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ON FUZZY INTUITIONISTIC LOGIC

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A first order fuzzy logic, called *Fuzzy Intuitionistic Logic* is introduced. This fuzzy logic is a generalization of the classical intuitionistic logic.

1. THE FOUNDATIONS OF FUZZY INTUITIONISTIC LOGIC

The starting point in Fuzzy Intuitionistic Logic is to fuzzify truth. We accept formulae that have different truth values. This corresponds to the use of sentences in everyday life; they may be true 'in different ways'. By accepting different truth values, we also break the true-false-dualism of classical logic. If we know the degree of truth of a sentence we do not necessarily know the degree of falsehood of the sentence. In Fuzzy Intuitionistic Logic a half true expression is not always half false.

Since we are not interested in the false sentences of a theory we let the falsehood be crisp. There is only one falsehood in Fuzzy Intuitionistic Logic. The negation of any formula being true in any degree is a false formula and the negation of any false formula is an absolutely true formula.

In everyday life we often experience sentences as being true 'in some degree' but we are not able to decide which of them is more true than the other. This kind of incomparable truth values are accepted in Fuzzy Intuitionistic Logic. We also accept the principle that for any set of truth values there exists a truth value which is at least as true as any of the truth values in the set under consideration and another truth value which is less or equally true to any of the truth values in the set under consideration. This leads to a state of affairs in which the set of degrees of truth consists of the largest element (the absolute truth, often marked by 1) and the smallest element (which differs from the truth value false of 0).

The set of truth values, composed of different truths and false, is always a finite set.

Similarly, as in classical first order logic, a set of well formed formulae $\mathcal{F}$ is composed of atomic formulae, containing the formula 'contradiction', and additional formulae obtained from the atomic formulae by means of logical connectives and, or, implies, not and quantors $\exists$ (read: there exists) and $\forall$ (read: for each).

Let L be some partially ordered set of truth values. Assume the binary operations $\wedge$ (meet), $\vee$ (join) and $\rightarrow$ (residuum with respect to $\wedge$) are defined in L. *A model of a theory* can be defined similarly as in classical logic; an *interpretation* T, which is roughly speaking the map

$$\mathsf{T}:\ \mathcal{F} \to \mathsf{L},$$

and has the following properties (see [4]):

$$\begin{aligned}&\mathsf{T}\ (\textsf{contradiction}) = 0 \text{ (the zero element of } \mathsf{L}),\\ &\mathsf{T}(\mathbf{a} \textsf{ and } \mathbf{b}) = \mathsf{T}(\mathbf{a}) \wedge \mathsf{T}(\mathbf{b}), \quad \mathsf{T}(\mathbf{a} \textsf{ or } \mathbf{b}) = \mathsf{T}(\mathbf{a}) \vee \mathsf{T}(\mathbf{b}),\\ &\mathsf{T}(\mathbf{a} \textsf{ implies } \mathbf{b}) = \mathsf{T}(\mathbf{a}) \to \mathsf{T}(\mathbf{b}), \quad \mathsf{T}(\textsf{not}\ \mathbf{a}) = \mathsf{T}(\mathbf{a}) \to 0,\\ &\mathsf{T}(\exists x \mathbf{a}(x)) = \vee \mathsf{T}(\mathbf{a}(x)), \quad \mathsf{T}(\forall x \mathbf{a}(x)) = \wedge \mathsf{T}(\mathbf{a}(x)).\end{aligned}$$

The value $\mathsf{T}(\mathbf{a}) \in \mathsf{L}$ is *the degree of truth* of **a** in interpretation T. If $\mathsf{T}(\mathbf{a}) = 1$ we say that T is a *model* of **a**. If $\mathsf{T}(\mathbf{a}) = 0$, then **a** is *false* in interpretation T. By *semantics* S_X with respect to some set of formulae X, we understand the set of all models of X.

Similarly, as in classical logic, it is reasonable to assume that formulae (**a and b**) **or c** and (**a or c**) **and** (**b or c**) have the same degree of truth. Also formulae (**a or b**) **and c** and (**a and c**) **or** (**b and c**) should have the same truth values. This implies that the truth value set L must be distributive, i. e.

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c), \qquad (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \quad \text{for each } a,\ b,\ c \in \mathsf{L}.$$

This is actually the case for operations $\wedge$ and $\vee$ in L. Also, the symmetry of these operations is a valuable property since it is natural to assume that formulae **a or b** (**a and b**) and **b or a** (**b and a**) have the same degree of truth.

In Fuzzy Intuitionistic Logic the Paradox of Bald (falakros) does not occur since we have more truth values than only one and since we set the following condition:

> The form of formula **a implies b** is absolutely true if and only if the degree of truth of **a** is less than or equal to the degree of truth of **b**. (1)

Between the connectives **and** and **implies** we set the following condition:

$\mathsf{T}(\mathbf{a} \textsf{ and } \mathbf{b}) \leq \mathsf{T}(\mathbf{c})$ if and only if $\mathsf{T}(\mathbf{a}) \leq \mathsf{T}(\mathbf{b} \textsf{ implies } \mathbf{c})$ for any interpretation T. (2)

This can be done if in value set L there exists the Galois connection

$$a \wedge b \leq c \text{ if and only if } a \leq b \to c \text{ for any } a,\ b,\ c \in \mathsf{L}. \tag{3}$$

By (2) we generalize (1).

We may combine the operations **not** and **implies** by

not a is equal to **a implies contradiction**. (4)

Rules of inference have a central role in proof theory. In classical logic a conclusion is connected with the premises in such a way that whenever the premises are true then the conclusion is also true. This is also the case in Fuzzy Intuitionistic Logic. We define a role of inference R in the following way:

$$\mathsf{R} = \left(\frac{\mathbf{a}_1, \ldots, \mathbf{a}_n}{\mathbf{b}}, \frac{a_1, \ldots, a_n}{b} \right),$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the premises and $\mathbf{b}$ is the conclusion. The values $a_1, \ldots, a_n,\ b \in \mathsf{K} \subseteq \mathsf{L} - \{0\}$ are the corresponding truth values.

In everyday life we sometimes hear reasoning like 'If you're not with me, then you're against me'. This is not accepted in Fuzzy Intuitionistic Logic. We assume that there exists formulae **a** such that **a or (not a)** is not absolutely true. Also such reasoning as 'The enemy of my enemy is my friend' is not generally valid in our logical system. This means that the formula **a** and **not (not) a** are not necessarily true to the same degree.

These two conditions imply that in set of truth values L

$$a \vee (a \rightarrow 0) \neq 1 \quad \text{for some } a \text{ in } \mathsf{L}, \tag{5}$$

and

$$a \neq (a \rightarrow 0) \rightarrow 0 \quad \text{for some } a \text{ in } \mathsf{L}. \tag{6}$$

If often happens that we associate truth value a (different to the absolute true 1 and false 0) to some phenomenon **a**. Then we receive new independent information about **a** and associate another truth value $b \neq 0,\ a,\ 1$ to the phenomenon **a**. Finally we conclude that the truth value c of **a** must be more than or equal to both a and b. This also characterizes Fuzzy Intuitionistic Logic.

2. FINITE BROUWERIAN LATTICES WITH EXACTLY ONE ATOM

As we saw in the previous section, logic is reducible to the structure of the set of truth values. We are looking for a finite residuated lattice L probably containing incomparable elements. The definition of a rule of inference implies that the set of non-false truth values should be closed with respect to the operations $\vee$, $\wedge$ and $\Rightarrow$, i. e. whenever $a,\ b \in \mathsf{L} - \{0\}$, then $a \wedge b,\ a \vee b,\ a \Rightarrow b \in \mathsf{L} - \{0\}$ too.

Any finite Brouwerian lattice (see [4]) has these properties. The residuum operation $\Rightarrow$ is defined by

$$a \Longrightarrow b = \bigvee \{c \mid a \wedge c \leq b\}. \tag{7}$$

3. SOME MORE DEFINITIONS

Let the truth value lattice L be fixed. A *fuzzy theory* X is composed of the set of the well formed formulae $\mathcal{F}$, containing a proper subset X_A of the *axioms* (formulae assumed to be true to some degree), a set R of the rules of inference and the set L of the truth values, i. e. X is a four tuple

$$\mathsf{X} = \langle \mathcal{F}, \mathsf{X}_A, \mathsf{R}, \mathsf{L} \rangle.$$

The *degree of* L*-validity* of formula $\mathbf{a} \in \mathcal{F}$ in the fuzzy theory X is defined by

$$\mathcal{C}_S\mathsf{X}(\mathbf{a}) = \wedge\{\mathsf{T}(\mathbf{a}) \mid \mathsf{T} \text{ is a model of } \mathsf{X}_A\}. \tag{8}$$

If $\mathcal{C}_S\mathsf{X}(\mathbf{a}) \neq 0$, then **a** is L*-valid* in X.

A L*-deduction* ω of formula a is a system

$$\begin{array}{ccc} \omega_1 & \omega_1 X & (B_1), \\ \vdots & \vdots & \vdots \\ \omega_{n-1} & \omega_{n-1} X & (B_{n-1}), \\ \omega_n & \omega_n X & (B_n), \end{array}$$

where $\omega_i : s$ $(i = 1, \ldots, n)$ are formulae and $\omega_i X : s$ the corresponding truth values so that $\omega_n = \mathbf{a}$ and each ω_i is an axiom or obtained by a rule of inference from some previous $\omega_j : s$ $(j < i)$. $B_i : s$ $(i = 1, \ldots n)$ are elucidating comments.

If formula **a** has an L-deduction, we say that it is L*-deducible* in X. The set of all L-deducible formulae in theory X will be marked by $\mathcal{F}^{\mathrm{ded}}$. Since L-deducible formula **a** may have different L-deductions, we define the *degree of* L*-deduction* of **a** by

$$\mathcal{C}_R X(\mathbf{a}) = \vee\{\omega X \mid \omega \text{ is an L-deduction of } \mathbf{a}\}. \tag{9}$$

The *subtheory* $\mathsf{X}^{\mathrm{sub}}$ of fuzzy theory X is the four tuple

$$\mathsf{X}^{\mathrm{sub}} = \langle \mathcal{F}^{\mathrm{ded}}, \mathsf{X}_A, \mathsf{R}, \mathsf{L} - \{0\} \rangle.$$

Fuzzy theory X is L*-consistent* if for any $a \in \mathsf{L}$ there exists $\mathbf{a} \in \mathcal{F}$ such that $\mathcal{C}_R X(\mathbf{a}) = a$. Finally, fuzzy theory X is L*-complete*, if for any $\mathbf{a} \in \mathcal{F}$

$$\mathbf{a} \text{ is L-valid if and only if } \mathbf{a} \text{ is L-deducible,} \tag{10}$$

and for any $\mathbf{a} \in \mathcal{F}^{\mathrm{ded}}$

$$\mathcal{C}_R\mathsf{X}^{\mathrm{sub}}(\mathbf{a}) = \mathcal{C}_S\mathsf{X}^{\mathrm{sub}}(\mathbf{a}). \tag{11}$$

4. THE *L*-COMPLETENESS OF FUZZY INTUITIONISTIC LOGIC

Completeness Theorem. Every L-consistent fuzzy theory X (containing X_A and R as defined above) is L-complete.

The proof of the theorem is rather long and can be found in [6] (see also [5]). □

Example. Assume we have five sentences **a**, **b**, **c**, **d** and **e**. We know that

b is absolutely true,
b implies d is very probably true,
b implies e is not out of question,
a and b is very probably true,
(**a and b**) **imply c** is quite sure.

Let the truth values absolutely true (1), very probably true (a), quite sure (b), not out of question (c) and false (0) form set L see the Diagram. To what degree, if any, is formula **c true**?

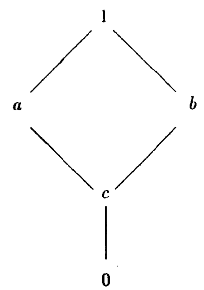

Diagram.

Define fuzzy theory X with the rules of inference R_1 – R_9, logical axioms (Ax. 1) – (Ax. 15) (cf. [6]) and special axioms

(Ax. 16)	**b**	(with truth value 1),
(Ax. 17)	**b implies d**	(with truth value a),
(Ax. 18)	**b implies e**	(with truth value c),
(Ax. 19)	**a and b**	(with truth value a),
(Ax. 20)	(**a and b**) **imply c**	(with truth value b),

We obtain an L–complete fuzzy theory. Using (Ax. 19), (Ax. 20) and R_1 we may construct the following L–deduction ω of **c**

$\omega_1 =$ **a and b**	$\omega_1 X = a$	((Ax. 19), assumption),
$\omega_2 =$ (**a and b**) **imply c**	$\omega_2 X = b$	((Ax. 20), assumption),
$\omega =$ **c**	$\omega = a \wedge b = c$	(apply R_1 to ω_1 and ω_2).

This means that

$$c \leq \mathcal{C}_R\mathsf{X}(\mathbf{c}). \tag{12}$$

Could there exist another L–deduction ω' of $\mathbf{c}$ so that $\omega' X > c$? Since X is L–complete this question can be solved semantically.

Let T_1 be such an interpretation that $\mathsf{T}_1(\mathbf{b}) = 1$, $\mathsf{T}_1(\mathbf{a}) = a$, $\mathsf{T}_1(\mathbf{d}) = a$, $\mathsf{T}_1(\mathbf{e}) = c$, $\mathsf{T}_1(\mathbf{c}) = a$. One easily verifies that $\mathsf{T}_1 \in S_X$. We conclude that $\mathcal{C}_S(\mathsf{X}(\mathbf{c})) \leq a$. Let T_2 be another interpretation so that $\mathsf{T}_2(\mathbf{b}) = 1$, $\mathsf{T}_2(\mathbf{a}) = 1$, $\mathsf{T}_2(\mathbf{d}) = 1$, $\mathsf{T}_2(\mathbf{e}) = c$, $\mathsf{T}_2(\mathbf{c}) = b$. Then $T_2 \in S_X$, too. We conclude that $\mathcal{C}_S\mathsf{X}(\mathbf{c}) \leq b$, but then we have

$$\mathcal{C}_S\mathsf{X}(\mathbf{c}) \leq c. \tag{13}$$

Because of the L–completeness of X and the equations (12) and (13) we conclude that

$$\mathcal{C}_S\mathsf{X}(\mathbf{c}) = c$$

i. e. **c** is not 'out of question.'

Exercise. Using the assumptions as above, define the degree of L–deduction of formulae **d** and **e**.

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