KYBERNETIKA — VOLUME 29 (1993), NUMBER 3, PAGES 210-221

AN IMPLICIT-FUNCTION THEOREM FOR A CLASS OF MONOTONE GENERALIZED EQUATIONS

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In this paper we prove an implicit-function theorem for a class of generalized equations defined by a monotone set-valued mapping in Hilbert spaces. We give applications to variational inequalities, single-valued functions and a class of nonsmooth functions.

1. INTRODUCTION

Implicit-function theorems for generalized equations play an important role in many applications, especially in the stability and sensitivity analysis of variational inequalities and optimization problems and in the convergence analysis of numerical algorithms solving such problems. We refer for instance to Fiacco [7] and Ioffe and Tihomirov [9] for applications of the classical implicit-function theorem in this context. Further results and some extensions of the classical implicit-function theorem can be found in Fiacco [8].

In [15] Robinson proved an implicit-function theorem for a class of generalized equations which he called strongly regular. This result has been widely used in the stability and sensitivity analysis of optimization and optimal control problems (see e.g. Robinson [15, 16], Alt [1, 2], Ito-Kunisch [10], Malanowski [13]) and in the convergence analysis of algorithms solving optimization problems and variational inequalities (see e.g. Robinson [16], Alt [1, 2]). In a recent paper [17], Robinson could extend his implicit-function theorem to a class of nonsmooth functions.

In [11, 12] Kassay and Kolumbán derived implicit-function theorems for a class of generalized equations defined by a monotone set-valued mapping. They have shown that from these theorems the classical implicit function theorem and Browder's surjectivity theorem can be easily derived. They also presented applications to variational inequalities.

The aim of the present paper is to further develop the rather general implicit-function theorem of Kassay and Kolumbán [12] in view of applications to variational inequalities and a class of generalized equations defined by nonsmooth functions.

[1] This research was performed while the second author was a visitor at the Institute of Applied Mathematics, University of Hamburg, Germany, supported by a grant from the Humboldt-Stiftung, Bonn, Germany.

Throughout the paper let W be a topological space, and H a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. We study generalized equations of the form

$$0 \in T(w, x), \tag{1.1}$$

where T is a set-valued map from $W \times H$ to H. Suppose $w_0 \in W$, and $x_0 \in H$ is a solution of the generalized equation (1.1) for $w = w_0$, i.e., $0 \in T(w_0, x_0)$. We give sufficient conditions on T such that there exists a neighborhood W_0 of w_0 and a function $x: W_0 \to H$ with

$$0 \in T(w, x(w)) \quad \text{for all} \quad w \in W_0,$$

and

$$x(w) \to x(w_0) \quad \text{for} \quad w \to w_0.$$

The main assumption will be uniform coercivity or uniform strong monotonicity of the mapping $T(w, \cdot)$. If the mapping T satisfies additional continuity or differentiabily assumptions, then it is shown that the mapping $x(\cdot)$ inherits some of these properties.

The paper is organized as follows. In Section 2 we introduce some basic definitions. Further we recall an implicit-function theorem due to Kassay and Kolumbán [12]. In Section 3 we prove an implicit-function theorem for a class of coercive multivalued mappings. In Section 4 we give some applications to variational inequalities. In Sections 5 and 6 the implicit-function theorem is applied to single-valued functions and to a class of nonsmooth functions.

2. DEFINITIONS AND AUXILIARY RESULTS

We use some usual notations and properties of set-valued maps which can be found e.g. in [3].

Let $T: H \rightsquigarrow H$ be a set-valued mapping. The *domain* of T is the set

$$\mathrm{Dom}(T) = \{x \in H \mid T(x) \neq \emptyset\}.$$

The *graph* of T is defined by

$$\mathrm{Graph}(T) = \{(x, y) \in H \times H \mid y \in T(x)\}.$$

The mapping T is called *monotone* if for all $x, y \in \mathrm{Dom}(T)$ and all $u \in T(x)$, $v \in T(y)$ the inequality

$$\langle u - v, x - y \rangle \geq 0 \tag{2.1}$$

holds. T is said to be *maximal monotone* if there is no other monotone set-valued map whose graph contains strictly the graph of T. T is said to be *coercive* if there exists an increasing function $\alpha: \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathrm{Dom}(T)$ and all $u \in T(x)$, $v \in T(y)$ the inequality

$$\langle u - v, x - y \rangle \geq \alpha(\|x - y\|)\, \|x - y\| \tag{2.2}$$

holds. T is said to be *strongly monotone* if for all $x, y \in \mathrm{Dom}(T)$ and all $u \in T(x)$, $v \in T(y)$ the inequality

$$\langle x - y, u - v \rangle \geq \beta \, \|x - y\|^2 \tag{2.3}$$

holds with a positive β. T is called *injective* if for each $x, y \in H$, $T(x) \cap T(y) \neq \emptyset$ implies $x = y$. Clearly, if T is coercive, it is injective, and if T is strongly monotone, it is coercive with $\alpha(t) = \beta t$.

A mapping $S: H \to H$ is called *nonexpansive*, if

$$\|S(x) - S(y)\| \leq \|x - y\|$$

for all $x, y \in H$, i.e., S is Lipschitz continuous with modulus 1.

The following fundamental characterization of maximal monotone maps is due to Minty [14] (see e.g. [3], Chap. 6, Sec. 7, Theorem 5 and Theorem 8).

Theorem 2.1. Let $T: H \rightsquigarrow H$ be a set-valued map. Then T is maximal monotone if and only if $1 + T$ maps $\mathrm{Dom}(T)$ onto H. In this case, $S = (1 + T)^{-1}$ is a single-valued map from H to H which is nonexpansive.

By $\mathrm{Fix}(S)$ we denote the set of fixed points of S.

Remark. Suppose $T: H \rightsquigarrow H$ is maximal monotone, and define $S = (1 + T)^{-1}$. Then $x \in H$ is a fixed point of S if and only if $x \in \mathrm{Dom}(T)$ and

$$x \in (1 + T)(x) = x + T(x),$$

which is equivalent to the fact that $0 \in T(x)$.

If $x \in H$ and $r > 0$, then we denote by $B(x, r)$ the closed ball with radius r around x. For a closed convex subset $C \subset H$, $P_C: H \to C$ denotes the metric projection.

Definition 2.2. Let $r > 0$, $x_0 \in H$, $S: H \to H$. One says that S is *retractible* on $B(x_0, r)$ if $\mathrm{Fix}(P_{B(x_0,r)} S) \subset \mathrm{Fix}(S)$.

The following lemma gives a sufficient condition for retractability.

Lemma 2.3. Let $T: H \to H$ be a maximal monotone set-valued map, $r > 0$, $x_0 \in H$, and $S = (1 + T)^{-1}$. Suppose that

(R) for each $x \in H$ with $\|x - x_0\| = r$, and $\|S(x) - x_0\| > r$ there exists $y \in B(x_0, r)$, such that the inequality

$$\langle x - z, x - y \rangle > 0 \tag{2.4}$$

holds for $z = S(x)$.

Then S is retractible on $B(x_0, r)$.

Proof. Let the assumptions of the lemma be satisfied, and suppose that S is not retractible on $B(x_0, r)$. Then there is a $x \in \mathrm{Fix}(P_{B(x_0,r)}S)$ with $x \notin \mathrm{Fix}(S)$. The characterization of the projection implies

$$\langle x - S(x), x - y\rangle \leq 0 \quad \text{for all} \quad y \in B(x_0, r). \tag{2.5}$$

In case $\|S(x) - x_0\| \leq r$, we obtain $x = P_{B(x_0,r)}S(x) = S(x)$. Hence $\|S(x) - x_0\| > r$, and therefore $P_{B(x_0,r)}S(x)$ is a boundary point of $B(x_0, r)$. This implies $\|x - x_0\| = \|P_{B(x_0,r)}S(x) - x_0\| = r$. By Assumption (R) there exists $y \in B(x_0, r)$ such that (2.4) holds, which contradicts (2.5). □

Remark. Let $w \in W$. Suppose $T(w, \cdot): H \rightsquigarrow H$ is maximal monotone, and define $S_w = (1 + T(w, \cdot))^{-1}$. Then by the remark preceding Definition 2.2 $x(w) \in H$ is a fixed point of S_w if and only if $0 \in T(w, x(w))$, i.e., $x(w)$ is a solution of the generalized equation (1.1).

Based on a fixed point theorem for nonexpansive maps due to Browder ([5], Theorems 8.2 and 8.5), Kassay and Kolumbán [12] proved the following implicit-function theorem ([12], Theorem 3.1).

Theorem 2.4. Let $T: W \times H \rightsquigarrow H$ be a set-valued map, $x_0 \in H$, $w_0 \in W$, and $d > 0$. Suppose that there exists a neighborhood W_0 of w_0 such that

(A1) $0 \in T(w_0, x_0)$, and $T(w, \cdot): H \rightsquigarrow H$ is maximal monotone and injective for all $w \in W_0$;

(A2) for each $r \in (0, d]$ there exists a neighborhood $W_r \subset W_0$ of w_0 such that $S_w = (1 + T(w, \cdot))^{-1}$ is retractible on $B(x_0, r)$ for all $w \in W_r$.

Then there exists a unique mapping $x: W_d \to B(x_0, d)$ continuous at w_0 such that $x(w_0) = x_0$ and $0 \in T(w, x(w))$ for all $w \in W_d$.

3. AN IMPLICIT-FUNCTION THEOREM FOR COERCIVE MAPS

In this section we state the main result of the paper, an implicit-function theorem for generalized equations described by a coercive mapping.

One of the main assumptions of Theorem 2.4 is the retractability of the operators S_w. We show that this assumption is satisfied, if the mappings $T(w, \cdot)$ are uniformly coercive and satisfy a consistency condition (compare Aubin and Frankowska [4], Definition 5.4.1).

Definition 3.1. Let $T: W \times H \rightsquigarrow H$ be a set-valued map, $x_0 \in H$, and $w_0 \in W$ such that $0 \in T(w_0, x_0)$. Then T is called *consistent in* w at (w_0, x_0), if there is a neighborhood W_0 of w_0 and a function $\beta: W_0 \to \mathbb{R}$ continuous at w_0 with $\beta(w_0) = 0$ such that for each $w \in W_0$ there exists $y_w \in T(w, x_0)$ with $\|y_w\| \leq \beta(w)$.

Definition 3.2. Let $T: W \times H \rightsquigarrow H$ be a set-valued map, and $W_0 \subset W$. The mappings $T(w, \cdot)$ are called *uniformly coercive on* W_0, if there exists an increasing function $\alpha:]0, \infty[\to]0, \infty[$ such that for all $w \in W_0$ and all $x_1, x_2 \in \mathrm{Dom}(T(w, \cdot))$, $x_1 \neq x_2$, $y_1 \in T(w, x_1)$, $y_2 \in T(w, x_2)$ the inequality

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \alpha(\|x_1 - x_2\|) \, \|x_1 - x_2\|$$

holds.

Based on Theorem 2.4 we can state the following implicit-function theorem for coercive mappings.

Theorem 3.3. Let $T: W \times H \rightsquigarrow H$ be a set-valued map, $x_0 \in H$, and $w_0 \in W$. Suppose there exists a neighborhood $W_0 \subset W$ of w_0 and $d > 0$ such that

(C1) $0 \in T(w_0, x_0)$,

(C2) T is consistent in w at (w_0, x_0),

(C3) The mappings $T(w, \cdot)$ are maximal monotone and uniformly coercive on W_0.

Then for any $d > 0$ there exist a neighborhood $W_d \subset W_0$ of w_0 and a unique mapping $x: W_d \to B(x_0, d)$ continuous at w_0 such that $x(w_0) = x_0$ and $0 \in T(w, x(w))$ for all $w \in W_d$.

Proof. Let $w \in W_0$ and an arbitrary $r > 0$ be given. Suppose $x \in H$ with $\|x - x_0\| = r$ and $\|S_w(x) - x_0\| > r$. Define $z = S_w(x)$. Then $x - z \in T(w, z)$. By Assumption (C2) there exist a function $\beta: W_0 \to \mathbb{R}$ continuous at w_0 with $\beta(w_0) = 0$ and $y_w \in T(w, x_0)$ with $\|y_w\| \leq \beta(w)$. Since $\|z - x_0\| > r$ we obtain by Assumption (C3)

$$\begin{aligned} \langle x - z, x - x_0 \rangle - \|x - z\|^2 &= \langle x - z, z - x_0 \rangle \\ &= \langle x - z - y_w, z - x_0 \rangle + \langle y_w, z - x_0 \rangle \\ &\geq \alpha(\|z - x_0\|)\|z - x_0\| - \beta(w)\|z - x_0\| \\ &\geq (\alpha(r) - \beta(w))\|z - x_0\| \,. \end{aligned} \tag{3.1}$$

By Assumption (C2) we can choose a neighborhood $W_r \subset W_0$ of w_0 such that

$$\alpha(r) - \beta(w) > 0$$

for all $w \in W_r$. But then by (3.1) Assumption (R) is satisfied for $T = T(w, \cdot)$ with $y = x_0$. Hence, by Lemma 2.3, S_w is retractible on $B(x_0, r)$ for all $w \in W_r$, i.e., Assumption (A2) of Theorem 2.4 is satisfied. Since (C1) and (C3) imply (A1), it follows from Theorem 2.4 that there exists a unique mapping $x: W_d \to B(x_0, d)$ with the desired properties. □

4. APPLICATION TO VARIATIONAL INEQUALITIES

For a closed convex subset $C \subset H$ we denote by $\partial\psi_C$ the normal cone operator

$$\partial\psi_C(x) = \begin{cases} \{\, y \in H \mid \langle y, c - x\rangle \le 0 \;\forall c \in C \,\}\,, & \text{if } x \in C, \\ \emptyset\,, & \text{if } x \notin C. \end{cases}$$

Let Ω be an open subset of H, and f a mapping from $W \times \Omega$ into H. Furthermore, let T be defined by

$$T(w, x) = f(w, x) + \partial\psi_C(x)\,, \tag{4.1}$$

where $\mathrm{Dom}(T(w,\cdot)) = \mathrm{Dom}(f(w,\cdot)) \cap \mathrm{Dom}(\partial\psi_C) = \Omega \cap C$.

By the definition of the normal cone operator, the generalized equation (1.1) is then equivalent to the variational inequality

$$x \in C \quad \text{and} \quad \langle f(w,x), c - x\rangle \ge 0 \quad \text{for all } c \in C. \tag{4.2}$$

We now show how Theorem 3.3 can be applied to variational inequalities of this type.

Definition 4.1. Let $W_0 \subset W$, $U_0 \subset \Omega$ and $f\colon W \times \Omega \to H$. Then f is called *uniformly coercive on* $W_0 \times U_0$, if there exists an increasing function $\alpha\colon [0,\infty] \to [0,\infty]$ such that for all $w \in W_0$ and all $x_1, x_2 \in U_0$, $x_1 \ne x_2$ the inequality

$$\langle f(w, x_1) - f(w, x_2), x_1 - x_2\rangle \ge \alpha(\|x_1 - x_2\|)\,\|x_1 - x_2\|$$

holds.

Theorem 4.2. Let $x_0 \in H$, $w_0 \in W$. Suppose there exist a neighborhood W_0 of w_0 and $\bar d > 0$ such that with $X := B(x_0, \bar d)$, the following assumptions are satisfied:

(V1) $0 \in T(w_0, x_0)$, i.e., x_0 is a solution of the variational inequality (4.2) for $w = w_0$.

(V2) f is continuous on $W_0 \times X$.

(V3) f is uniformly coercive on $W_0 \times (C \cap X)$.

Then there exist a neighborhood $W_1 \subset W_0$ of w_0 and a unique mapping $x\colon W_1 \to X$ continuous at w_0 such that $x(w_0) = x_0$ and $0 \in T(w, x(w))$ for all $w \in W_1$, i.e., $x(w)$ is a solution of the variational inequality (4.2).

P r o o f. We apply Theorem 3.3 to the mapping

$$F(w, x) = f(w, x) + \partial\psi_{C\cap X}(x)\,,$$

For this mapping, (V1) implies (C1). Furthermore, by Assumption (V1) we have $-f(w_0, x_0) \in \partial\psi_{C\cap X}(x_0)$. This implies

$$y_w = f(w, x_0) - f(w_0, x_0) \in F(w, x_0)\,.$$

By (V2), the mapping $\beta(w) := \|y_w\|$ is continuous at w_0. This shows that (C2) is satisfied. Furthermore, by Theorem 3 of Rockafellar [18], the mappings $F(w,\cdot)$ are maximal monotone. Since the subdifferential $\partial\psi_C$ is monotone, (V3) implies (C3). Therefore, applying Theorem 3.3 to the mapping F with $d = \bar{d}$, we obtain a neighborhood $W_1 \subset W_0$ of w_0 and a unique mapping $x: W_1 \to B(x_0, d)$ continuous at w_0 such that $x(w_0) = x_0$ and $0 \in F(w, x(w))$ for all $w \in W_1$. Since $x(w) \in \operatorname{int} X$, we have

$$\partial\psi_{C\cap X}(x(w)) = \partial\psi_C(w)(x(w)).$$

Hence $F(w, x(w)) = T(w, x(w))$, i.e., $x(w)$ is a solution of the variational inequality (4.2). □

Dafermos ([6], Theorem 2.1) proved a similar result where in addition the set C may depend on the parameter. However, Dafermos requires Lipschitz continuity of f in x while we only need continuity in x. Moreover, strong monotonicity can be replaced by coercivity.

As we shall see in the following section, mappings of the type (4.1) naturally arise in connection with single-valued equations, especially in connection with equations considered by Robinson [15, 17].

If we impose some more restrictive conditions we can show that $x(\cdot)$ is locally Lipschitz continuous.

Definition 4.3. Let $W_0 \subset W$, $U_0 \subset \Omega$ and $f: W \times \Omega \to H$. Then f is called *uniformly strongly monotone on* $W_0 \times U_0$, if for all $w \in W_0$ and all $x_1, x_2 \in U_0$,

$$\langle x_1 - x_2, f(w, x_1) - f(w, x_2)\rangle \geq \alpha\|x_1 - x_2\|^2 \tag{4.3}$$

holds with a positive α.

Corollary 4.4. Let the assumptions of Theorem 4.2 be satisfied. Suppose that in addition W is a subset of a normed linear space and that $f(\cdot, x_0)$ is Lipschitz continuous on W_0 with modulus λ, i.e.,

$$\|f(w, x_0) - f(w_0, x_0)\| \leq \lambda\,\|w - w_0\|$$

for all $w \in W_0$. Suppose further that f is uniformly strongly monotone on $W_0 \times (C \cap X)$. Then the mapping $x(\cdot)$ is Lipschitz continuous on W_1 with modulus $\alpha^{-1}\lambda$.

Proof. By Theorem 4.2 there exists a neighborhood $W_1 \subset W_0$ of w_0 and a unique mapping $x: W_1 \to U_0$ continuous at w_0 such that $x(w_0) = x_0$ and $0 \in T(w, x(w)) = 0$ for all $w \in W_1$. Now let $w \in W_1$. Since the subdifferential $\partial\psi_C$ is monotone, we obtain for arbitrary $z_w \in \partial\psi_C(x(w))$ and $z_0 \in \partial\psi_C(x_0)$

$$\begin{aligned}\alpha\,\|x(w) - x_0\|^2 &\leq \langle f(w, x(w)) - f(w, x_0), x(w) - x_0\rangle \\ &\leq \langle f(w, x(w)) + z_w - (f(w, x_0) + z_0), x(w) - x_0\rangle.\end{aligned}$$

Choosing $z_w = -f(w, x(w))$ and $z_0 = -f(w_0, x_0)$ we obtain

$$\|x(w) - x_0\| \leq \alpha^{-1}\|f(w_0, x_0) - f(w, x_0)\|.$$

This proves the assertion. □

5. APPLICATION TO SINGLE-VALUED FUNCTIONS

We consider now the special case of equations defined by a single-valued monotone map. To this end let Ω be an open subset of H, and $G: W \times \Omega \to H$ a single-valued map. Let $w_0 \in H$ and suppose $x_0 \in \Omega$ is a solution of the equation

$$G(w, x) = 0 \tag{5.1}$$

for $w = w_0$. Then we investigate the solvability of (5.1) for w close to w_0. From Theorem 4.2 we obtain the following result.

Theorem 5.1. Let $x_0 \in \Omega$, $w_0 \in W$. Suppose that there exist neighborhoods W_0 of w_0 and $U_0 \subset \Omega$ of x_0 such that the following holds:

(S1) $G(w_0, x_0) = 0$.

(S2) G is continuous on $W_0 \times U_0$.

(S3) G is uniformly coercive on $W_0 \times U_0$.

Then there exists a neighborhood $W_1 \subset W_0$ of w_0 and a unique mapping $x: W_1 \to U_0$ continuous at w_0 such that $x(w_0) = x_0$ and $G(w, x(w)) = 0$ for all $w \in W_1$.

Proof. We choose $\bar{d} > 0$ such that $X := B(x_0, \bar{d}) \subset U_0$. Define $C = H$, $f(w, x) = G(w, x)$ and $T(w, x) = f(w, x) + \partial\psi_C(x)$. Since $\partial\psi_C(x) = \{0\}$ for all $x \in H$, equation (5.1) is equivalent to the generalized equation $0 \in T(w, x)$. Since Assumptions (V1), (V2), (V3) are satisfied, the assertion follows from Theorem 4.2. □

As in the previous section, by imposing more restrictive conditions we can show that $x(\cdot)$ is locally Lipschitz continuous.

Corollary 5.2. Let the assumptions of Theorem 5.1 be satisfied. Suppose that in addition W is a subset of a normed linear space and that for each $x \in U_0$, $G(\cdot, x)$ is Lipschitz continuous on W_0 with modulus λ, i.e.,

$$\|G(w_1, x) - G(w_2, x)\| \le \lambda \|w_1 - w_2\|$$

for all $w_1, w_2 \in W_0$. Suppose further that G is uniformly strongly monotone on $W_0 \times U_0$. Then the mapping $x(\cdot)$ is Lipschitz continuous on W_1 with modulus $\alpha^{-1}\lambda$.

Proof. By Theorem 5.1 there exists a neighborhood $W_1 \subset W_0$ of w_0 and a unique mapping $x: W_1 \to U_0$ continuous at w_0 such that $x(w_0) = x_0$ and $G(w, x(w)) = 0$ for all $w \in W_1$. Now let $w_1, w_2 \in W_1$. Since $G(w_1, x(w_1)) = G(w_2, x(w_2)) = 0$ we obtain from (4.3)

$$\begin{aligned}\alpha \|x(w_1) - x(w_2)\|^2 &\le \langle G(w_1, x(w_1)) - G(w_1, x(w_2)), x_1 - x_2\rangle \\ &= \langle G(w_2, x(w_2)) - G(w_1, x(w_2)), x_1 - x_2\rangle,\end{aligned}$$

which implies

$$\|x(w_1) - x(w_2)\| \le \alpha^{-1}\|G(w_1, x(w_2)) - G(w_2, x(w_2))\|.$$

This proves the assertion. □

6. APPLICATION TO NONSMOOTH FUNCTIONS

In a recent paper [17], Robinson proved an implicit-function theorem for a class of nonsmooth functions. We show how Robinson's implicit-function theorem can be derived from Theorem 5.1.

Throughout this section let X be a real Hilbert space, Z, W normed linear spaces, $x_0 \in X$ and $w_0 \in W$. Further, let W_0 be a neighborhood of w_0 and U_0 a neighborhood of x_0, and suppose F is a function from $W_0 \times U_0$ to Z, and f is a function from U_0 to Z. Suppose x_0 is a solution of the equation

$$F(w, x) = 0 \tag{6.1}$$

for $w = w_0$. Then we investigate the solvability of (6.1) for w close to w_0. We use the concept of a *strong approximation* introduced by Robinson [17].

Definition 6.1 *f strongly approximates F in x at (w_0, x_0)* if for each $\varepsilon > 0$ there exist neighborhoods V of w_0 and U of x_0 such that whenever w belongs to V and x_1, x_2 belong to U we have

$$\|[F(w, x_1) - f(x_1)] - [F(w, x_2) - f(x_2)]\| \leq \varepsilon \|x_1 - x_2\| .$$

For a $A \subset U_0$ let

$$\delta(f, A) = \inf\{ \|f(x_1) - f(x_2)\|/\|x_1 - x_2\| \mid x_1 \neq x_2,\, x_1, x_2 \in A \} .$$

Then we can state the following implicit-function theorem.

Theorem 6.2. Suppose that $f(x_0) = 0$ and $F(w_0, x_0) = 0$. Assume further that

(a) f strongly approximates F in x at (w_0, x_0);

(b) $F(\cdot, x_0)$ is continuous at w_0;

(c) $f(U_0) \supset B(0, \rho)$ for some $\rho > 0$;

(d) $\delta(f, U_0) =: d_0 > 0$.

Then there are neighborhoods V of w_0 and U of x_0 and a function $x: V \to U$ such that

(i) $x(\cdot)$ is continuous at w_0;

(ii) $x(w_0) = x_0$, and for each $w \in V$, $x(w)$ is the unique solution in U of $F(w, x) = 0$.

Proof. Choose positive numbers ε, α, and a neighborhood W_1 of w_0 such that $W_1 \subset V_0$,

$$0 < \varepsilon < d_0, \quad 0 < \alpha < \rho \tag{6.2}$$

and such that for each $x_1, x_2 \in B(x_0, d_0^{-1}\alpha)$ and each $w \in W_1$

$$\|[F(w, x_1) - f(x_1)] - [F(w, x_2) - f(x_2)]\| \leq \varepsilon \|x_1 - x_2\| , \tag{6.3}$$

and

$$\varepsilon \|x_1 - x_0\| + \|F(w, x_0) - F(w_0, x_0)\| < \rho\,. \tag{6.4}$$

Now define $U = B(x_0, d_0^{-1}\alpha)$. By Assumptions c) and d), $f^{-1}: B(0,\rho) \to U_0$ is Lipschitz continuous with modulus d_0^{-1}. For $x \in U$, $w \in W_1$ we have by (6.3) and (6.4)

$$\begin{aligned}\|f(x) - F(w,x)\| &\leq \|[f(x) - F(w,x)] - [f(x_0) - F(w,x_0)]\| \\ &\quad + \|F(w,x_0) - F(w_0,x_0)\| \\ &\leq \varepsilon\|x - x_0\| + \|F(w,x_0) - F(w_0,x_0)\| < \rho\,.\end{aligned}$$

Therefore, the mapping $G: W_1 \times U \to X$,

$$G(w,x) = x - f^{-1}[f(x) - F(w,x)]\,, \tag{6.5}$$

is well-defined and single-valued. Moreover, for $(w,x) \in W_1 \times U$ we have

$$F(w,x) = 0 \iff G(w,x) = 0\,.$$

We show that G satisfies assumptions (S1)–(S3). Then the assertion follows from Theorem 5.1. (S1) follows from the definition of G. For $x \in U$, $w \in W_1$ we have by (6.3)

$$\begin{aligned}&\|G(w,x) - G(w_0,x_0)\| \\ &\quad\leq \|x - x_0\| + d_0^{-1}\|[f(x) - F(w,x)] - [f(x_0) - F(w_0,x_0)]\| \\ &\quad\leq \|x - x_0\| + d_0^{-1}\|[f(x) - F(w,x)] - [f(x_0) - F(w,x_0)]\| \\ &\qquad + d_0^{-1}\|F(w,x_0) - F(w_0,x_0)\| \\ &\quad\leq (1 + d_0^{-1}\varepsilon)\|x - x_0\| + d_0^{-1}\|F(w,x_0) - F(w_0,x_0)\|\,.\end{aligned}$$

By Assumption b) this implies (S2). For fixed $w \in W_1$ and $x_1, x_2 \in U$ we have

$$\begin{aligned}&\langle G(w,x_1) - G(w,x_2), x_1 - x_2\rangle \\ &\quad= \|x_1 - x_2\|^2 - \langle f^{-1}[f(x_1) - F(w,x_1)] - f^{-1}[f(x_2) - F(w,x_2)], x_1 - x_2\rangle\,.\end{aligned}$$

Since by (6.3)

$$\begin{aligned}&\|f^{-1}[f(x_1) - F(w,x_1)] - f^{-1}[f(x_2) - F(w,x_2)]\| \\ &\quad\leq d_0^{-1}\|[f(x_1) - F(w,x_1)] - [f(x_2) - F(w,x_2)]\| \\ &\quad\leq d_0^{-1}\varepsilon\|x_1 - x_2\|\,,\end{aligned}$$

this implies

$$\langle G(w,x_1) - G(w,x_2), x_1 - x_2\rangle \geq \bar{\alpha}\|x_1 - x_2\|^2\,, \tag{6.6}$$

where $\bar{\alpha} = 1 - d_0^{-1}\varepsilon > 0$ by (6.2). This shows that (V3) is satisfied. □

Again by imposing a more restrictive continuity condition on F we can show that $x(\cdot)$ is locally Lipschitz continuous.

Corollary 6.3. Let the assumptions of Theorem 6.2 be satisfied. Suppose that in addition for each $x \in U_0$, $F(\cdot, x)$ is Lipschitz continuous on W_0 with modulus ϕ. Then the mapping $x(\cdot)$ is Lipschitz continuous on V.

Proof. Choose positive numbers ε, α, and neighborhoods W_0 of w_0 and U of x_0 as in the proof of Theorem 6.2, and let the mapping G be defined by (6.5). Let $x \in U_0$ and $w_1, w_2 \in V_0$. By Lipschitz continuity of $F(\cdot, x)$ we obtain

$$\begin{aligned}
&\|G(w_1,x) - G(w_2,x)\| \\
&\quad = \|f^{-1}[f(x) - F(w_1,x)] - f^{-1}[f(x) - F(w_2,x)]\| \\
&\quad \le d_0^{-1}\|[f(x) - F(w_1,x)] - [f(x) - F(w_2,x)]\| \\
&\quad = d_0^{-1}\|F(w_1,x) - F(w_2,x)\| \\
&\quad \le d_0^{-1}\phi\,\|w_1 - w_2\|\,.
\end{aligned}$$

Together with (6.6) this implies by Corollary 5.2 that $x(\cdot)$ is Lipschitz continuous with modulus

$$\lambda = \bar{\alpha}^{-1} d_0^{-1}\phi = (d_0 - \varepsilon)^{-1}\phi\,.$$

This proves the assertion. □

(Received May 7, 1992.)

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