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A UNIFIED OPTIMALITY CONDITION FOR EIGENVALUE PROBLEMS

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An optimality condition is derived for the problem of maximization of the first eigenvalue of an abstract elliptic differential operator in a Hilbert space. This condition unifies some known criteria in optimal design and gives some new ones as well. The result is obtained by using some techniques of nonsmooth optimization.

1. INTRODUCTION

In structural optimization eigenvalue problems (we mean those of maximization of the first eigenvalue of a differential operator over admissible design) are very important, see for example, [4], [5], [8] and [12]. One of the main difficulties is to obtain useful necessary conditions for the optimal design. The work on such problems in optimal design began long time ago (see [13]). Some of the early results, however, proved wrong because the possible presentation of the multiple eigenvalues was neglected (see [13]). It was not until 1977 that Olhoff and Rasmussen, in the study of the Lagrange problem, realized that the first eigenvalue does not vary smoothly with design parameters at the points where its multiplicity exceeds one and therefore the formal differentiation in the early work would be hard to justify. It turns out that simple eigenvalues are Gateaux differentiable with respect to design but repeated eigenvalues can only be expected to be directionally differentiable in general (see [8]). Although design sensitivity analysis of eigenvalue variations and explicit directional derivatives of repeated eigenvalues were present in eighty's (see [8] and [14] for example), they do not directly give useful optimality conditions for the optimal design if repeated eigenvalues appear. For example, it does not seems easy at all from them to derive the optimality condition which was proposed by [1] and [11] for the problem of maximization of a column's Euler buckling load under the clamped-champed boundary conditions (see also [5]), and is useful in optimal design. The main reason seems that in general, directional derivatives of repeated eigenvalues are not linear functional in direction arguments. Recently this condition has been rigorously justified in [5] by the method of explicitly calculating the generalized gradient of Clarke for some related functionals. A similar work can also be found in [2]. It is of interests to derive some optimality conditions for an abstract eigenvalue problem which includes as many as possible concrete problems and then to unify some known conditions. It should be noted that the approaches used in [2] and [5] do not seem suitable for this purpose though they could apparently be applied to many concrete problems in optimal design.

It is the purpose of this paper to obtain some optimality condition for an abstract eigenvalue problem. We achieve this by exploring the well known formula of semi-derivative of the first eigenvalues. Our result includes some known results in [2] and [5]. Our approach seems applicable to a wide range of problems including nonlinear ones.

Let H and V be two Hilbert spaces such that $V \subset H$ and the imbedding operator is continuous and compact. Let U_{ad} be a closed and bounded convex set in U, another Banach space.

We assume that the function $a(u,\cdot,\cdot)$ is a continuous bilinear form on $V \times V$ for any $u \in U_{ad}$ and that $a(u,z,z) \geq c\,\|z\|_V^2$ for any $z \in V$, where c is independent of u. Further we assume that the functional $(u,z,y) \longrightarrow a(u,z,y) \in C^1\,(U_{ad} \times V \times V)$. We will suppose that there is a Banach space W such that $U = W^*$, and that for any $(u,z) \in U \times V$ the linear functional $v \longrightarrow a'_u(u,z,z)\,(v)$ is in W.

Let $b(\cdot,\cdot)$ be a symmetric bilinear and continuous form on $H \times H$. We further suppose that $b(h,h) > 0$ if $h \neq 0$. We are in the position of considering the following abstract first eigenvalues problem:

$$\sup_{u \in U_{ad} \cap G} \ \inf_{b(z,z)=1,\, z \in V} a(u,z,z), \tag{AEP}$$

where $G = \{u \in U : f(u) = 0 \text{ and } g(u) \leq 0\}$ with $f,\, g \in C^1(U)$ and we will always suppose that $U_{ad} \cap G$ is nonempty.

For existence of optimal controls of (AEP) one can refer to [5]. We here only try to find optimality conditions for it. In other words we would like to find some necessary conditions for an optimal control u^* satisfying that

$$\sup_{u \in U_{ad} \cap G} \ \inf_{b(z,z)=1,\, z \in V} a(u,z,z) = \inf_{b(z,z)=1,\, z \in V} a(u^*,z,z).$$

In the next section we will present our optimality condition for (AEP).

2. THE OPTIMALITY CONDITION

Before giving our main result let us recall a result in convex function theory (see [7]), which will be used later.

Lemma 2.1. Let E be a compact convex set in a locally convex space D and let $\{f_t\}$ $(t \in T)$ be a class of convex and lower semicontinuous functionals on E. If the inequality system:

$$f_t(e) \leq 0, \quad t \in T$$

has no solutions in E, then there is a sequence of finite numbers $\{\lambda_i\}$ $(1 \leq i \leq m)$ such that $\lambda_i \geq 0$ $(1 \leq i \leq m)$ and $\sum_{i=1}^{m} \lambda_i = 1$, and a sequence of indexes $\{t_i\}$ in

T $(1 \le i \le m)$ such that

$$\sum_{i=1}^{m} \lambda_i f_{t_i}(e) > 0, \quad \text{for any } e \in E.$$

If T is finite, then the condition that E is compact can be further removed.

Let $u \in U_{ad}$ and let $W(u) = \{z \in V : b(z,z)=1,\ a(u,z,z)\} = \inf_{b(y,y)=1} a(u,y,y)$. From the compactness of the imbedding operator from V to H and the continuity of b on $H \times H$, the set $\{z \in V : b(z,z) = 1\}$ is weakly closed. Therefore $W(u)$ is a nonempty closed set for any $u \in U_{ad}$. We are now in the position to give our main result:

Theorem 2.1. Let u^* be an optimal control of (AEP). Assume that the eigenfunction space correspondent to the first eigenvalue $\lambda^* = \lambda(u^*) = \inf_{b(z,z)=1} a(u^*, z, z,)$ is n-dimensional. Then there are n elements $z_1, z_2, \ldots, z_n$ (normalized eigenfunctions, see below) in $W(u^*)$, $\theta_1 \ge 0$, $\sum\limits_{i=1}^{3} \theta_i^2 > 0$, $\sum\limits_{i=1}^{n} d_{ii}^2 = 1$ and $d_{ii}d_{jj} \ge d_{ij}^2$ $(1 \le i,\ j \le n)$, such that for any $u \in U_{ad}$

$$\theta_1 \sum_{i,j=1}^{n} d_{ij} a'_u(u^*, z_i, z_j)\,(u - u^*) \le (\theta_2 \nabla f(u^*) + \theta_3 \nabla g(u^*))\,(u - u^*). \tag{2.1}$$

If further u^* is not a boundary points of U_{ad}, and ∇f and ∇g are linearly independent, then

$$\sum_{i,j=1}^{n} d_{ij} a'_u(u^*, z_i, z_j) = \theta_2 \nabla f(u^*) + \theta_3 \nabla g(u^*). \tag{2.2}$$

Proof. The idea is that instead of using the well known necessary conditions in [3] for optimization problems we apply those in [9] in terms of some directional derivatives other than the Clarke's, which seem to give sharper conditions here, and then linearize the conditions obtained by Lemma 2.1. Let u^* be a solution of (AEP) and $F(u) = \inf_{b(z,z)=1} a(u,z,z)$, which is locally Lipschitz from [5]. Then F achieves its maximum at u^* over $U_{ad} \cap G$. By the Ioffe's necessary conditions in [9] (cf. Theorem 3), there are θ_1, θ_2 and θ_3 such that $\theta_1 \ge 0$, $\theta_2 \ge 0$, $\sum_{i=1}^{3} \theta_i^2 > 0$ and

$$0 \in N_{U_{ad}}(u^*) + \theta_1\, \partial\psi_1(0) + \theta_2\, \partial\psi_2(0) + \theta_3\, \nabla f(u^*), \tag{2.3}$$

where $N_{U_{ad}}(u^*)$ is the normal cone of U_{ad} at u^*, ψ_1 and ψ_2 are two convex first order approximation of $H(u) = \sup_{b(z,z)=1} -a(u,z,z) = -F(u)$ and $g(u)$ at u^* (see [9]) and $\partial\psi_i$ $(i = 1, 2)$ is the subgradient of ψ_i in convex analysis. Select $\psi_2(h) = \nabla g(u^*)\,h$ or $\partial\psi_2(0) = \{\nabla g(u^*)\}$. From (2.3) there is $\xi \in \partial\psi_1(0)$ such that for any $u \in U_{ad}$,

$$\theta_1 \xi(u - u^*) \ge -[\theta_2\, \nabla g(u^*) + \theta_3\, \nabla f(u^*)]\,(u - u^*). \tag{2.4}$$

We now apply Theorem 4.3 in [10] to prove that

$$\varliminf_{t\to 0}\ [F(u+tu)-F(u)]/t \geq \inf_{z\in W(u)} a'_u(u,z,z)\,(v). \tag{WS}$$

To see this one only needs to show that the weak stability condition in [10] holds for every $u \in U_{ad}$ here and this can be seen by choosing the element x_0 in $W(u)$ and curve $x\,[0,\delta) \to V$ in the weak stability condition of [10] as x_0 being any element in $W(u)$ and $x(t) \equiv x_0$ since F and $a(\cdot,z,z)$ (for any $z \in V$) are Lipschitz. It follows from (WS) that the convex and continuous function $H'(u^*,\cdot)$ is a first order approximation of H at u^* (as it is a convex lower semicontinuous function with finite values, see below), where $H'(u^*,v)$ is defined by

$$H'(u^*,v) = \sup_{z\in W(u^*)} -a'_u(u^*,z,z)\,(v). \tag{2.5}$$

Consequently for any $u \in U_{ad}$,

$$\theta_1\,H'(u^*,u-u^*) \geq \theta_1\,\psi(u-u^*) \geq -[\theta_2\,\nabla g(u^*)+\theta_3\,\nabla f(u^*)]\,(u-u^*).$$

Note that $b(z,z)>0$ if $z\neq 0$. Thus one can find $\{z_i\}$ $(i=1,2,\ldots,n)$ in $W(u^*)$ such that they are linearly independent and $b(z_i,z_j)=\delta_{ij}$ $(1\leq i,\ j\leq n)$, where $\delta_{ij}=1$ if $i=1$ and $\delta_{ij}=0$ if $i\neq j$. Then $W(u^*)=\left\{\sum a_i z_i\ (1\leq i\leq n) \text{ with } \sum_{i=1}^n a_i^2=1\right\}$. From (2.4) and (2.5) one infers that for any $u\in U_{ad}$ there is a $z(u)=\sum a_i(u)\,z_i$ with $\sum_{i=1}^n a_i^2=1$ such that

$$\theta_1\,a'_u(u^*,z(u),z(u))\,(u-u^*) \leq [\theta_2\,\nabla g(u^*)+\theta_3\,\nabla f(u^*)]\,(u-u^*). \tag{2.6}$$

Now note that the bounded closed convex set U_{ad} is U^* weakly compact (U is the dual space of W and thus one can define the weak star $(\sigma(V,W))$ topology on U) and the functional $v \longrightarrow a'_u(u,z,z)\,(v)$ is U^* weakly continuous. Thus one can apply Lemma 2.1 to this case. Therefore for any $\varepsilon>0$ there are $\{\mu_j(\varepsilon)\}$ $(j=1,2,\ldots,m(\varepsilon))$ such that $\mu_j(\varepsilon)\geq 0$, $\sum_{j=1}^m \mu_j(\varepsilon)=1$ and

$$\begin{aligned}\theta_1\sum_{j=1}^m\left(\mu_j(\varepsilon)\,a'_u\left(u^*,\sum_{i=1}^n a_i^j z_i,\sum_{i=1}^n a_i^j z_i\right)\,(u-u^*)\right) &\leq\\ \leq\ [\theta_2\,\nabla g(u^*)+\theta_3\,\nabla f(u^*)]\,(u-u^*)+\varepsilon&\end{aligned}$$

for any $u\in U_{ad}$. Thus for any $u\in U_{ad}$,

$$\begin{aligned}\theta_1\sum_{i=1,j=1}^n\left(\sum_{k=1}^m\left(\mu_k(\varepsilon)\,a_i^k a_j^k\right)a'_u(u^*,z_i,z_j)\,(u-u^*)\right) &\leq\\ \leq\ [\theta_2\,\nabla g(u^*)+\theta_3\,\nabla f(u^*)]\,(u-u^*)+\varepsilon.&\end{aligned}$$

Note that $\left|\sum_{k=1}^m \mu_k(\varepsilon)\,a_i^k a_j^k\right| \leq \sum_{k=1}^m \mu_k(\varepsilon)\,\left|a_i^k a_j^k\right| \leq 1$. Let $\varepsilon\to 0$ and suppose that $\left(\sum_{k=1}^m \mu_k(\varepsilon)\,a_i^k a_j^k\right) \longrightarrow d_{ij}$ (possibly a subsequence) as $\varepsilon\to 0$. From the Holder inequality $d_{ii}d_{jj}\geq d_{ij}^2$ $(1\leq i,\ j\leq n)$ so that one obtains (2.1). If u^* is not a boundary point of U_{ad}, then the set $\{(v-u^*),\ v\in U_{ad}\}$ contains zero point as its interior point so that (2.1) is equivalent with (2.2). Note that ∇f and ∇g are independent and one concludes that $\theta_1\neq 0$. □

Remark 2.1. It follows from the proof of Theorem 2.1 that the condition that U_{ad} is bounded can be removed. To see this, let us consider (AEP) with $F(u)$ replaced by $F(u) - R\|u - u_0\|_U^2$, where u_0 is a solution of (AEP), and U_{ad} replaced by $U_{ad} \cap B_R$, where $B_R = \{u \in U : \|u\| \le R + \|u\|_0\}$. With some large R one will find that the problem has the same solution u_0 and therefore one has (2.1) with U_{ad} replaced by $U_{ad} \cap B_R$. This prove our conclusion. Thus if $U_{ad} = U$, (2.2) will hold. If we replace the vector $u - u^*$ in (2.1) by an element of a suitable tangent cone of U_{ad} at u^*, then we can even manage to remove the convexity condition of U_{ad}.

Remark 2.2. The condition that there is a Banach space W such that $U = W^*$ can naturally be replaced by some others that can ensure that U_{ad} is a weakly precompact set in U. In this cases the condition that $a_i' \in W$ can be removed.

It is frequently occurred in optimal design that b depends also on u. It is very interesting to know some similar conditions in such case (see [2]). The ideas used in this paper seem possible to generalize to such case. For example, suppose that b depends on u and the problem: $F(u) = \inf_{b(u,y,y)=1} a(u,y,y)$ is weakly stable in the sense given in [10]. Then one has that $\underline{\lim}_{t\to 0^+}[F(u+tv) - F(u)] \ge \inf_{z\in W(u)} a_u'(u,z,z)(v) - \lambda(u)\, b_u'(u,z,z)(v)$ (see [10] and [14]). Therefore one can choose $\psi_1(h) = -\inf_{z\in W(u^*)} a_u'(u^*,z,z)(v) - \lambda(u^*)\, b_u'(u^*,z,z)(v)$ and then obtains (2.2) but with $a_u'(u^*,z_i,z_j)(u-u^*)$ being replaced by $a_u'(u^*,z_i,z_j)(u-u^*) - \lambda(u^*)$ $b_u'(u^*,z_i,z_j)(u-u^*)$. For example it is known from [8] that this inequality (actually is an equality in those cases) is true for all the problems given in [2] and [8] (note it is given in another equivalent form). Therefore all the results in [2] can be viewed as special cases of these results. We will study this problem in another paper. From the proof of Theorem 2.1 it can also be seen that one can prove the theorem directly from (2.4) by calculating $\partial\psi_1(0)$ using Lemma 2.1. The proof is very much similar to that of Theorem 2.1.

In the following section we will give some applications of this abstract result just to show that it can indeed unify some known results.

3. SOME APPLICATIONS

Throughout this section we adopt the standard notation $W^{m,p}(\Omega)$ $(p \ge 1)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. We denote $W^{2,2}(\Omega)$ by $H^2(\Omega)$ and $W_0^{2,2}(\Omega)$ by $H_0^2(\Omega)$. We will assume that Ω is a bounded open set in R^ℓ with a Lipschitz boundary $\partial\Omega$.

First let $\Omega = (0,1)$, $V = H_0^2(0,1)$, $H = H_0^1(0,1), U = L^\infty(0,1)$, and $U_{ad} = \{\sigma \in U : 0 < \alpha \le \sigma \le \beta\}$ where α and β are constants. Then we take $a(\sigma,z,y) = \int_0^1 \sigma^p z''y''\,\mathrm{d}x$ with $p > 0$, $b(h,t) = \int_0^1 h't'\mathrm{d}x$, $g = 0$ and $f(u) = \int_0^1 u\,\mathrm{d}x - 1$. In this case (AEP) will be the following eigenvalue problem: (see [2] and [8])

$$\sup_{\sigma\in U_{ad}} \lambda(\sigma), \qquad \text{(AEP)}_{\text{CBC}}$$

where $\lambda(\sigma)$ is the first eigenvalue of the following problem:

$$\int_0^1 \sigma^p z'' y'' \mathrm{d}x = \lambda \int_0^1 y' z' \mathrm{d}x, \quad \text{for any } y \in H_0^2(0,1).$$

This problem is nothing but that of optimal design of columns against buckling under the clamped-clamped boundary conditions. Note that the well-known fact that $\lambda(\sigma) = \inf_{b(z,z)=1} a(\sigma, z, z)$ so that Theorem 2.1 can be applied here since all the conditions required in it hold. From Theorem 2.1 one has the following well known result (see [5]):

Proposition 3.1. Let σ be a solution of $(\mathrm{AEP})_{\mathrm{CBC}}$. Suppose that the eigenfunction space correspondent to the first eigenvalue $\lambda(\sigma)$ is 2-dimensional. Then there are two elements z_1 and z_2 (normalized eigenfunctions) in $W(\sigma)$ such that for any $\eta \in U_{ad}$

$$\int_0^1 \left[\theta_1 \left(\sum_{i,j=1}^{2} d_{ij} \sigma^{p-1} z_i'' z_j'' \right) - 1 \right] (\eta - \sigma) \,\mathrm{d}x \le 0, \tag{3.1}$$

where $\theta_1 > 0$, $d_{ij} = d_{ij}$ and $d_{ii} d_{jj} \ge d_{ij}^2$ $(1 \le i,\ j \le 2)$. If further σ is not a boundary point of U_{ad}, then

$$\theta_1 \sum_{i,j=1}^{2} d_{ij} \sigma^{p-1} z_i'' z_j'' = 1. \tag{3.2}$$

Proof. All the conditions required in Theorem 2.1 can be easily confirmed here. From a physical consideration (see [5]) $\theta_1 \theta_2 \ne 0$. By the well known procedure we can obtain a pointwise form of (3.1), which leads to (3.2). □

One can see from Section 2 that Theorem 2.1 can be easily used to other problems in optimal design.

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REFERENCES

[1] A.S. Bratus and S. Seiranian: Bimodel solutions in eigenvalue optimization problems. Appl. Math. Mech. *43* (1983), 451–457.

[2] A.S. Bratus: Condition of extremum for eigenvalues of some elliptic boundary value problems. J. Optim. Theory Appl. *68* (1991), 423–437.

[3] F. Clarke: Optimization and Nonsmooth Analysis. J. Wiley, New York 1983.

[4] K. Choi and E.J. Haug: Optimization of structures with repeated eigenvalues. In: Optimal Design of Distributed Structures (E.J. Haug and J. Cea, eds.), Sijthoff–Noordhoff, Leyden 1981, pp. 219–277.

[5] S.J. Cox and M.L. Overton: On the optimal design of columns against buckling. Preprint of Rice university 1990, TR90-13.

[6] M. Delfour and J.P. Zolesio: Shape sensitivity analysis via min max differentiability. SIAM J. Control Optim. *26* (1988), 835–863.

[7] Z. Z. Guang and G. C. Zhang: Linear Functional Analysis. Science Press of Shangai, Shanghai 1979 (in Chinese).

[8] E. J. Haug and B. Rousselet: Design sensitivity of eigenvalue variations. In: Optimal Design of Distributed Structures (E. J. Haug and J. Cea, eds.), Sijthoff–Noordhoff, Leyden 1981, pp. 1370–1396.

[9] A. D. Ioffe: Necessary and sufficient conditions for a local minimum I. SIAM J. Control Optim. *17* (1979), 245–250.

[10] F. Lempio and H. Maurer: Differential stability in infinite-dimensional nonlinear programming. Appl. Math. Optim. *6* (1980), 139–152.

[11] E. Masur: Optimal structural design under multiple eigenvalue constraints. Internat. J. Solids and Structures *20* (1984), 211–231.

[12] N. Olhoff and S. Rasmussen: On single and bimodal optimum buckling loads of clamped columns. Internat. J. Solids and Structures *13* (1977), 605–614.

[13] I. Tadjbakhsh and J. Keller: Strongest columns and isoperimetric inequalities for eigenvalues. J. Appl. Mech. *29* (1962), 159–164.

[14] J. P. Zalesio: Semi-derivatives of repeated eigenvalues. In: Optimal Design of Distributed Structures (E. J. Haug and J. Cea, eds.), Sijthoff–Noordhoff, Leyden 1981, pp. 1457–1473.

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