SUPPLEMENT TO KYBERNETIKA VOLUME 28 (1992), PAGES 65-71

MINIMAL AXIOMATIC SYSTEM OF FUZZY LOGICAL ALGEBRA

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This paper presents seven axioms of fuzzy logical algebra based on an axiomatic treatment of system $\langle U, \star, 0, 1\rangle$. This system will make a research into fuzzy logical algebra much more rigorous than before.

1. INTRODUCTION

One of the most important tools in modern mathematics is the theory of sets. Fuzzy set theory, introduced by L. A. Zadeh in 1965 [1], is a generalization of abstract set theory, while operations of fuzzy sets are obvious extensions of the corresponding definitions for ordinary sets. A year later, BCK-algebra, introduced by Y. Imai and K. Iseki in 1966 [2], is a generalization of set algebra based on six properties of the relative complement of a set with respect to the other. However there is a question between the two theories, whether exists a connection or not, and what it implies, this not problem seems to have been put forward so far.

As a matter of fact, fuzzy logical algebra [3] which is based on fuzzy set theory is special case of BCK-algebra, and from this, minimal axiomatic system in fuzzy logical algebra is obtained.

2. ABCD-ALGEBRA

Definition 1. ABCD-algebra is a system

$$S = \langle U, \star, 0, 1\rangle,$$

where U is a partially ordered set and it has at least two constant elements 0 and 1,

$$\star : U \times U \longrightarrow U$$

and for $\forall\, x,\, y,\, z \in U$, system S satisfies the following set of axioms:

a_1 Order:
$x \star y = 0 \Longleftrightarrow x \leq y.$

a_2 Equivalence:
$x \star y = 0, \quad y \star x = 0 \Longrightarrow x = y.$

a_3 0 Element:
$0 \star x = 0.$

a_4 Associativity:
$x \star (x \star (z \star (z \star y))) = z \star (z \star (x \star (x \star y))).$

a_5 Boundedness:
$x \star 1 = 0.$

a_6 Collocation:
$((x \star y) \star (x \star z)) \star (z \star y) = 0.$

a_7 Distributivity:
$((x \star (x \star D)) \star (x \star (x \star z))) \star ((x \star (x \star y)) \star (x \star (x \star z))) = 0,$

where

$$D = 1 \star ((1 \star y) \star ((1 \star y) \star (1 \star z))).$$

Theorem 1. Let a_1, a_2, a_3, a_4, a_5 and a_6 be the set of axioms. Then

b_0 $0 \star 0 = 0.$

b_1 $x \star x = 0.$

b_2 $x \star (x \star 0) = 0.$

b_3 $x \star 0 = x.$

b_4 $(x \star (x \star y)) \star y = 0.$

b_5 $x \star (x \star y) = y \star (y \star x)).$

Proof.

(b_0) Let $x = 0$. Then $0 \star 0 = 0$, since a_3

(b_1) Let $y = 0$, $z = 0$. Then

$$((x \star 0) \star (x \star 0)) \star (0 \star 0) = 0$$

by a_6, and since b_0

$$((x \star 0) \star (x \star 0)) \star 0 = 0,$$

while since a_3

$$0 \star ((x \star 0) \star (x \star 0)) = 0.$$

Hence by a_2

$$(x \star 0) \star (x \star 0) = 0.$$

If $u = x \star 0$, then

$$u \star u = 0.$$

Thus we obtain b_1.

(b_2) Let $z = 0$. Then

$$x \star (x \star (0 \star (0 \star y))) = 0 \star (0 \star (x \star (x \star y)))$$

by a_4, and since a_3, b_0, we have b_2.

(b_3) Let $y = 0$, $z = x$. Then

$$((x \star 0) \star (x \star x)) \star (x \star 0) = 0,$$

by a_6, and since b_1

$$((x \star 0) \star 0) \star (x \star 0) = 0,$$

while since b_2

$$(x \star 0) \star ((x \star 0) \star 0) = 0.$$

Hence by a_2

$$(x \star 0) \star 0 = (x \star 0).$$

Similarly, we obtain b_3.

(b_4) Let $y = 0$. Then

$$((x \star 0) \star (x \star z)) \star (z \star 0) = 0$$

by a_6, and since b_3

$$(x \star (x \star z)) \star z = 0.$$

Hence b_4.

(b_5) Let $y = 1$. Then

$$x \star (x \star (z \star (z \star 1))) = z \star (z \star (x \star (x \star 1)))$$

by a_4, and by a_5, b_3, we have b_5. □

A system $\langle U, \star, 0\rangle$ is a BCK–algebra, if U has at least one constant element 0 and it satisfies six axioms: a_1, a_2, a_3, a_6, b_1 and b_4. A system $\langle U, \star, 0, 1\rangle$ is a boundary commutative BCK–algebra, if it satisfies six axioms: a_1, a_2, a_3, a_5, a_6 and b_5.

Above Theorem 1 shows that an ABCD–algebra is a special case of the BCK–algebra class.

Theorem 2. Suppose $U = [0, 1]$, and $\forall\, x, y \in [0, 1]$;

$$x \star y = \begin{cases} x - y, & \text{if } x > y; \\ 0, & \text{if } x \le y, \end{cases}$$

then the system $\langle [0, 1], \star\rangle$ is the ABCD–algebra.

The proof of this theorem is evident from the above definition and is thus omitted.

3. FUZZY LOGICAL ALGEBRA

Definition 2. A fuzzy logical algebra is a system

$$Z = \langle U, +, \cdot, {}', 0, 1\rangle$$

where $U = [0,1]$, and for $\forall\, x,\ y,\ z \in U$, system Z satisfies the following set of axioms:

(A_1) Indempotency:
$x + x = x, \quad x \cdot x = x.$

(A_2) Commutativity:
$x + y = y + x, \quad x \cdot y = y \cdot x.$

(A_3) Associativity:
$(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).$

(A_4) Distributivity:
$x + y \cdot z = (x + y) \cdot (x + z), \quad x \cdot (y + z) = x \cdot y + x \cdot z.$

(A_5) Complement:
$x'' = x.$

(A_6) Identifies:
$x + 0 = x, \quad x \cdot 1 = x.$

(A_7) 0–1 Laws:
$x + 1 = 1, \quad x \cdot 0 = 0.$

(A_8) Absorption:
$x + x \cdot y = x, \quad x \cdot (x + y) = x.$

(A_9) De Morgan Laws:
$(x + y)' = x' \cdot y', \quad (x \cdot y)' = x' + y'.$

(A_{10}) Complementation:
$x + x' = \sup\{x, x'\},$
$x \cdot x' = \inf\{x, x'\}.$

In particular, $\forall\, x \in \{0,1\}$

$$x + x' = 1, \qquad x \cdot x' = 0.$$

Theorem 3. Let $S = \langle U, \star, 0, 1\rangle$ be an ABCD–algebra. If $U = [0,1]$ and for $\forall\, x,\, y \in U$,

$$x' = 1 \star x,$$
$$x \cdot y = y \star (y \star x),$$
$$x + y = 1 \star ((1 \star y) \star ((1 \star y) \star (1 \star x))).$$

Then the operations "+", "·", "′" satisfy the axioms $A_1 - A_{10}$.

4. THE LEMMAS FOR PROVING THEOREM 3

L_1 $x \le y \Longrightarrow z \star y \le z \star x, \qquad \forall\, z \in U.$

L_2 $x \le y,\ y \le z \Longrightarrow x \le z.$

L_3 $(x \star y) \star z = (x \star z) \star y.$

L_4 $x \star y \le z \Longrightarrow x \star z \le y.$

L_5 $x \le y \Longrightarrow x \star z \le y \star z.$

L_6 $x' \star y' = y \star x.$

L_7 $x \star (y + z) = (x \star z) \star (y \star z).$

L_8 $x\,y \le x, \qquad x\,y \le y.$

L_9 $x \le x + y, \qquad y \le x + y.$

L_{10} $u \le x,\ u \le y \Longrightarrow u \le x\,y,$ i.e. $x\,y = \inf\{x,\ y\}.$

L_{11} $x \le v,\ y \le v \Longrightarrow x + y \le v,$ i.e. $x + y = \sup\{x,\ y\}.$

L_{12} $x \le y \Longrightarrow x\,z \le y\,z.$

L_{13} $x\,y + x\,z \le x(y + z).$

The proofs of the lemmas L_1 – L_{13} are based on the definitions of the operations "+", "·", "′", and the axioms a_1 – a_6 (cf. [4, 5]).

5. PARTIAL PROOF OF THEOREM 3

$$A_1 \quad x \cdot x = x \star (x \star x) \quad \text{def.}$$
$$= x \star 0 \quad b_1$$
$$= x. \quad b_3$$

$$A_2 \quad x \cdot y = y \star (y \star x) \quad \text{def.}$$
$$= x \star (x \star y) \quad b_5$$
$$= y\,x. \quad \text{def.}$$

$$A_3 \quad (x \cdot y) \cdot z = (y \cdot x) \cdot z \quad A_2$$
$$= z \star (z \star (x \star (x \star y))) \quad \text{def.}$$
$$= x \star (x \star (z \star (z \star y))) \quad a_4$$
$$= (y \cdot z) \cdot x \quad \text{def.}$$
$$= x \cdot (y \cdot z). \quad A_2$$

$$A_4 \quad (x \cdot (y + z)) \star (x \cdot y + x \cdot z) =$$
$$= (x \cdot (y + z) \star x \cdot z) \star (x \cdot y \star x \cdot z) \quad L_7$$
$$= 0, \quad a_7$$

and $(x \cdot y + x \cdot z) \star (x \cdot (y + z)) = 0.$ L_{13}

Hence $x \cdot (y+z) = x \cdot y + x \cdot z.$ a_2

$$
\begin{array}{lll}
A_5 & x'' = 1 \star (1 \star x) & \text{def.} \\
 & = x \star (x \star 1) & b_5 \\
 & = x \star 0 & a_5 \\
 & = x. & b_3 \\
A_6 & x \cdot 1 = 1 \star (1 \star x) & \text{def.} \\
 & = x'' & \text{def.} \\
 & = x. & A_5 \\
A_7 & x \cdot 0 = 0 \star (0 \star x) & \text{def.} \\
 & = 0 \star 0 & a_3 \\
 & = 0. & b_0 \\
A_8 & x + x \cdot y = x \cdot 1 + x \cdot y & A_6 \\
 & = x \cdot (1+y) & A_4 \\
 & = x \cdot 1 & A_7 \\
 & = x. & A_6 \\
A_9 & (x \cdot y)' = (x'' \cdot y'')' & A_5 \\
 & = ((1 \star x)'(1 \star y)')' & \text{def.} \\
 & = (1 \star x) + (1 \star y) & \text{def.} \\
 & = x' + y'. & \text{def.} \\
A_{10} & x \cdot x' = \inf\{x, x'\}. & L_{10}.
\end{array}
$$

The proof of dual part for Theorem 3 is omitted. □

6. CONCLUSION

Theorem 2 and 3 show that the ABCD-algebra $\langle [0,1], \star \rangle$ is exactly the fuzzy logical algebra $\langle [0,1], +, \cdot, ' \rangle$. Hence the axioms a_1 – a_7 of $\langle [0,1], \star \rangle$ become the minimal axiomatic system of fuzzy logical algebra $\langle [0,1], +, \cdot, ' \rangle$. This system will make a research into fuzzy logical algebra much more rigorous than before.

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