

Let t be a real parameter, such that $t \in [0,\sigma), \sigma > 0$. We denote by $\mathbf{V}(\cdot,\cdot) : [0,\sigma) \times R^2 \to R^2$ enough regular vector field:

$$\mathbf{V}(t,\cdot) \in C^2(R^2,R^2)\,\forall t \in [0,\sigma), \quad \mathbf{V}(\cdot,x) \in C([0,\sigma),R^2)\,\forall x \in R^2 \tag{11}$$

By $T_t(\mathbf{V}) : R^2 \ni X \to x(t,X) \in R^2$ we denote the family of mappings depending on the parameter $t \in [0,\sigma)$ where the vector function $x(\cdot,X) = x(\cdot)$ satisfies the ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau,X) = \mathbf{V}(\tau,x(\tau,X)) \quad \tau \in [0,\sigma), \quad x(0,X) = X \quad X \in R^2 \tag{12}$$

The family Ω_t of domains depending on the parameter $t \in [0,\sigma)$ is defined as follows:

$$\begin{aligned}&\Omega_0 = \Omega \text{ and}\\ &\Omega_t = T_t(\mathbf{V})(\Omega) = \{x \in R^2 \;:\; \exists X \in R^2 \text{ such that } x = x(t) \text{ where}\\ &\qquad \text{the function } x(\cdot) \text{ satisfies equation (12) for } \; 0 \le \tau \le t\}\end{aligned} \tag{13}$$

We shall assume that for a given value of the parameter t domain Ω_t is bounded, simple connected and has Lipschitz continuous boundary Γ_t. The variational problem (9)–(10) in the domain Ω_t takes the form: *for given element $g \in L^2(R^2)$ and parameter $\lambda \in R$ find functions $w_t \in H_0^2(\Omega_t)$ and $f_t \in H_0^2(\Omega_t)$ satisfying*:

$$a_t(w_t,\phi_t) = b_t(f_t,w_t,\phi_t) + \lambda\tilde{b}_t(w_t,\phi_t) + l_t(\phi_t) \quad \forall \phi_t \in H_0^2(\Omega_t) \tag{14}$$

$$a_t(f_t,\eta_t) = -b_t(w_t,w_t,\eta_t) \quad \forall \eta_t \in H_0^2(\Omega_t) \tag{15}$$

The forms $a_t(\cdot,\cdot)$, $\tilde{b}_t(\cdot,\cdot)$, $b_t(\cdot,\cdot,\cdot)$, $l_t(\cdot)$ are given, respectively, by (5), (6), (7), (8) where the integrals are taken over domain Ω_t instead of Ω. Moreover $\phi_t = \phi(t, x(t))$. For each $t \in [0,\sigma), \sigma > 0$ and λ satisfying (31) the system (14) – (15) has a unique solution $(w_t, f_t) \in H_0^2(\Omega_t) \times H_0^2(\Omega_t)$.

We shall consider the following shape optimization problem for system (14) – (15): *find domain* $\Omega_t \in U$ *minimizing the cost functional:*

$$J(\Omega_t) = \int_{\Omega_t} w_t^2 \, \mathrm{d}x \tag{16}$$

where U *is the set of admissible bounded domains.*
The shape optimization problem (16) consists in finding such configuration of the domain Ω_t which minimizes the stiffness of the plate. For engineering motivation of this optimization problem see [2, 14, 15].

3.1. Existence of optimal solutions

In order to assure the existence of an optimal solution to the problem (16) we have to select the family U of the admissible domains compact in an appriopriate sense [8]. Let D be a given bounded set in R^2 and moreover assume $\Omega_0 \subset D$, $\Omega_t \subset D$ for all $t \in [0,\sigma)$, $\sigma > 0$, $\Omega_0 \in U$. We assume that a minimizing sequence of domains $\{\Omega_{t_k}\} \subset U$ satisfies for $k \to \infty$ the condition [8]:

> there exists a subsequence $\{\Omega_{t_k'}\} \subset \{\Omega_{t_k}\}$, denoted further by
> $\{\Omega_{t_k}\}$ such that the sequence of characteristic functions χ_k of $\Omega_{t_k} \subset U$ (17)
> converges in $L^2(D)$ to a characteristic function $\hat{\chi}$ of a subset $\hat{\Omega} \subset U$.

i.e., U is assumed to be compact for the strong $L^2(D)$ topology of the characteristic functions of its elements. For details concerning the selection of the family U see [8]. We can prove:

Lemma 1. There exists an optimal domain $\hat{\Omega} \in U$ to the problem (16).

Proof. The proof follows from [17] and [8]. □

Note that Lemma 1 does not assure the existence of a unique optimal domain.

Remark 1. In applications [14, 15, 19, 20, 21] the reference domain Ω_0 is selected as a rectangle and the vector field $\mathbf{V}$ is selected in the following way:

$$\mathbf{V} = (V_1, V_2) \quad V_1 = x_1 k(x_2) \quad V_2 = 0 \tag{18}$$

For (18) the set U of the admissible domains is equivalent to the following set $\tilde{U}$ of admissible functions describing the boundary of the optimized domain: [19, 20]:

$$\tilde{U} = \{k \in C^{0,1}(0,1) \; : \; 0 < c_1 \le k(x_2) \le c_2, \left|\frac{\mathrm{d}k}{\mathrm{d}x_2}\right| \le c_3, \int_0^1 k(x_2)\,\mathrm{d}x_2 = c_4\} \tag{19}$$

where $C^{0,1}(0,1)$ denotes the set of Lipschitz continuous functions on the segment $(0,1)$ and c_1, c_2, c_3, c_4 are given positive constants. The set (19) is assumed to be nonempty. The integral condition in (19) denotes that the mass of the optimized plate is constant [2]. The existence of optimal solutions to the problem (16) with $\mathbf{V}$ and $\tilde{U}$ selected according to (18),(19), respectively, was shown in [19]. Note, that to obtain the existence of optimal solutions to the problem (16) with $\mathbf{V}$ given by (18) and $\tilde{U}$ given by (19) but without the bound on the derivative of the function k, we can also use the regularization technique. For details see [22, 24].

4. NECESSARY OPTIMALITY CONDITION

In this section we shall calculate the derivative of the cost functional (16). Let us recall the definition of the Euler derivative [24]:

Definition 1. The Euler derivative $\mathrm{d}J(\Omega, \mathbf{V})$ of the cost functional $J(\Omega)$ at a point Ω in the direction of the vector field $\mathbf{V}$ is determined by:

$$\mathrm{d}J(\Omega, \mathbf{V}) = \limsup_{t \to 0} [J(\Omega_t) - J(\Omega)]/t \tag{20}$$

Before we calculate the derivative of the cost functional (16) we shall show the Lipschitz continuity of the solutions to the system (14)-(15).

Lemma 2. Let the pair $(w_t, f_t) \in H_0^2(\Omega_t) \times H_0^2(\Omega_t)$, $t \in [0, \sigma)$, $\sigma > 0$ be a solution to the system (14)-(15). We denote by: $w^t = w_t \circ T_t \in H_0^2(\Omega)$, $f^t = f_t \circ T_t \in H_0^2(\Omega)$, $w = w_0 \in H_0^2(\Omega)$, $f = f_0 \in H_0^2(\Omega)$. Then there exists constant $e > 0$, independent on t such that for $t > 0$ small enough:

$$\|w^t - w\|_{H^2(\Omega)} \leq et \qquad \|f^t - f\|_{H^2(\Omega)} \leq et \tag{21}$$

Proof. The proof consists of two parts. First we write the system (14)-(15) transported to the reference domain Ω in canonical form [3, 6, 11]. Next we show, using the implicit function theorem [17] that the mapping:

$$[0, \sigma) \ni t \to w^t \in H_0^2(\Omega) \tag{22}$$

is Frechet differentiable at a point $t = 0$. This implies that the condition (21) is satisfied.

Let us consider the system (14)-(15) transported to the reference domain Ω. Let us introduce the operators $L^t : H_0^2(\Omega) \to H_0^2(\Omega)$ and $B^t : H^2(\Omega) \times H^2(\Omega) \to H_0^2(\Omega)$ defined for all $\phi \in H_0^2(\Omega)$ by:

$$\tilde{b}^t(w^t, \phi) = (L^t w^t, \phi)_{H_0^2(\Omega)} \tag{23}$$

$$b^t(f^t, w^t, \phi) = (B^t(f^t, w^t), \phi)_{H_0^2(\Omega)} \tag{24}$$

where $(\cdot,\cdot)_{H_0^2(\Omega)}$ denotes scalar product in $H_0^2(\Omega)$ [1]. Note, that the forms $b^t(\cdot,\cdot,\cdot)$ and $\tilde{b}^t(\cdot,\cdot)$ are given by (52), (53). By $C^t(w^t) : H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) \to H_0^2(\Omega)$ we denote the operator:

$$C^t(w^t) = B^t(B^t(w^t, w^t), w^t) \tag{25}$$

Let $D^t \in H_0^2(\Omega)$ be a solution of the equation:

$$a^t(D^t, \phi) = \int_\Omega J_t g \phi \,\mathrm{d}x \quad \forall \phi \in H_0^2(\Omega) \tag{26}$$

where $J_t = \det DT_t$ and the form $a^t(\cdot,\cdot)$ is given by (51). It can be shown that for all $t \in [0, \sigma), \sigma > 0$ the operators L^t, B^t, C^t have the following properties:

L^t is linear, compact and selfadjoint operator
B^t is bilinear, compact and symmetric operator
C^t is compact operator such that

$$\|C^t(w) - C^t(v)\|_{H_0^2(\Omega)} \le \text{const } \|w - v\|_{H_0^2(\Omega)} \tag{27}$$

$$(C^t(w), w)_{H_0^2(\Omega)} = \|B^t(w, w)\|^2_{H_0^2(\Omega)} > 0 \ \forall w \in H_0^2(\Omega), \ w \neq 0 \tag{28}$$

Taking into account (23)–(26) we can write the system (14)–(15) transported to domain Ω in the canonical form [3, 6, 11]:

$$P(w^t, t) \stackrel{\text{def}}{=} (I - \lambda L^t) w^t + C^t(w^t) - D^t = 0 \tag{29}$$

$$f^t = -B^t(w^t, w^t) \tag{30}$$

Using the same arguments as in [3, 5, 11] it can be shown that for $t \in [0, \sigma), \sigma > 0$, $\lambda \in R$ satisfying:

$$\lambda < \lambda_1 (1 - (\alpha^{-1} \|B^t\| \, \|D^t\|_{H^2(\Omega)})^{2/3}) \tag{31}$$

there exists a unique solution $(w^t, f^t) \in H_0^2(\Omega) \times H_0^2(\Omega)$ to the system (29)–(30). λ_1 in (31) denotes the inverse of the largest eigenvalue of the operator L^t, $\alpha \in (0, 1)$ is a real number, the norm of the operator B^t is given by:

$$\|B^t\| = \tag{32}$$

$$= \sup\{\|B^t(f^t, w^t)\|_{H_0^2(\Omega)} \mid (f^t, w^t) \in H_0^2(\Omega) \times H_0^2(\Omega), \|f^t\|_{H_0^2(\Omega)} = \|w^t\|_{H_0^2(\Omega)} = 1\}$$

Moreover we can evaluate:

$$\|w^t\|_{H_0^2(\Omega)} \le \frac{\lambda_1}{\lambda_1 - \lambda} \|D^t\|_{H_0^2(\Omega)} \tag{33}$$

Let us consider the equation (29) only. To show differentiability of the mapping (22) at a point $t = 0$ we shall use the implicit function theorem [17]. We verify assumptions of this theorem. From (31) it follows that the equation (29) has a unique solution $w^t \in H_0^2(\Omega)$ for $t \in [0, \sigma), \sigma > 0$. From continuity of operators

L^t, B^t, C^t as well as from (11)-(12) it follows the existence of continuous partial derivatives P'_{w^t} and P'_t of the mapping (22) with respect to w^t and t respectively. The derivative P'_{w^t} in the direction ϕ has the form:

$$P'_{w^t}\phi = (I - \lambda L^t)\phi + 2B^t(B^t(w^t, \phi), w^t) + B^t(B^t(w^t, w^t), \phi) \tag{34}$$

From (28) and (34) it follows:

$$\begin{aligned}(P'_{w^t}\phi, \phi)_{H_0^2(\Omega)} = {} & \|\phi\|^2_{H_0^2(\Omega)} - \lambda(L^t\phi, \phi)_{H_0^2(\Omega)} + \\ & 2\|B^t(w^t, \phi)\|^2_{H_0^2(\Omega)} + (B^t(w^t, w^t), B^t(\phi, \phi))_{H_0^2(\Omega)} \stackrel{\text{def}}{=} R_1\end{aligned} \tag{35}$$

Using the boundedness of the operator L^t as well as (28) we can evaluate (35) from below:

$$R_1 \geq [1 - \frac{\lambda}{\lambda_1} - \|B^t\|^2 \|w^t\|^2_{H_0^2(\Omega)}] \|\phi\|^2_{H_0^2(\Omega)} \stackrel{\text{def}}{=} R_2 \tag{36}$$

Taking into account (33) we obtain from (36):

$$R_2 \geq \left[1 - \frac{\lambda}{\lambda_1} - \left(\frac{\lambda_1}{\lambda_1 - \lambda}\right)^2 \|B^t\|^2 \|D^t\|^2_{H_0^2(\Omega)}\right] \|\phi\|^2_{H_0^2(\Omega)} \stackrel{\text{def}}{=} R_3 \tag{37}$$

Using (31) we can evaluate (37):

$$\begin{aligned}R_3 \geq \left[(1 - \frac{\lambda}{\lambda_1}) - \alpha^2 \left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\right] \|\phi\|^2_{H_0^2(\Omega)} \geq \\ \min\left\{1, 1 - \frac{\lambda}{\lambda_1}\right\}(1 - \alpha^2)\|\phi\|^2_{H_0^2(\Omega)} \stackrel{\text{def}}{=} K\|\phi\|^2_{H_0^2(\Omega)} > 0\end{aligned} \tag{38}$$

From (35)-(38) we obtain:

$$(P'_{w^t}\phi, \phi)_{H_0^2(\Omega)} \geq K\|\phi\|^2_{H_0^2(\Omega)} > 0 \quad \forall \phi \in H_0^2(\Omega) \tag{39}$$

From (39) it follows the existence of the continuous inverse of the operator P'_{w^t}. Hence by the implicit function theorem [17] follows Frechet differentiability of the mapping (22), i.e. Lipschitz continuity of w^t. Moreover from (11)-(12), (30) follows the condition (21). □

Using Lemma 2 we are able to prove:

Lemma 3. The derivative $\mathrm{d}J(\Omega, \mathbf{V})$ of the cost functional (16) at a point Ω in a direction $\mathbf{V}$, defined by (20) is given by:

$$\mathrm{d}J(\Omega, \mathbf{V}) = 2\int_\Omega wW \,\mathrm{d}x \tag{40}$$

where $W = \frac{\partial w}{\partial t}$ is a shape derivative of the function w_t defined by (42).

Proof. From (16) and (20) it results:

$$\mathrm{d}J(\Omega, \mathbf{V}) = \limsup_{t \to 0} \left[\int_{\Omega_t} w_t^2 \, \mathrm{d}x - \int_{\Omega} w^2 \, \mathrm{d}x \right] / t$$

Using formulae for transport the integrals from domain Ω_t to the reference domain Ω [14, 24] we obtain:

$$\mathrm{d}J(\Omega, \mathbf{V}) = \limsup_{t \to 0} \left\{ \int_{\Omega} [(w_t \circ T_t)^2 - w^2] \det DT_t \, \mathrm{d}x + \int_{\Omega} w^2 (\det DT_t - 1) \mathrm{d}\, x \right\} / t \tag{41}$$

where DT_t is the Jacobian of the mapping T_t, $\det DT_t$ is determinant of DT_t. Passing to the limit in (41) with $t \to 0$, using (3) as well as (21) we obtain (40). □

4.1. Sensitivity analysis of solutions to the state system

In order to calculate the derivative (40) we have to calculate the shape derivative W of the solution w_t to the system (14)–(15). Let us recall the definition [24]:

Definition 2. The shape derivative $W \in H^2(\Omega)$ of the function $w_t \in H_0^2(\Omega_t)$ is determined by:

$$(\Pi w_t)_{|\Omega} = w + tW + o(t) \tag{42}$$

where $\|o(t)\|_{H^2(\Omega)}/t \to 0$ for $t \to 0$, $w = w_0 \in H^2(\Omega)$ and $\Pi w_t \in H^2(R^2)$ is an extension of the function $w_t \in H_0^2(\Omega_t)$ to an open neighbourhood of $\Omega_t \subset R^2$ such that the restriction $(\Pi w_t)_{|\Omega} \in H_0^2(\Omega)$ for $t > 0$, t small enough.

To calculate the shape derivative we need also the notion of a material derivative [24]:

Definition 3. The material derivative $\dot{w} \in H^2(\Omega)$ of the function $w_t \in H^2(\Omega_t)$ at a point $X \in \Omega$ is defined by:

$$\lim_{t \to 0} \|(w_t \circ T_t - w_0)/t - \dot{w}\|_{H^2(\Omega)} = 0 \tag{43}$$

where $w = w_0 \in H^2(\Omega)$, the function $w^t = w_t \circ T_t \in H^2(\Omega)$ is an image of the function $w_t \in H^2(\Omega_t)$ in the space $H^2(\Omega)$.

Let us recall [24] that if the shape derivative $W \in H^2(\Omega)$ of the function $w_t \in H_0^2(\Omega_t)$ exists, then the following condition holds:

$$W = \dot{w} - \nabla w \mathbf{V}(0) \tag{44}$$

Using Lemma 2 as well as Definition 3 we can prove:

Lemma 4. The material derivatives $\dot{w} \in H_0^2(\Omega)$ and $\dot{f} \in H_0^2(\Omega)$ of the functions $w_t \in H_0^2(\Omega_t)$ and $f_t \in H_0^2(\Omega_t)$ satisfying the system (14)–(15) are given by:

$$\begin{aligned} a(\dot{w},\phi) + a'(w,\phi) &= b(f,\dot{w},\phi) + b(\dot{f},w,\phi) + b'(f,w,\phi) + \\ &\quad \lambda\tilde{b}(\dot{w},\phi) + \lambda\tilde{b}'(w,\phi) + (\dot{g},\phi) \quad \forall\phi \in H_0^2(\Omega) \end{aligned} \tag{45}$$

$$a(\dot{f},\eta) + a'(f,\eta) = -b(\dot{w},w,\eta) - b(w,\dot{w},\eta) - b'(w,w,\eta) \quad \forall\eta \in H_0^2(\Omega) \tag{46}$$

where

$$(\dot{g},\phi) = \int_\Omega \operatorname{div}(g\mathbf{V}(0))\phi \, \mathrm{d}x \tag{47}$$

The derivatives $a'(\cdot,\cdot)$, $b'(\cdot,\cdot,\cdot)$, $\tilde{b}'(.,.)$ of the forms $a_t(\cdot,\cdot)$, $b_t(\cdot,\cdot,\cdot)$, $\tilde{b}_t(\cdot,\cdot)$ with respect to t at $t=0$ are given by:

$$\begin{aligned} a'(w,\phi) &= \int_\Omega [-\operatorname{div}\mathbf{V}(0)]\triangle w \triangle\phi \, \mathrm{d}x + \\ &\quad \int_\Omega \operatorname{div}[\operatorname{div}\mathbf{V}(0)I - (D\mathbf{V}(0) + {}^T D\mathbf{V}(0))](\triangle w\nabla\phi + \nabla w\triangle\phi)\,\mathrm{d}x \end{aligned} \tag{48}$$

$$\begin{aligned} b'(f,w,\phi) &= \int_\Omega \nabla^T w[\nabla(\nabla f)\operatorname{div}\mathbf{V}(0) - D\mathbf{V}(0)\nabla(\nabla f) - \\ &\quad {}^T D\mathbf{V}(0)\nabla(\nabla f) - \nabla({}^T D\mathbf{V}(0)\nabla f) - \nabla(\nabla f)^T D\mathbf{V}(0)]\nabla\phi\,\mathrm{d}x - \\ &\quad \int_\Omega \operatorname{div}[\operatorname{div}\mathbf{V}(0)I - (D\mathbf{V}(0) + {}^T D\mathbf{V}(0))]\nabla f\,\mathrm{d}x + \\ &\quad \int_\Omega \triangle f\nabla^T w(D\mathbf{V}(0) + {}^T D\mathbf{V}(0))]\nabla w\nabla\phi\,\mathrm{d}x \end{aligned} \tag{49}$$

$$\tilde{b}'(w,\phi) = \int_\Omega [\operatorname{div}\mathbf{V}(0)I - (D\mathbf{V}(0) + {}^T D\mathbf{V}(0))]\nabla w\nabla\phi\,\mathrm{d}x \tag{50}$$

where $\mathbf{V}(0) = \mathbf{V}(0,X)$, $D\mathbf{V}(0)$ denotes the Jacobian matrix of the matrix $\mathbf{V}(0)$, ${}^T D\mathbf{V}(0)$ denotes transpose matrix of the matrix $D\mathbf{V}(0)$, I is an identity matrix.

P r o o f. Using formulae for transport the gradient and the Laplacian of the function into the fixed domain [24] we transform the forms $a_t(\cdot,\cdot)$, $b_t(\cdot,\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$ from the domain Ω_t to the fixed domain Ω:

$$\begin{aligned} & a^t(w^t,\phi^t) = \\ &= \int_\Omega J_t^{-1}\operatorname{div}(J_t DT_t^{-1}\,{}^T DT_t^{-1}\nabla w^t)\operatorname{div}(J_t DT_t^{-1}\,{}^T DT_t^{-1}\nabla\phi^t)\,\mathrm{d}x \quad \forall w^t,\phi^t \in H_0^2(\Omega) \end{aligned} \tag{51}$$

$$\begin{aligned} & b^t(f^t,w^t,\phi^t) = \\ &= \int_\Omega \nabla^T w^T DT_t^{-1}\,{}^T DT_t^{-1}\nabla({}^T DT_t^{-1}\nabla f^t)^T DT_t^{-1}\nabla\phi^t J_t\,\mathrm{d}x - \int_\Omega \operatorname{div}(J_t DT_t^{-1} \\ & \quad {}^T DT_t^{-1}\nabla w^t)(\nabla^T f^t DT_t^{-1}\,{}^T DT_t^{-1}\nabla\phi^t)\,\mathrm{d}x \quad \forall f^t,w^t,\phi^t \in H_0^2(\Omega) \end{aligned} \tag{52}$$

$$\tilde{b}^t(w^t,\phi^t) = \int_\Omega^T DT_t^{-1}\nabla w^t DT_t^{-1}\nabla\phi^t J_t\,\mathrm{d}x \quad \forall w^t,\phi^t \in H_0^2(\Omega) \tag{53}$$

where $J_t = \det DT_t$, DT_t is the Jacobian matrix of the mapping T_t, DT_t^{-1} is the inverse of DT_t and ${}^TDT_t$ is a transpose of DT_t. Since the mapping T_t and the inverse mapping T_t^{-1} have continuous second derivatives with respect to the spatial variables [24] hence for all functions $\phi_t \in H_0^2(\Omega_t)$, $\phi \in H_0^2(\Omega)$, $(w^t, f^t) \in H_0^2(\Omega) \times H_0^2(\Omega)$ we have:

$$\begin{aligned} a_t(w_t, \phi_t) &= a^t(w^t, \phi^t) = a^t(w^t, \phi) \\ b_t(f_t, w_t, \phi_t) &= b^t(f^t, w^t, \phi^t) = b^t(f^t, w^t, \phi) \\ \tilde{b}_t(w_t, \phi_t) &= \tilde{b}^t(w^t, \phi^t) = \tilde{b}^t(w^t, \phi) \end{aligned} \tag{54}$$

Subtracting from the system (14) – (15) the system (9) – (10) and using (51) – (54) we obtain

$$\begin{aligned} & a(w^t - w, \phi) + a^t(w^t, \phi) - a(w^t, \phi) = \qquad (55) \\ = {} & b(f^t, w^t - w, \phi) + b(f^t - f, w^t, \phi) + b^t(f^t, w^t, \phi) - \\ - {} & b(f^t, w^t, \phi) + \lambda\tilde{b}(w^t - w, \phi) + \lambda\tilde{b}^t(w^t, \phi) - \lambda\tilde{b}(w^t, \phi) + (g^t - g, \phi) \ \ \forall \phi \in H_0^2(\Omega) \\ & a(f^t - f, \eta) + a^t(f^t, \eta) - a(f^t, \eta) = \qquad (56) \\ = {} & -\{b(w^t - w, w, \eta) + b(w^t, w^t - w, \eta) + b^t(w^t, w^t, \eta) - b(w^t, w^t, \eta)\} \ \ \forall \eta \in H_0^2(\Omega) \end{aligned}$$

Using (11) – (13), Lemma 2, dividing both sides of the system (55) – (56) by t as well as passing to the limit with $t \to 0$ in (55) – (56) we obtain that the limits of the subsequences $\{(w^t - w)/t\}$ and $\{(f^t - f)/t\}$ satisfy (45) – (46). Since for $\lambda \in R$ satisfying condition (26) the system (45) – (46) has a unique solution $(\dot{w}, \dot{f}) \in H_0^2(\Omega) \times H_0^2(\Omega)$ by Lemma 2 follows the existence of the limits of the whole sequences $\{(w^t - w)/t\}$ and $\{(f^t - f)/t\}$ satisfying (45) – (46). □

In [11] it was shown that the solution (w, f) to the system (9) – (10) has regularity $(w, f) \in \{H_0^2(\Omega) \cap H^4(\Omega)\} \times \{H_0^2(\Omega) \cap H^4(\Omega)\}$. Hence as well as from (11) it follows

$$\nabla w \mathbf{V}(0) \in H^2(\Omega) \quad \nabla f \mathbf{V}(0) \in H^2(\Omega) \tag{57}$$

Integrating by parts the system (45) – (46) two times, eliminating the terms containing the derivatives of $\mathbf{V}(0)$ and taking into account (44) as well as (57) we obtain the system of equations determining the shape derivative $(W, F) \in H^2(\Omega) \times H^2(\Omega)$ of the solution $(w_t, f_t) \in H_0^2(\Omega) \times H_0^2(\Omega)$ of the system (14) – (15):

$$a(W, \phi) = b(F, w, \phi) + b(f, W, \phi) + \lambda\tilde{b}(W, \phi) + l_1(W, w, \phi) \quad \forall \phi \in H_0^2(\Omega) \tag{58}$$

$$a(F, \eta) = -2b(W, w, \eta) + l_1(F, f, \eta) \ \ \forall \eta \in H_0^2(\Omega) \tag{59}$$

where

$$l_1(W, w, \phi) = \int_\Gamma W \frac{\partial}{\partial n}(\Delta\phi)\, d\Gamma - \int_\Gamma \left(\frac{\partial W}{\partial n} + \Delta w \mathbf{V}(0)\mathbf{n} \right) \Delta\phi \, d\Gamma \tag{60}$$

Note that in (60), by Sobolev trace theorem [1], we have on Γ: $W \in H^{3/2}(\Gamma)$, $\frac{\partial W}{\partial n} \in H^{1/2}(\Gamma)$, $\Delta\phi \in H^{-1/2}(\Gamma)$, $\frac{\partial}{\partial n}\Delta\phi \in H^{-3/2}(\Gamma)$. Asuming ϕ more regular, i.e.,

$\phi \in \{H_0^2(\Omega) \cap H^4(\Omega)\}$ we have traces in $\Delta\phi \in H^{3/2}(\Gamma)$, $\frac{\partial}{\partial n}\Delta\phi \in H^{1/2}(\Gamma)$. For definition of Sobolev trace spaces $H^s(\Gamma)$, s real number, see [1].

Let us determine the boundary conditions for the derivatives W and F. Note that from (3) it follows that on the boundary Γ:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial s} = 0 \tag{61}$$

$$\frac{\partial^2 w}{\partial s^2} = \frac{\partial^2 f}{\partial s^2} = 0 \qquad \frac{\partial^2 w}{\partial s \partial n} = \frac{\partial f}{\partial s \partial n} = 0 \tag{62}$$

where $\frac{\partial w}{\partial s}$ and $\frac{\partial^2 w}{\partial s^2}$ denote, respectively, the first and second derivatives of the function w in the tangent direction s to the boundary Γ. From $w_t = f_t = 0$ on the boundary Γ_t and from (3) it results that $\dot{w} = \dot{f} = 0$ on the boundary Γ. Hence and from (44), (61) it follows that on the boundary Γ:

$$W = F = 0 \tag{63}$$

Using the same arguments as above for the gradients of the functions w_t and f_t as well as taking into account (3),(44) and (62) we obtain on the boundary Γ:

$$\frac{\partial W}{\partial n} = -\frac{\partial^2 w}{\partial n^2}\mathbf{V}(0)\mathbf{n}, \quad \frac{\partial F}{\partial n} = -\frac{\partial^2 f}{\partial n^2}\mathbf{V}(0)\mathbf{n} \tag{64}$$

Note, that the boundary conditions (63),(64) imply vanishing of the integral (60).

4.2. The form of the directional derivative of the cost functional

In order to eliminate W from (40) we introduce an adjoint state $(p, q) \in H_0^2(\Omega) \times H_0^2(\Omega)$ satisfying the following system of equations:

$$a(p,\phi) = b(p,f,\phi) + \lambda\tilde{b}(p,\phi) - 2b(q,w,\phi) - 2\int_\Omega w\phi \,\mathrm{d}x + \int_\Gamma \Delta p \frac{\partial \phi}{\partial n}\,\mathrm{d}\Gamma \;\forall\, \phi \in H(\Omega) \tag{65}$$

$$a(q,\eta) = b(p,w,\eta) + \int_\Gamma \Delta q \frac{\partial \eta}{\partial n} d\Gamma \quad \forall\, \eta \in H(\Omega) \tag{66}$$

where

$$H(\Omega) = \{\phi \in H^2(\Omega) \; : \; \phi = 0 \text{ on } \Gamma\} \tag{67}$$

and $(w, f) \in H_0^2(\Omega) \times H_0^2(\Omega)$ denotes the solution to the system (9)–(10). For λ suitably small, it follows from [11], the system (65),(66) has a unique solution $(p, q) \in H_0^2(\Omega) \times H_0^2(\Omega)$. Moreover this solution has regularity $(p, q) \in \{H_0^2(\Omega) \cap H^4(\Omega)\} \times \{H_0^2(\Omega) \cap H^4(\Omega)\}$. Hence, by Sobolev trace theorem [1], we have on Γ: $\Delta p \in H^{3/2}(\Gamma)$, $\Delta q \in H^{3/2}(\Gamma)$.

Lemma 5. The directional derivative $\mathrm{d}J(\Omega, \mathbf{V})$ of the cost functional (16) at a point Ω in the direction $\mathbf{V}$ is given by:

$$\mathrm{d}J(\Omega, \mathbf{V}) = -\int_\Gamma (\Delta w \Delta p + \Delta f \Delta q)\mathbf{V}(0)\mathbf{n}\,\mathrm{d}\Gamma \tag{68}$$

where $(w,f) \in H_0^2(\Omega) \times H_0^2(\Omega)$ and $(p,q) \in H_0^2(\Omega) \times H_0^2(\Omega)$ satisfy, respectivly, systems (9) – (10) and (65) – (66).

Proof. Setting $\phi = W$ in (65) and taking into account (40) we obtain:

$$2\int_\Omega Ww\,\mathrm{d}x = b(p,f,W) + \lambda\tilde{b}(p,W) - a(p,W) - 2b(q,w,W) - \int_\Gamma \triangle w \triangle p \mathbf{V}(0)\mathbf{n}\,\mathrm{d}\Gamma \quad (69)$$

Using (64) and setting $\eta = F$ in (66) we obtain from (66):

$$a(q,F) = b(p,w,F) - \int_\Gamma \triangle f \triangle q \mathbf{V}(0)\mathbf{n}\,\mathrm{d}\Gamma \tag{70}$$

Setting $\phi = p$ in (58) and $\eta = q$ in (59) we obtain:

$$\begin{aligned} a(W,p) &= b(F,w,p) + b(f,W,p) + \lambda\tilde{b}(W,p) \\ a(F,q) &= -2b(W,w,q) \end{aligned} \tag{71}$$

From (69) – (71) it results (68). □

The necessary optimality condition for the problem (16) has the standard form:

Lemma 6. For all vector fields $\mathbf{V}$ defined by (11) – (12) an optimal solution $\hat{\Omega} \in U$ to the problem (16) satisfies the following condition:

$$\mathrm{d}J(\hat{\Omega}, \mathbf{V}) \geq 0 \tag{72}$$

where $\mathrm{d}J(\hat{\Omega}, \mathbf{V})$ is given by (68).

Proof. The proof is standard [7]. □

5. CONCLUDING REMARKS

In the paper the shape optimization problem for von Kármán plate is considered. The small deflections of the plate are assumed. The conditions of the existence of the optimal domain are discussed. Using the material derivative method the directional derivative of the cost functional with respect to the variation of the domain occupied by the plate is calculated. This derivative depends only on the normal component $\mathbf{V}(0)\mathbf{n}$ of the vector field $\mathbf{V}(t,x)$ at t=0 to the boundary Γ of the domain Ω. Necessary optimality condition is derived.

Selecting the vector field $\mathbf{V}$ and the family of admissible domains $\tilde{U}$ in the form (18), (19), respectively, we can solve problem (16) numerically. Finite element approximation for this problem is proposed in [19]. The optimization algorithms for solving such optimization problems are discussed in [2, 14, 15, 20, 22, 23]. The calculated directional derivative (68) of the cost functional (16) can be used in the optimization algorithms for calculating a descent direction [22]. To accelerate the convergence of these algorithms we have to use second order derivatives of the cost functional. The first attempts in this direction are reported in [12, 16].

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