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EXISTENCE, UNIQUENESS AND EVALUATION OF LOG-OPTIMAL INVESTMENT PORTFOLIO[1]

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It is proved that if and only if the stock market return vector $\mathbf{X} = (X_1, \dots, X_m)$ satisfies the condition $\mathsf{E}\,|\log \sum_{j=1}^m X_j| < \infty$ a log-optimal portfolio exists in a reasonable sense. Its uniqueness is guaranteed under the assumption that the underlying distribution of $\mathbf{X}$ is not concentrated on a hyperplane in $\mathbb{R}^m$ containing the diagonal $D = \{(d, \dots, d) \in \mathbb{R}^m : d \in \mathbb{R}\}$. Under these assumptions, approximations of log-optimal portfolio by means of more easily evaluated portfolios are considered. In particular, a strongly consistent estimate of log-optimal portfolio based on independent observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ of $\mathbf{X}$ is obtained.

1. INTRODUCTION

We consider the stock market and the log-optimal investment portfolio introduced and systematically studied in Chap. 15 of Cover and Thomas [5]. In other words, we consider a random *return vector* $\mathbf{X} = (X_1, \dots, X_m)$ for one stock market day, distributed by $F(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$. To avoid bankruptcy, let the

basic assumption: $F(\mathbf{x}) = 0$ whenever at least one $x_j = 0$, $j \in \{1, \dots, m\}$,

be satisfied — without mentioning it any further. Thus it holds $\mathbf{X} \geq \mathbf{0}$ a.s. where $\geq$ is the usual partial ordering in $\mathbb{R}^m$. Each component $X_j \geq 0$ represents the ratio of closing to opening price for stock j.

We shall assume without loss of generality the validity of the following condition:

The distribution F is not concentrated on a hyperplane in $\mathbb{R}^m$ containing the diagonal $D = \{(d, \dots, d) \in \mathbb{R}^m : d \in \mathbb{R}\}$. (1)

Otherwise one can reduce the dimension m by choosing Euclidean coordinates $(\tilde{x}_1, \dots, \tilde{x}_m)$ so that the distribution is concentrated on the hyperplane $\tilde{x}_m = 0$.

An *investment portfolio* $\mathbf{b} = (b_1, \dots, b_m)$ is an allocation of wealth across the stocks, i.e. b_j is the fraction of one's wealth invested in stock j. Therefore the simplex $\mathbb{B} = \{\mathbf{b} = (b_1, \dots, b_m) : b_j \geq 0,\ \sum_{j=1}^m b_j = 1\}$ is the set of all portfolios $\mathbf{b}$.

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If one uses $\mathbf{b} \in \mathbb{B}$ and the return vector is $\mathbf{X}$, the wealth at the end of the stock market day is

$$W = \mathbf{b}\,\mathbf{X} \triangleq \sum_{j=1}^{m} b_j\, X_j$$

in units of the wealth invested at the beginning of the day. If the whole wealth is reinvested each day $i \in \mathbb{N}$ according to a portfolio $\mathbf{b}_i$ then the wealth after n days is

$$W_n = \prod_{i=1}^{n} \mathbf{b}_i\, \mathbf{X}_i$$

in units of the initially invested wealth, where $\mathbf{X}_i$ denotes the return vector on day i. The stock market is i. i. d. if the return vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots$ are i. i. d., $\mathbf{X}_i \sim F$.

We wish to maximize W_n in some sense. But W_n is a random variable, so there is controversy over the choice of best investment strategy $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Not being clairvoyants we restrict ourselves to *causal strategies*, where each portfolio $\mathbf{b}_i$ may depend (in a measurable way) on the past return vectors $\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}$, but is independent of the future values $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots$. One reasonable possibility is to maximize the expected utility

$$\mathsf{E}\; U(W_n) = \int_{\mathbb{R}^{mn}} U\left(\prod_{i=1}^{n} \mathbf{b}_i\, \mathbf{x}_i\right)\, \mathrm{d}F(\mathbf{x}_1) \cdots \mathrm{d}F(\mathbf{x}_n),$$

where $U : [0, \infty) \longrightarrow [-\infty, \infty)$ is a nondecreasing *utility function*. As shown in Lemma 15.3.1 on p. 466 of Cover and Thomas [5], for the utility function $U(x) = \log x$ there may exist a portfolio $\mathbf{b}^* \in \mathbb{B}$ such that the wealth W_n resulting from an arbitrary causal strategy $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and the wealth W_n^* resulting from the constant strategy $\mathbf{b}^*, \ldots, \mathbf{b}^*$ satisfy the relation

$$\mathsf{E}\; U(W_n) \le \mathsf{E}\; U(W_n^*) \tag{2}$$

or, equivalently,

$$\sum_{i=1}^{n} \mathsf{E}\; \log \mathbf{b}_i\, \mathbf{X}_i \le n\, \mathsf{E}\; \log \mathbf{b}^*\, \mathbf{X}. \tag{3}$$

This is one reason why we restrict ourselves to the logarithmic utility. Before discussing other reasons we introduce basic notations and the conventions $\log x = -\infty$ for $x \le 0$ and $\log(0/0) = 0$ considered throughout this paper.

Consider for portfolios $\mathbf{b} \in \mathbb{B}$ the *doubling rate*

$$\phi(\mathbf{b}) = \mathsf{E}_F \log \mathbf{b}\, \mathbf{X}$$

and let furthermore

$$\psi(\mathbf{x}) = \log \sum_{j=1}^{m} x_j, \quad \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{4}$$

This function will allow us to characterize both the *domain of* ϕ

$$\operatorname{dom}\phi = \{\mathbf{b} \in \mathbb{B} : \mathsf{E}(\log \mathbf{b}\,\mathbf{X})^+ < \infty \text{ or } \mathsf{E}(\log \mathbf{b}\,\mathbf{X})^- < \infty\}$$

i.e. the subset of $\mathbb{B}$ where ϕ is well defined, and the *effective domain of* ϕ

$$\operatorname{effdom}\phi = \{\mathbf{b} \in \operatorname{dom}\phi : \phi(\mathbf{b}) \in \mathbb{R}\}$$

i.e. the subset of the domain of ϕ where $\phi(\mathbf{b})$ is finite.

The motivation of the term **doubling rate** follows from the following considerations. If the stock market is i. i. d. (or stationary and ergodic) then, by the law of large numbers, the wealth W_n resulting from the constant strategy $\mathbf{b}, \ldots, \mathbf{b}$ satisfies the relation

$$\lim_{n\to\infty} \frac{1}{n} \log W_n = \phi(\mathbf{b}) \quad \text{a.s.}.$$

This means that, for large n, the wealth will be a.s. close to $2^{n\phi(\mathbf{b})}$ provided log is taken to the base 2. Therefore $\phi(\mathbf{b}) = 1$ means that the wealth is doubled each stock market day.

Definition. An investment portfolio $\mathbf{b}^* \in \mathbb{B}$ is said to be *log-optimal* if $\operatorname{dom}\phi = \mathbb{B}$ and if

$$\phi^* \triangleq \sup\{\,\phi(\mathbf{b}) : \mathbf{b} \in \mathbb{B}\} \in \mathbb{R} \qquad \text{and} \qquad \phi(\mathbf{b}^*) = \phi^*\,. \tag{5}$$

ϕ^* is called the optimal doubling rate. (Note that, by definition, $\mathbf{b}^*$ is an element of $\operatorname{effdom}\phi$.)

The log-optimal portfolio has been introduced by Kelly [6], and later studied by many authors (cf. Latané [7], Breiman [2], Cover [3,4], Cover and Thomas [5], and others cited there). A strong argument in favour of considering this portfolio as a solution to the optimum investment problem is the fact that it maximizes not only the expected one-day utility $\mathsf{E}\ U(W)$, but also the expected long term utility $\mathsf{E}\ U(W_n)$ (cf. (2) and (3)). It is known (cf. Samuelson [13]), that $U_\gamma(x) = x^\gamma/\gamma$, $\gamma \in \mathbb{R} - \{0\}$, are the only non-logarithmic utilities for which there exists a portfolio $\mathbf{b}^*$ satisfying (2). Other favourable arguments can be found in Cover and Thomas [5], p. 466 (Theorem 15.3.1, asymptotic optimality) and p. 472 (Theorem 15.6.1, competitive optimality).

On the other hand, some discouraging arguments were presented by Samuelson [12,13]. His criticism points to the fact that the log-optimal portfolio, while always maximizing the expected growth rate (the doubling rate), need not always maximize the expected utility. This takes place only if the utility is logarithmic, which is for many investors evidently an unrealistic assumption. E. g. for those who prefer, in the fashion recommended already by Pascal, to maximize the expected wealth $\mathsf{E}\ W_n$.

In all the cited papers we missed a rigorous analysis of conditions under which log-optimal portfolio exist. In this paper we present an easily verifiable necessary and sufficient condition for the existence of log-optimal portfolio in the sense of (5).

The second problem addressed in this paper is the evaluation of the optimal doubling rate ϕ^* and the log-optimal portfolio $\mathbf{b}^*$. We consider approximations $\mathbf{b}^n$

to $\mathbf{b}^*$ obtained as solutions of (5) with $\mathbb{B}$ replaced by appropriate subsets $\mathbb{B}_n$ of $\mathbb{B}$ or with F replaced by more easily available distributions F_n. In the particular case where $F_n = \hat{F}_n$ are distributions statistically derived from collections of observations $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$, we obtain in this manner causal portfolios $\mathbf{b}^n = \hat{\mathbf{b}}_n$ which are strongly consistent statistical estimates of $\mathbf{b}^*$. In this case the optimal doubling rate ϕ^* can be asymptotically achieved without the knowledge of F (adaptively).

The asymptotic achievability of ϕ^* means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\, \phi(\hat{\mathbf{b}}_i) = \phi^*$$

or the stronger property

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi(\hat{\mathbf{b}}_i) = \phi^* \quad \text{a.s.}$$

Results of the last type can be found in Cover and Thomas [5] and also in Sec. 7 of Cover [4] and in Morvai [8, 9]. We shall consider in addition the even stronger results of the type $\phi(\hat{\mathbf{b}}_n) \longrightarrow \phi^*$ a.s. and also $\hat{\mathbf{b}}_n \longrightarrow \mathbf{b}^*$ a.s.

Note that a pragmatic investor will only be interested in the optimal doubling rate ϕ^* and in causal portfolios $\hat{\mathbf{b}}_n$ asymptotically achieving ϕ^* and not in whether they satisfy the consistency relation

$$\lim_{n\to\infty} \hat{\mathbf{b}}_n = \mathbf{b}^* \quad \text{a.s.}$$

But such an investor will presumably not be interested in the log-optimal portfolio $\mathbf{b}^*$ at all. However, as soon as one enters the world of log-optimal portfolio, the problem of achieving this portfolio becomes to him or her probably at least as challenging as the problem of achieving ϕ^*.

2. EXISTENCE AND UNIQUENESS

Using the monotonicity of $\log x$ one easily obtains for every $\mathbf{b} \in \mathbb{B}$ and $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ (in view of our extension of the logarithm, it suffices to consider $\mathbf{x} \in [0,\infty)^m$)

$$\log \mathbf{b}\,\mathbf{x} \le (\log \mathbf{b}\,\mathbf{x})^+ \le \sum_{j=1}^{m} (\log x_j)^+ \quad \text{and} \quad -\log \mathbf{b}\,\mathbf{x} \le (\log \mathbf{b}\,\mathbf{x})^- \le \sum_{j=1}^{m} (\log x_j)^-$$

and hence

$$|\log \mathbf{b}\,\mathbf{x}| \le \sum_{j=1}^{m} |\log x_j|.$$

Thus if

$$\sum_{j=1}^{m} \mathsf{E}\,|\log X_j| < \infty \tag{6}$$

then, using Lebesgue's dominated convergence theorem and the strict concavity of $\log x$, one easily obtains that $\phi(\mathbf{b})$ is a finite, continuous and strictly concave function on $\mathbb{B}$, with the unique log-optimal portfolio $\mathbf{b}^* = \operatorname{argmax} \phi(\mathbf{b}) \in \mathbb{B}$. Unfortunately, (6) is not satisfied by as typical an example of the stock market as the horse race.

Example 1 (Horse race). Assume that the random variable $\mathbf{X}$ takes on m values according to the following table

Table 1. Horse race stock market.

value of $\mathbf{X}$	probability
$(a_1, 0, 0, \ldots, 0)$	p_1
$(0, a_2, 0, \ldots, 0)$	p_2
$\vdots$	$\vdots$
$(0, 0, 0, \ldots, a_m)$	p_m

where $a_j > 0$ and $\mathbf{p} = (p_1, \ldots, p_m)$ is a probability distribution with positive p_j. This table completely describes a discrete distribution function F on $\mathbb{R}^m$ satisfying the assumptions of the present paper. This stock market model describes the situation where m horses run in a race and the j-th horse wins with probability p_j. The gambler is supposed to put one dollar down. If he bets on one horse, say j, he will receive a_j dollars after the race if his horse wins, and will receive nothing otherwise. Let $I(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{m} p_j \log(p_j/b_j)$ be the I-divergence of $\mathbf{p}$ and $\mathbf{b}$ then

$$\phi(\mathbf{b}) = \sum_{j=1}^{m} p_j \log a_j\, b_j = \sum_{j=1}^{m} p_j \log a_j\, p_j - I(\mathbf{p}, \mathbf{b}), \quad \mathbf{b} \in \mathbb{B}, \tag{7}$$

$$\text{and} \quad \mathsf{E}\,|\log X_j| = p_j |\log a_j| + (1 - p_j)|\log 0| = \infty, \quad j \in \{1, \ldots, m\}.$$

Hence (6) is not satisfied but, as we see from (7), ϕ is continuous and concave on $\mathbb{B}$, strictly concave and finite on the interior $\mathbb{B}^0$ of $\mathbb{B}$, and minus infinity on the boundary $\mathbb{B} - \mathbb{B}^0$. Owing to $I(\mathbf{p}, \mathbf{b}) \geq 0$ for all $\mathbf{b} \in \mathbb{B}$, with equality iff $\mathbf{b} = \mathbf{p}$, $\mathbf{b}^*$ equals $\mathbf{p}$ and $\phi(\mathbf{b}^*) = \sum_{j=1}^{m} p_j \log a_j p_j$. This example illustrates and to some extent also motivates the following general theory.

Let for $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{B}$

$$S(\mathbf{b}) = \{j \in \{1, \ldots, m\} : b_j > 0\}$$

be the support of $\mathbf{b}$ and let $\mathbf{b}_{\min} = \min\{b_j : j \in S(\mathbf{b})\}$. Hence the interior of $\mathbb{B}$ can be written as $\mathbb{B}^0 = \{\mathbf{b} \in \mathbb{B} : S(\mathbf{b}) = \{1, \ldots, m\}\}$. Further, define in accordance with (4), for every $S \subset \{1, \ldots, m\}$

$$\psi_S(\mathbf{x}) = \log \sum_{j \in S} x_j, \quad \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m,$$

so that, in particular, $\psi_{\{j\}}(\mathbf{x}) = \log x_j$ and $\psi_{\{1,\ldots,m\}}(\mathbf{x}) = \psi(\mathbf{x})$. Next follows an auxiliary result.

Lemma 1. Let $\mathbf{x} = (x_1, \dots, x_m) \in [0,\infty)^m$. Then for every $\mathbf{b} \in \mathbb{B}$ and for every subset $S \subset \{1,\dots,m\}$ such that $S(\mathbf{b}) \subset S$

$$(\log \mathbf{b}\,\mathbf{x})^+ \le \sum_{j \in S(\mathbf{b})} (\log x_j)^+, \tag{8}$$

$$\psi_{S(\mathbf{b})}(\mathbf{x})^+ + \log \mathbf{b}_{\min} \le (\log \mathbf{b}\,\mathbf{x})^+ \le \psi_{S(\mathbf{b})}(\mathbf{x})^+ \le \psi_S(\mathbf{x})^+, \tag{9}$$

and

$$\psi_S(\mathbf{x})^- \le \psi_{S(\mathbf{b})}(\mathbf{x})^- \le (\log \mathbf{b}\,\mathbf{x})^- \le \psi_{S(\mathbf{b})}(\mathbf{x})^- - \log \mathbf{b}_{\min}. \tag{10}$$

P r o o f. Since the functions $\log x$ and $(x)^+ = \max(x, 0)$ are increasing on the domain $x \in [0,\infty)$,

$$\log \sum_{j \in S(\mathbf{b})} b_j x_j \le \log(\max_{j \in S(\mathbf{b})} x_j) \le \max_{j \in S(\mathbf{b})} (\log x_j)^+ \le \sum_{j \in S(\mathbf{b})} (\log x_j)^+$$

and hence (8). Since, in addition, $(x)^- = \max(-x, 0)$ is decreasing on the domain $x \in [0,\infty)$, we see that the chains of inequalities (9) and (10) follow from

$$b_{\min} \cdot \sum_{j \in S(\mathbf{b})} x_j \le \mathbf{bx} \le \sum_{j \in S(\mathbf{b})} x_j \le \sum_{j \in S} x_j,$$

and from the fact that, for every $c > 0$, $\log x\,c = \log x + \log c$, $(x-c)^+ \ge (x)^+ - c$, $(x-c)^- \le (x)^- + c$. □

The next lemma characterizes the function ϕ on $\mathbb{B}$ by means of $\mathsf{E}\,\psi(\mathbf{X})^+$ and $\mathsf{E}\,\psi(\mathbf{X})^-$. The results are summarized in Table 2 below.

Lemma 2. Let S_0, S be two nonempty subsets of $\{1,\dots,m\}$ such that $S_0 \subset S$. Then
(i)

$$\mathsf{E}\,\psi_{S_0}(\mathbf{X})^+ \quad \le \quad \mathsf{E}\,\psi_S(\mathbf{X})^+ \tag{11}$$

and

$$\mathsf{E}\,\psi_{S_0}(\mathbf{X})^- \quad \ge \quad \mathsf{E}\,\psi_S(\mathbf{X})^-. \tag{12}$$

Now let $\mathbf{b}^0$ be any element of $\mathbb{B}$. Then
(ii) the following four statements are equivalent

$$\mathsf{E}\,\psi_S(\mathbf{X})^+ < \infty, \qquad \emptyset \ne S \subset S(\mathbf{b}^0), \tag{13}$$

$$\mathsf{E}\,(\log \mathbf{bX})^+ < \infty, \qquad \mathbf{b} \in \mathbb{B} :\ S(\mathbf{b}) \subset S(\mathbf{b}^0), \tag{14}$$

$$\mathsf{E}\,(\log \mathbf{b}^0\mathbf{X})^+ < \infty, \tag{15}$$

$$\mathsf{E}\,\psi_{S(\mathbf{b}^0)}(\mathbf{X})^+ < \infty, \tag{16}$$

and
(iii) the following three statements are equivalent

$$\begin{aligned} &\mathsf{E}\,(\log \mathbf{bX})^- < \infty, \qquad \mathbf{b} \in \mathbb{B} : S(\mathbf{b}) = S(\mathbf{b}^0), &(17)\\ &\mathsf{E}\,(\log \mathbf{b}^0\mathbf{X})^- < \infty, &(18)\\ &\mathsf{E}\,\psi_{S(\mathbf{b}^0)}(\mathbf{X})^- < \infty. &(19)\end{aligned}$$

By choosing $\mathbf{b}^0 \in \mathbb{B}^0$ particularly it holds
(iv)

$$\begin{aligned} \mathsf{E}\,\psi(\mathbf{X})^+ &< \infty \iff \mathsf{E}\,(\log \mathbf{bX})^+ < \infty, \quad \mathbf{b} \in \mathbb{B}, \\ &\iff \mathsf{E}\,\psi_S(\mathbf{X})^+ < \infty, \quad \emptyset \neq S \subset \{1,\dots,m\}, \qquad (20)\\ \mathsf{E}\,\psi(\mathbf{X})^- &< \infty \iff \mathsf{E}\,(\log \mathbf{bX})^- < \infty, \quad \mathbf{b} \in \mathbb{B}^0, \qquad (21)\\ \mathsf{E}\,\psi(\mathbf{X})^- &= \infty \iff \mathsf{E}\,(\log \mathbf{bX})^- = \infty, \quad \mathbf{b} \in \mathbb{B}.\end{aligned}$$

P r o o f. (i) (11) and (12) are consequences of the third inequality in (9) respectively the first inequality in (10).

(ii) Choose $\mathbf{b}$ according to (14) and select in (13) all one-point-sets $S = \{j\}, j \in S(\mathbf{b})$. Then taking into account $\psi_{\{j\}}(\mathbf{X}) = \log X_j$ (14) follows from (13) by virtue of (8). (14) trivially implies (15). (16) follows from (15) owing to the first inequality in (9). The implication (16) $\Longrightarrow$ (13) is an immediate consequence of (11).

(iii) (17) trivially implies (18). (19) follows from (18) by virtue of the second inequality in (10). (17) follows from (19) owing to the third inequality in (10).

(iv) Since for $\mathbf{b}^0 \in \mathbb{B}^0$ $S(\mathbf{b}^0) = \{1,\dots,m\}$ the statements (20) and (21) are the specifications of assertion (ii) respectively (iii) of this Lemma for $\mathbf{b}^0 \in \mathbb{B}^0$. Owing to (12) applied to $S = \{1,\dots,m\}$ and the equivalence (18) $\iff$ (19),

$$\begin{aligned}\mathsf{E}\,\psi(\mathbf{X})^- = \infty &\iff \mathsf{E}\,\psi_{S(\mathbf{b})}(\mathbf{X})^- = \infty, \quad \mathbf{b} \in \mathbb{B}\\ &\iff \mathsf{E}\,(\log \mathbf{bX})^- = \infty, \quad \mathbf{b} \in \mathbb{B}. \qquad \square\end{aligned}$$

The essence of Lemma 2 is summarized in the following table. We see that in the first three cases the log-optimal portfolio $\mathbf{b}^*$ does not exist ($\operatorname{dom}\phi \neq \mathbb{B}$ and/or $\phi^* \notin \mathbb{R}$).

Table 2. Characterization of $\operatorname{dom}\phi$, $\operatorname{effdom}\phi$ and of $\phi(\mathbf{b})$ on $\operatorname{dom}\phi - \operatorname{effdom}\phi$.

$\mathsf{E}\,\psi(\mathbf{X})^+$	$\mathsf{E}\,\psi(\mathbf{X})^-$	$\operatorname{dom}\phi$	$\operatorname{effdom}\phi$	$\phi(\mathbf{b})$ is on $\operatorname{dom}\phi - \operatorname{effdom}\phi$
$= \infty$	$= \infty$	$\subset \mathbb{B} - \mathbb{B}^0$	$= \emptyset$	$= -\infty$
$= \infty$	$< \infty$	$\supseteq \mathbb{B}^0$	$\subset \mathbb{B} - \mathbb{B}^0$	$= \infty$ if $\mathbf{b} \in \mathbb{B}^0$, $\in \{-\infty, \infty\}$ otherwise
$< \infty$	$= \infty$	$= \mathbb{B}$	$= \emptyset$	$= -\infty$
$< \infty$	$< \infty$	$= \mathbb{B}$	$\supseteq \mathbb{B}^0$	$= -\infty$

Example 2. Examples for the case

$$\mathsf{E}\ \psi(\mathbf{X})^+ = \infty, \quad \mathsf{E}\ \psi(\mathbf{X})^- = \infty$$

and an empty or a nonempty domain of ϕ are easily found. Let e. g. $m = 2$ and

(i) let in the first case $\{(0,0)\} \cup \{1\} \times [1,\infty) \cup [1,\infty) \times \{1\}$ be the support of F and let $\mathsf{E}\,(\log X_1)^+ = \mathsf{E}\,(\log X_2)^+ = \infty$. Then $\mathsf{E}\ \psi(\mathbf{X})^+ = \mathsf{E}\ \psi(\mathbf{X})^- = \infty$ and $\mathrm{dom}\,\phi = \emptyset$.

(ii) Let $\{(0,0)\} \cup \{1\} \times [1,\infty)$ be the support of F and let $\mathsf{E}\,(\log X_2)^+ = \infty$. Then $\mathsf{E}\ \psi(\mathbf{X})^+ = \mathsf{E}\ \psi(\mathbf{X})^- = \infty$ and owing to $\mathsf{E}\,(\log X_1)^+ = 0$ $\mathrm{dom}\ \phi = \{1_{\{1\}}\}$.

Examples for the case

$$\mathsf{E}\ \psi(\mathbf{X})^+ = \infty, \quad \mathsf{E}\ \psi(\mathbf{X})^- < \infty$$

and an empty respectively a nonempty effective domain of ϕ are equally easily found. Again let e. g. $m = 2$ and

(i) let $\{(0,1)\} \cup \{1\} \times [1,\infty)$ be the support of F and let $\mathsf{E}\,(\log X_2)^+ = \infty$. Then $\mathsf{E}\,(\log X_1)^- = \infty, \mathsf{E}\,(\log X_1)^+ = \mathsf{E}\,(\log X_2)^- = 0, \mathsf{E}\ \psi(\mathbf{X})^+ = \infty, \mathsf{E}\ \psi(\mathbf{X})^- = 0$ and hence $\mathrm{dom}\,\phi = \mathbb{B}$, $\mathrm{effdom}\,\phi = \emptyset$, $\phi(1_{\{1\}}) = -\infty$ and $\phi(\mathbf{b}) = \infty$ for all $\mathbf{b} \in \mathbb{B}$ such that $S(\mathbf{b}) \supset \{2\}$.

(ii) Let $\{1\} \times [1,\infty)$ be the support of F and let $\mathsf{E}\,(\log X_2)^+ = \infty$. Then $\mathsf{E}\ \psi(\mathbf{X})^+ = \infty, \mathsf{E}\ \psi(\mathbf{X})^- = 0$ and owing to $\mathsf{E}\,(\log X_1)^+ = 0$ $\mathrm{dom}\,\phi = \mathbb{B}$, $\mathrm{effdom}\,\phi = \{1_{\{1\}}\}$, $\phi(1_{\{1\}}) = 0$ and $\phi(\mathbf{b}) = \infty$ for all $\mathbf{b} \in \mathrm{dom}\,\phi - \mathrm{effdom}\,\phi$.

In the remainder of the paper we restrict ourselves to the last case considered in Table 2, i.e. we assume

$$\mathsf{E}\ |\ \psi(\mathbf{X})\ | < \infty. \tag{22}$$

The next auxiliary result is helpful in characterizing $\mathrm{effdom}\,\phi$.

Lemma 3. Assume (22) and let

$$\mathcal{C}_F = \{S \subset \{1,\dots,m\} : \mathsf{E}\ |\ \psi_S(\mathbf{X})\ | < \infty\}.$$

(i) The class $\mathcal{C}_F$, which can be equivalently expressed by

$$\mathcal{C}_F = \{S \subset \{1,\dots,m\} : \mathsf{E}\ \psi_S(\mathbf{X})^- < \infty\}, \tag{23}$$

is *hereditary* in the sense that it contains with any S_0 all sets $S : S_0 \subset S \subset \{1,\dots,m\}$.

(ii) The effective domain of ϕ

$$\mathrm{effdom}\,\phi = \{\mathbf{b} \in \mathbb{B} : S(\mathbf{b}) \in \mathcal{C}_F\} \tag{24}$$

is a convex subset of $\mathrm{dom}\,\phi = \mathbb{B}$ containing $\mathbb{B}^0$. In addition

$$\phi(\mathbf{b}) \le \mathsf{E}\ \psi(\mathbf{X})^+ < \infty, \quad \mathbf{b} \in \mathbb{B}, \tag{25}$$

and

$$\phi(\mathbf{b}) = -\infty, \quad \mathbf{b} \in \mathbb{B} - \operatorname{effdom} \phi . \tag{26}$$

Proof. Owing to (22) $\mathsf{E}\,\psi(\mathbf{X})^+ < \infty$. Therefore (20) implies both (23) and $\sup\{\phi(\mathbf{b}) : \mathbf{b} \in \mathbb{B}\} < \infty$ and hence (26). Taking into account (23) and applying the equivalence (17) $\iff$ (19) to every element of $\mathcal{C}_F$ yields

$$\mathsf{E}\,(\log \mathbf{bX})^- < \infty, \quad \mathbf{b} \in \mathbb{B} : S(\mathbf{b}) \in \mathcal{C}_F,$$

and hence (24). The second and third inequality in (9) and assumption (22) yield

$$\log \mathbf{bX} \leq \psi(\mathbf{X})^+, \quad \mathbf{b} \in \mathbb{B}, \qquad \text{and} \qquad \mathsf{E}\,\psi(\mathbf{X})^+ < \infty \tag{27}$$

and hence (25). The hereditarity of $\mathcal{C}_F$ is an immediate consequence of (23) and (12).

The convexity of effdom ϕ is seen as follows: Let $\mathbf{b}^1 \in \operatorname{effdom}\phi$ and thus $S(\mathbf{b}^1) \in \mathcal{C}_F$, let, more generally, $\mathbf{b}^2 \in \mathbb{B}$ and $\alpha \in (0,1)$. Then, owing to $S(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) = S(\mathbf{b}^1) \cup S(\mathbf{b}^2) \supset S(\mathbf{b}^1)$ and the hereditarity of $\mathcal{C}_F$, $S(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \in \mathcal{C}_F$. □

Example 3. If (6) holds then $\mathcal{C}_F$ contains by definition all one-point-sets $\{j\}$, $j \in \{1, \ldots, m\}$ and hence by the hereditarity of $\mathcal{C}_F$ all nonempty subsets of $\{1, \ldots, m\}$. Therefore in this case the effective domain is as large as possible: effdom $\phi = \mathbb{B}$. For the horse race which has been investigated in Example 1 the class $\mathcal{C}_F$ contains only the set $\{1, \ldots, m\}$ and therefore the effective domain is as small as possible: effdom $\phi = \mathbb{B}^0$. These are the extreme cases for effdom ϕ.

In order to characterize the effective domain of ϕ for the general stock market model in more geometric terms let us consider

$$\mathbb{B}_S = \{\mathbf{b} \in \mathbb{B} : S(\mathbf{b}) = S\}, \quad \emptyset \neq S \subset \{1, \ldots, m\}.$$

Lemma 4. It holds

$$\operatorname{effdom}\phi = \mathbb{B}^0 \cup \bigcup_{S \in \mathcal{C}_F - \{1,\ldots,m\}} \mathbb{B}_S,$$

where the sets $\mathbb{B}_S$, $S \in \mathcal{C}_F - \{1, \ldots, m\}$, are the faces of the boundary of the simplex $\mathbb{B}$ for which ϕ is defined and finite.

Proof. Clear from (24). □

Lemma 5. If (22) holds then the function $\phi(\mathbf{b})$ is continuous and concave on $\mathbb{B}$. Furthermore, it is finite on the effective domain of ϕ and $-\infty$ elsewhere. If, moreover, (1) holds then $\phi(\mathbf{b})$ is strictly concave on effdom ϕ.

Proof. The statements concerning the domain and the range of ϕ are clear from Lemma 3.

(I) *Concavity.* We have to show that for every $\mathbf{b}^1, \mathbf{b}^2 \in \mathbb{B}$ and every $\alpha \in (0,1)$

$$\alpha\phi(\mathbf{b}^1) + (1-\alpha)\phi(\mathbf{b}^2) \leq \phi(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \tag{28}$$

and, provided $\mathbf{b}^1, \mathbf{b}^2 \in \operatorname{effdom} \phi$ and (1) is satisfied, that equality holds if and only if $\mathbf{b}^1 = \mathbf{b}^2$. To this end we consider the following three possible cases:

(i) In the case that none of the elements $\mathbf{b}^1$, $\mathbf{b}^2$ belong to $\operatorname{effdom} \phi$ (28) follows immediately from (26).

(ii) In the case that exactly one of the elements $\mathbf{b}^1$, $\mathbf{b}^2$ belongs to $\operatorname{effdom} \phi$, and hence — by the argument at the end of the proof of Lemma 3 — $\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2$ also belongs to $\operatorname{effdom} \phi$, (26) implies strict inequality in (28).

(iii) Finally let both $\mathbf{b}^1$ and $\mathbf{b}^2$ belong to $\operatorname{effdom} \phi$. Since the function $\log x$, $x \in [0, \infty)$, is concave, Jensen's inequality yields

$$\log\left(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2\right)\mathbf{X} - \left(\alpha \log \mathbf{b}^1\mathbf{X} + (1-\alpha)\log \mathbf{b}^2\mathbf{X}\right) \geq 0 \tag{29}$$

and hence

$$\mathsf{E}\left[\log\left(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2\right)\mathbf{X} - \left(\alpha \log \mathbf{b}^1\mathbf{X} + (1-\alpha)\log \mathbf{b}^2\mathbf{X}\right)\right] \geq 0,$$

where — since the latter is finite — equality holds if and only if equality holds in (29) a.s. Owing to the strict concavity of $\log x$, this is equivalent to

$$(\mathbf{b}^1 - \mathbf{b}^2)\mathbf{X} = 0 \quad \text{a.s.}$$

Provided condition (1) is satisfied, this holds if and only if $\mathbf{b}^1 = \mathbf{b}^2$.

(II) *Continuity.* In order to show continuity, let us first prove that the concave function ϕ is upper-semicontinuous and hence, by definition, is closed.

To this end let $\mathbf{b}^n \in \operatorname{effdom} \phi$, $n \in \mathbb{N}$, be a sequence tending to $\mathbf{b}^0 \in \mathbb{B}$ and, consequently,

$$\lim_{n\to\infty} \log \mathbf{b}^n\mathbf{X} = \log \mathbf{b}^0\mathbf{X} \quad \text{a. s.}. \tag{30}$$

Since in addition (27) holds, the application of Fatou's lemma yields

$$\lim_{n\to\infty} \sup \phi(\mathbf{b}^n) \leq \phi(\mathbf{b}^0). \tag{31}$$

Therefore, by definition, ϕ is a closed concave function.

Finally we have to distinguish the two cases $\mathbf{b}^0 \in$ and $\notin \operatorname{effdom} \phi$ respectively.
(i) Since for $\mathbf{b}^0 \in \mathbb{B} - \operatorname{effdom} \phi$ it holds $\phi(\mathbf{b}^0) = -\infty$, (31) trivially implies

$$\lim_{n\to\infty} \phi(\mathbf{b}^n) = \phi(\mathbf{b}^0). \tag{32}$$

(ii) Now let $\mathbf{b}^0 \in \operatorname{effdom} \phi$ and consider for any $\varepsilon > 0$ the set

$$(\mathbf{b}^0)^\varepsilon = \{\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{B} : b_j \geq b_j^0 - \varepsilon \text{ for all } j \in S(\mathbf{b}^0)\}$$

which evidently is a simplex containing the element $\mathbf{b}^0$. Provided $\varepsilon \in (0, \mathbf{b}^0_{\min})$, moreover,

$$(\mathbf{b}^0)^\varepsilon \subset \operatorname{effdom} \phi.$$

To see this let $\mathbf{b} \in (\mathbf{b}^0)^\varepsilon$. Then $b_j \geq b_j^0 - \varepsilon \geq \mathbf{b}^0_{\min} - \varepsilon > 0$ for all $j \in S(\mathbf{b}^0)$ yields $S(\mathbf{b}) \supseteq S(\mathbf{b}^0)$ and thus, since $S(\mathbf{b}^0) \in \mathcal{C}_F$ and $\mathcal{C}_F$ is hereditary, $S(\mathbf{b}) \in \mathcal{C}_F$. Owing to (24) this implies $\mathbf{b} \in \operatorname{effdom} \phi$. Hence we have shown that there is a simplex containing $\mathbf{b}^0$ which is entirely included in $\operatorname{effdom} \phi$, $\operatorname{effdom} \phi$ is a locally simplicial set. Together with the fact that ϕ is a closed concave function this implies by virtue of Theorem 10.2 in Rockafellar [11] that ϕ is continuous on $\operatorname{effdom} \phi$. □

Remark 1. The continuity of ϕ on every set $\mathbb{B}_S$, $S \in \mathcal{C}_F$, may also be shown as follows: Let $\mathbf{b}^n \in \mathbb{B}_S$, $n \in \mathbb{N}$, tend to $\mathbf{b}^0 \in \mathbb{B}_S$. Then for every $\varepsilon \in (0, \mathbf{b}^0_{\mathrm{min}})$ there exists a number $n_0 \in \mathbb{N} : \| \mathbf{b}^n - \mathbf{b}^0 \|_\infty = \max\{| b_j^n - b_j^0 | : j \in S\} < \varepsilon$ for all $n > n_0$ and thus, by the second inequality in (9) and the third inequality in (10),

$$| \log \mathbf{b}^n \mathbf{X} | \le | \psi_S(\mathbf{X}) | - \log(\mathbf{b}^0_{\mathrm{min}} - \varepsilon), \quad n \ge n_0 .$$

Owing to $\mathsf{E}\,|\psi_S(\mathbf{X})| < \infty$ and (30), the application of Lebesgue's dominated convergence theorem yields (32).

Theorem 1. If (22) holds then a log-optimal portfolio $\mathbf{b}^*$ exists. If, moreover, (1) holds then $\mathbf{b}^*$ is unique.

Proof. First of all (25) and $\mathrm{effdom}\,\phi \ne \emptyset$ imply $\phi^* \in \mathbb{R}$. Since, owing to Lemma 5, ϕ is a continuous function on the compact set $\mathbb{B}$ there exists an element $\mathbf{b}^* \in \mathbb{B}$ satisfying $\phi(\mathbf{b}^*) = \phi^*$ and hence being log-optimal. $\phi^* \in \mathbb{R}$ and (26) finally imply $\mathbf{b}^* \in \mathrm{effdom}\,\phi$. Now assume the validity of condition (1). Then the uniqueness of $\mathbf{b}^*$ is seen as follows: Let $\mathbf{b}^0 \in \mathrm{effdom}\,\phi$ be any log-optimal portfolio and let $\alpha \in (0,1)$. Then (28) applied to $\mathbf{b}^0$ and $\mathbf{b}^*$ and $\phi(\mathbf{b}) \le \phi^*$ for all $\mathbf{b} \in \mathbb{B}$ yield

$$\alpha\phi(\mathbf{b}^0) + (1-\alpha)\phi(\mathbf{b}^*) = \phi(\alpha\mathbf{b}^0 + (1-\alpha)\mathbf{b}^*)$$

and hence, due to the strict concavity of ϕ, $\mathbf{b}^0 = \mathbf{b}^*$. □

3. DETERMINISTIC APPROXIMATION

The following result can be employed in the theory of approximations of log-optimal portfolios.

Lemma 6. Let (22) hold and let ϕ^* be the optimal doubling rate and $\mathbf{b}^*$ be any log-optimal portfolio. Furthermore, let $\mathbf{b}^n$, $n \in \mathbb{N}$, be any sequence in $\mathbb{B}$. Then the following property (33) implies (34). If (1) holds, then $\mathbf{b}^*$ is unique and the two properties (33) and (34) are equivalent.

$$\lim_{n\to\infty} \mathbf{b}^n = \mathbf{b}^*, \tag{33}$$

$$\lim_{n\to\infty} \phi(\mathbf{b}^n) = \phi^* . \tag{34}$$

Proof. Since, by Lemma 5, ϕ is under (22) continuous and since $\phi(\mathbf{b}^*) = \phi^*$ holds, (33) $\Longrightarrow$ (34). Now suppose that (1) and (34) hold. Since $\mathbb{B}$ is compact, the sequence $\mathbf{b}^n$ has a limit point $\mathbf{b}^0$. Let us consider a subsequence $\mathbf{b}^{n_k}$ such that

$$\lim_{k\to\infty} \mathbf{b}^{n_k} = \mathbf{b}^0.$$

Then, another application of the continuity of ϕ yields

$$\lim_{k\to\infty} \phi(\mathbf{b}^{n_k}) = \phi(\mathbf{b}^0)$$

and hence, by virtue of (34), $\phi(\mathbf{b}^0) = \phi(\mathbf{b}^*)$. Therefore $\mathbf{b}^0$ is log-optimal. Since, by Theorem 1, the log-optimal portfolio is unique, it holds $\mathbf{b}^0 = \mathbf{b}^*$. Thus $\mathbf{b}^*$ is the only possible accumulation point of $\mathbf{b}^n$. Therefore (33) holds. □

The next result guarantees that the optimization method of sieves (cf. e. g. van de Geer [14]) leads asymptotically to the log-optimal portfolio. It follows from this result that, in particular, the evaluation of log-optimal portfolio is possible by using the iterative optimization algorithms from standard software packages.

Lemma 7. Let (22) hold and let $\mathbb{B}_0 = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ be a subset of $\mathbb{B}$ the closure $\overline{\mathbb{B}}_0$ of which contains $\mathbf{b}^*$. Furthermore, let $\mathbf{b}^n$ be the log-optimal portfolios with respect to the subset $\mathbb{B}_n = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, i.e. $\phi(\mathbf{b}^n) = \max\{\phi(\mathbf{b}) : \mathbf{b} \in \mathbb{B}_n\}$. Then the sequence $\mathbf{b}^n$ satisfies the relation (34).

Proof. This holds since, by definition,

$$\phi(\mathbf{b}^n) \nearrow \sup\{\phi(\mathbf{b}) : \mathbf{b} \in \mathbb{B}_0\}$$

and since, because of the continuity of ϕ, the latter equals ϕ^*. □

In the sequel let F and F_n, $n \in \mathbb{N}$, be probability distribution functions satisfying the basic assumption of Section 1. Furthermore, let f, f_n denote the associated Radon–Nikodym derivatives of the corresponding probability distributions with respect to a dominating σ-finite measure μ on $\mathbb{R}^m$ (i.e. f, f_n are the derivatives of F, F_n if these functions are absolutely continuous, and probabilities of the corresponding vectors $\mathbf{x} \in [0, \infty)^m$ if F, F_n are discrete).

Moreover, let

$$\text{ess sup}_F(F_n) \triangleq \text{ess sup}_F\,(f_n/f)\ ,$$

be the essential supremum of the ratio f_n/f taken with respect to the probability distribution corresponding to F and remind that

$$I(F, F_n) = \mathsf{E}_F \log \frac{f(\mathbf{X})}{f_n(\mathbf{X})}\ ,$$

are the I-divergences of F and F_n.

Theorem 2. Let F and F_n, $n \in \mathbb{N}$, be probability distribution functions satisfying (22) and let ϕ^* be the optimal doubling rate with respect to F. Furthermore, let

$$\lim_{n\to\infty} I(F, F_n) = 0\ . \tag{35}$$

Then any sequence $\mathbf{b}_n$ of log-optimal portfolio with respect to F_n satisfies the relation (34).

Proof. Let $\mathbf{b}^*$ be any log-optimal portfolio with respect to F. Then the application of Theorem 15.4.1 on p. 467 of Cover and Thomas [5] implies

$$0 \le \phi(\mathbf{b}^*) - \phi(\mathbf{b}_n) \le I(F, F_n).$$

and hence, in view of (35) and $\phi^* = \phi(\mathbf{b}^*)$, the assertion. □

The following lemma provides a simple but rather strong sufficient condition for the existence of log-optimal portfolios with respect to F_n.

Lemma 8. Let F and F_n be probability distribution functions and let

$$\text{ess sup}_F(F_n) < \infty\,. \tag{36}$$

Then, provided F satisfies (22), F_n satisfies (22) as well.

Proof. Note that, owing to $0 = \mathsf{E}_F\,(f_n/f - 1) \leq \text{ess sup}_F\,(f_n/f) - 1$ and hence $|\,f_n/f - 1\,| \leq \max(f_n/f, 1) \leq \text{ess sup}_F\,(f_n/f)$, it holds

$$|\mathsf{E}_{F_n}|\psi(\mathbf{X})| - \mathsf{E}_F|\psi(\mathbf{X})|| \;\leq\; |\mathsf{E}_F|\psi(\mathbf{X})|\,|f_n/f - 1|| \;\leq\; \mathsf{E}_F|\psi(\mathbf{X})| \times \text{ess sup}_F(F_n)\,.$$

Therefore (36) and the validity of (22) for F yield the validity of (22) for F_n. □

4. STATISTICAL APPROXIMATION

Let F be a probability distribution function satisfying the basic assumption of Section 1 and the conditions (1) and (22) and let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be i.i.d. random vectors distributed according to $F(\mathbf{x})$. Furthermore let, for every $\mathbf{x} \in \mathbb{R}^m$, $k_n(\mathbf{x})$ be the number of integers $i \in \{1, \ldots, n\}$ such that $\mathbf{X}_i < \mathbf{x}$. Then, for every $n > 1$,

$$\tilde{F}_n(\mathbf{x}) = \frac{k_{n-1}(\mathbf{x})}{n-1} \tag{37}$$

is the *empirical distribution function*. Note that, since F satisfies the basic assumption of Section 1 then consequently $\tilde{F}_n$ does as well. Moreover, as $n \to \infty$, $\tilde{F}_n$ satisfies condition (1) with a probability tending to 1.

An arbitrary sequence of probability distribution functions $\hat{F}_n$ measurably depending on the observations $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$ is said to be a *random estimator of* F. It is called *consistent in I-divergence* if

$$\lim_{n\to\infty} I(F, \hat{F}_n) = 0 \quad \text{a.s.}$$

If F is discrete (e.g. as for the horse race, Example 1), then the empirical distribution function (37) is a random estimator of F. It follows from the strong law of large numbers that it is consistent in I-divergence.

Barron, Győrfi and van der Meulen [1] found relatively mild restrictions on general F under which some modifications $\hat{F}_n$ of the empirical distribution function (37) are consistent in I-divergence. Their estimator satisfies the basic assumption of Section 1. Another interesting property of $\hat{F}_n$ is that if condition (1) is satisfied with respect to F then it is a. s. satisfied with respect to $\hat{F}_n$. In other words, their estimator $\hat{F}_n$ then satisfies a.s. all basic assumptions imposed on F in Section 1.

Theorem 3. Let F satisfy (1) and (22) and let $\mathbf{b}^*$ and ϕ^* be the corresponding log-optimal portfolio and the optimal doubling rate respectively. Moreover, let $\hat{F}_n$ be a random estimator of F which satisfies (1), which is consistent in I-divergence and satisfies

$$\text{ess sup}_F(\hat{F}_n) < \infty \quad \forall n \in \mathbb{N}. \tag{38}$$

Then the log-optimal portfolios $\hat{\mathbf{b}}_n$ with respect to $\hat{F}_n$ exist, are unique, and measurably depend on $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$. Furthermore they satisfy the equivalent relations

$$\lim_{n\to\infty} \hat{\mathbf{b}}_n = \mathbf{b}^* \quad \text{a.s.}, \tag{39}$$

$$\lim_{n\to\infty} \phi(\hat{\mathbf{b}}_n) = \phi^* \quad \text{a.s.} \tag{40}$$

Proof. By virtue of Lemma 8, (38) implies that property (22) holds for every $\hat{F}_n$ and every realization of $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$. The validity of property (1) is given by assumption. Thus the existence and uniqueness of the log-optimal portfolios $\hat{\mathbf{b}}_n$ follow from Theorem 1. The measurability follows from the continuity of

$$\phi_n(\mathbf{b}) = \mathsf{E}_{\hat{F}_n} \log \mathbf{b}\,\mathbf{X}$$

on $\mathbb{B}$ for every realization of $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$ (cf. Lemma 5 with F replaced by $\hat{F}_n$), the compactness of $\mathbb{B}$, and from Theorem 1.9 of Pfanzagl [10]. The limit relations (39) and (40) are clear from Theorem 2. □

Remark 2. An immediate consequence of (40) is the convergence of the corresponding sequence of Cesaro means, namely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi(\hat{\mathbf{b}}_i) = \phi^* \quad \text{a.s.} \tag{41}$$

Cover in Theorem 7.1. of [4] and Morvai in [8, 9] established under weaker assumptions somewhat weaker result. They found portfolios $\hat{\mathbf{b}}_n$ depending on the empirical distributions (37) and satisfying (41).

5. APPLICATION TO FINITE STOCK MARKET

Let $\mathbf{X}$ be a random variable with finite range $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\} \subset [0, \infty)^m - \{(0, \ldots, 0)\}$ and probability distribution $\mathbf{p} = (p_1, \ldots, p_M)$ satisfying condition (1) and $\mathrm{p}_{\min} = \min\{p_k : k \in \{1, \ldots, M\}\} > 0$. Note that, since the range of $\mathbf{X}$ does not contain the null-vector of $\mathbb{R}^m$, condition (22) also holds. Hence this general finite stock market model satisfies all conditions considered in Section 1.

As said in the previous section, the empirical distribution function $\tilde{F}_n$ considered in (37) also satisfies the basic assumption of Section 1. But condition (1) need not be satisfied by $\tilde{F}_n$ for every $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$. However, the following modification satisfies both these conditions

$$\hat{F}_n = (1 - \varepsilon_n)\, \tilde{F}_n + \varepsilon_n \left(\frac{1}{M} \sum_{k=1}^{M} \delta_{\mathbf{x}_k} \right), \tag{42}$$

where $0 < \varepsilon_n \leq 1$ and $\delta_{\mathbf{x}}$ denotes the distribution function with all probability concentrated at the point $\mathbf{x} \in \mathbb{R}^m$. In the sequel we additionally assume $\varepsilon_n \to 0$.

The Radon–Nikodym densities f and $\hat{f}_n$ of F and $\hat{F}_n$ respectively are zero outside the range $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$. On this set they are given by the formulas

$$f(\mathbf{x}_k) = p_k, \qquad \hat{f}_n(\mathbf{x}_k) = (1 - \varepsilon_n)\frac{\kappa_{n-1}(\mathbf{x}_k)}{n-1} + \frac{\varepsilon_n}{M}, \tag{43}$$

where $\kappa_n(\mathbf{x}_k)$ is the number of integers $i \in \{1, \ldots, n\}$ such that $\mathbf{X}_i = \mathbf{x}_k$. Since f, $\hat{f}_n$ are probabilities,

$$\text{ess sup}(\hat{f}_n/f) \leq \mathrm{p}_{\min}^{-1}$$

and moreover,

$$| \hat{f}_n(\mathbf{x}_k) - f(\mathbf{x}_k) | \leq | \frac{\kappa_{n-1}(\mathbf{x}_k)}{n-1} - p_k | + \varepsilon_n .$$

Therefore the law of large numbers applied to $\kappa_{n-1}(\mathbf{x}_k)/(n-1)$ yields together with $\varepsilon_n \to 0$

$$\hat{f}_n(\mathbf{x}_k) \to f(\mathbf{x}_k), \quad k \in \{1, \ldots, M\} \quad \text{a.s.} \tag{44}$$

and thus

$$I(F, \hat{F}_n) = \sum_{k=1}^{M} f(\mathbf{x}_k) \log \frac{f(\mathbf{x}_k)}{\hat{f}_n(\mathbf{x}_k)} \to 0 \quad \text{a.s.}$$

Hence all assumptions of Theorem 3 are satisfied. Consequently, for all $n \in \mathbb{N}$ there exist unique log-optimal portfolios

$$\hat{\mathbf{b}}_n = \operatorname{argmax} \sum_{k=1}^{M} \hat{f}_n(\mathbf{x}_k) \log(\mathbf{b}\, \mathbf{x}_k) \tag{45}$$

measurably depending on $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$ and satisfying the asymptotic relations (39) and (40).

In Example 1 this conclusion can be verified by an explicit evaluation of the portfolios (45). Namely, under the assumptions considered there,

$$\hat{\mathbf{b}}_n = (\hat{f}_n(\mathbf{x}_1), \ldots, \hat{f}_n(\mathbf{x}_m)).$$

Therefore the equivalent relations (39) and (40) follow immediately from (43) and (44).

Remark 3. The mixture of distributions (42) is analogical to the mixture of portfolios

$$\hat{\mathbf{b}}_n = (1 - \varepsilon_n)\, \tilde{\mathbf{b}}_n + \varepsilon_n \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$$

in Morvai [9], where $\tilde{\mathbf{b}}_n$ is log-optimal for the empirical distribution $\tilde{F}_n$. It seems that there are two possibilities how to avoid being ruined in the horse betting based on empirical distributions. One possibility is to mix random noise with the empirical distribution over the horses, and to employ the corresponding log-optimal rule. The other possibility is to mix random noise with the log-optimal rule resulting from the empirical distribution. This is one of the interesting situations where the random numbers help to make money.

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