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ON ENDOMORPHISM SEMIGROUPS OF A FUZZILY STRUCTURED SET

VENIAMIN SHTEINBUK AND ALEXANDER ŠOSTAK

Category $FS(\mathcal{L})$ of $\mathcal{L}$-fuzzily structured sets (fs-sets) (X, L, τ) is introduced. $FS(\mathcal{L})$ contains, for appropriately chosen category $\mathcal{L}$ of lattices, various categories of fuzzy topological spaces. The problem of definability of fs-sets by means of $\mathcal{L}$-endomorphism semigroups is discussed. However the tool of usual endomorphism semigroups used successfully in topology appears to be completely inadequate for this purpose: there are essentially different "good" fs-sets with isomorphic endomorphism semigroups. This difficulty is overcome by using a richer semigroup $S_{\mathcal{L}}(X, L, \tau)$ defined on the basis of the usual endomorphism semigroup $C_{\mathcal{L}}(X, L, \tau)$.

Let $\mathcal{L}$ be a category whose objects are complete lattices with 0 and 1 and whose morphisms are mapping of some kind between the lattices. By an $(\mathcal{L})$-fuzzily structured set (or an fs-set for short) we call a triple (X, L, τ) where X is a set, $L \in \mathrm{Ob}(\mathcal{L})$ and $\tau \subset L^X$. Let $FS(\mathcal{L})$ be the category, the objects of which are fs-sets and the morphisms are pairs $(f, \mu) : (X_1, L_1, \tau_1) \longrightarrow (X_2, L_2, \tau_2)$, where $f \in \mathrm{Mor}_{\mathrm{Set}}(X_1, X_2)$ (i.e. $f : X_1 \to X_2$ is a mapping), $\mu \in \mathrm{Mor}_{\mathcal{L}}(L_2, L_1)$ and $\mu \circ V \circ f \in \tau_1$ for each $V \in \tau_2$.

Notice that (as it will be specified to some extent below) various categories of fuzzy topological spaces considered in [1], [2], [7] e.g. are in fact full subcategories of the categories $FS(\mathcal{L})$ for appropriately chosen $\mathcal{L}$.

Let $FT(\mathcal{L})$ denote the complete subcategory of $FS(\mathcal{L})$ whose objects are fs-sets (X, L, τ) where τ is an L-fuzzy topology on X [2] (i.e. (1) $0, 1 \in \tau$, (2) if $U, V \in \tau$, then $U \wedge V \in \tau$, and (3) if $U_\gamma \in \tau$ for all $\gamma \in \Gamma$, then $\bigvee_\gamma U_\gamma \in \tau$). For a lattice $L \in \mathrm{Ob}(\mathcal{L})$ let $FS_L(\mathcal{L})$ (resp. $FT_L(\mathcal{L})$) denote the complete subcategory of $FS(\mathcal{L})$ (resp. of $FT(\mathcal{L})$) the objects of which are fs-sets (X, L, τ) where L is the given lattice.

Extending standard topological terminology to the situation under discussion, the morphisms of $FS(\mathcal{L})$ will be called $\mathcal{L}$-continuous mappings. For an fs-set (X, L, τ) let $C_{\mathcal{L}}(X, L, \tau)$ denote the semigroup of all its endomorphisms (= $\mathcal{L}$-continuous mappings of (X, L, τ) into itself) in the category $FS(\mathcal{L})$. Two fs-sets are called $\mathcal{L}$-homeomorphic if they are isomorphic as objects of $FS(\mathcal{L})$. We emphasize that the relation of $\mathcal{L}$-homeomorphism essentially depends on the choice of the category $\mathcal{L}$. Two fs-sets $(X_1, L_1, \tau_1$ and (X_2, L_2, τ_2) are called quasihomeomorphic if there exists a pair (f, μ) such that $f : X_1 \to X_2$ and $\mu : L_2 \to L_1$ are bijections and $\mu \circ V \circ f \in \tau_1$ iff $V \in \tau_2$.

The main problem considered in the paper is to reveal the possibility of definability up to $\mathcal{L}$-homeomorphism of an fs-set by means of its $\mathcal{L}$-endomorphism semigroup. We shall restrict ourselves here to two specific categories $\mathcal{L} = \mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_2$ introduced below. However, the tool of usual endomorphism semigroups which is successfully used in General Topology (see e.g. [4], [9]) appears to be completely inadequate for our purposes: there are many essentially different (in $FS(\mathcal{L})$) "good" fs-sets with equal endomorphism semigroups. We overcome these difficulties by using a richer semigroup $S_{\mathcal{L}}(X, L, \tau)$ introduced below instead of the semigroup $C_{\mathcal{L}}(X, L, \tau)$.

By the Plotkin endomorphism semigroup (with respect to the category $FS(\mathcal{L})$) of an fs-set (X, L, τ) we call the product $S_{\mathcal{L}}(X, L, \tau) = C_{\mathcal{L}}(X, L, \tau) \times L^X$ equipped with operation "$\cdot$" defined as follows

$$(f_1, \mu_1, U_1) \cdot (f_2, \mu_2, U_2) = (f_2 \circ f_1,\ \mu_1 \circ \mu_2,\ U_2 \circ f_1).$$

(A similar semigroup first appeared in [5] in connection with the theory of algebraic automata.) In the sequel we write sometimes $S_{\mathcal{L}}(X)$ instead of $S_{\mathcal{L}}(X, L, \tau)$.

Notice that apart from the binary operation "$\cdot$" there are two additional structures on the semigroup $S_{\mathcal{L}}(X, L, \tau)$. The first one is the subset τ of the lattice L^X and the second one is the partial order relation "$\prec$" introduced as follows: $(f_1, \mu_1, U_1) \prec (f_2, \mu_2, U_2)$ iff $f_1 = f_2$, $\mu_1 = \mu_2$ and $U_1 \leq U_2$ (i.e. $U_1(x) \leq U_2(x)$ for each $x \in X$). According to these structures we consider the following three kinds of isomorphism for Plotkin semigroups. We say that Plotkin semigroups $S_{\mathcal{L}}(X_1, L_1, \tau_1)$ and $S_{\mathcal{L}}(X_2, L_2, \tau_2)$ are

(1) isomorphic, if they are isomorphic in the category of semigroups;

(2) τ-isomorphic, if there exists an isomorphism $\sigma : S_{\mathcal{L}}(X_1, L_1, \tau_1) \longrightarrow S_{\mathcal{L}}(X_2, L_2, \tau_2)$ such that $\sigma(C_{\mathcal{L}}(X_1) \times \tau_1) = C_{\mathcal{L}}(X_2) \times \tau_2$;

(3) ω-isomorphic, if there exists a τ-isomorphism $\sigma : S_{\mathcal{L}}(X_1, L_1, \tau_1) \longrightarrow S_{\mathcal{L}}(X_2, L_2, \tau_2)$ such that $(f, \mu, U_1) \prec (f, \mu, U_2)$ iff $\sigma(f, \mu, U_1) \prec \sigma(f, \mu, U_2)$.

To formulate the main results we have first to specify the category $\mathcal{L}$. Namely, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be categories whose objects are complete lattices with 0 and 1, $\mathrm{Mor}(\mathcal{L}_1)$ consists of all mappings $f : L_1 \to L_2$ preserving arbitrary non-empty suprema and finite infima and $\mathrm{Mor}(\mathcal{L}_2)$ consists of identical mappings $\varepsilon_L : L \to L$ only (i.e. $\mathcal{L}_2$ is a discrete category). (Here L_1, L_2, $L \in \mathrm{Ob}(\mathcal{L}_1) = \mathrm{Ob}(\mathcal{L}_2)$.)

Notice that $FT(\mathcal{L}_1)$ is in fact a slight enlargement of Rodabaugh's category $\mathbf{T}$ [8] (cf. also the category FUZZ from [7]). It is easy to notice also that $FT_L(\mathcal{L}_2)$ is just the category of L-fuzzy topological spaces as they are defined by Goguen [2]; specifically, $FT_I(\mathcal{L}_2)$, where $I = [0,1]$, is the category of Chang fuzzy topological spaces [1] and $FT_Z(\mathcal{L}_2)$, where $Z = \{0,1\}$, in an obvious way can be identified with the category Top of topological spaces.

We shall need also the next notion. An fs-set (X, L, τ) is called laminated if τ contains constant mappings $\alpha_X : X \to L$ for all $\alpha \in L$ (cf. Lowen's definition of a fuzzy topology; see e.g. [3]).

Theorem 1. For laminated fs-sets (X_1, L_1, τ_1) and (X_2, L_2, τ_2) the following conditions are equivalent:

(1) the semigroups $S_{\mathcal{L}_1}(X_1)$ and $S_{\mathcal{L}_1}(X_2)$ are ω-isomorphic;

(2) the semigroups $S_{\mathcal{L}_2}(X_1)$ and $S_{\mathcal{L}_2}(X_2)$ are ω-isomorphic;

(3) fs-sets (X_1, L_1, τ_1) and (X_2, L_2, τ_2) are $\mathcal{L}_1$-homeomorphic.

Theorem 2. Laminated fs-sets (X_1, L_1, τ_1) and (X_2, L_2, τ_2) are quasihomeomorphic iff the semigroups $S_{\mathcal{L}_i}(X_1)$ and $S_{\mathcal{L}_i}(X_2)$ are τ-isomorphic ($i = 1, 2$).

To restore a laminated fs-set up to $\mathcal{L}_2$-homeomorphism by means of its Plotkin endomorphism semigroup we need the following special kind of ω-isomorphism:
A τ-isomorphism $\sigma : S_{\mathcal{L}}(X_1, L, \tau_1) \longrightarrow S_{\mathcal{L}}(X_2, L, \tau_2)$ is called tough, if $\sigma(\varepsilon_{X_1}, \varepsilon_L, \alpha) = (\varepsilon_{X_2}, \varepsilon_L, \alpha)$ for each $\alpha \in L$. One can prove that each tough isomorphism of laminated fs-sets is an ω-isomorphism.

Theorem 3. Laminated fs-sets (X_1, L, τ_1) and (X_2, L, τ_2) are $\mathcal{L}_2$-homeomorphic iff the semigroups $S_{\mathcal{L}_i}(X_1)$ and $S_{\mathcal{L}_i}(X_2)$ are toughly isomorphic ($i = 1, 2$).

These theorems immediately imply analogous results for laminated fuzzy topological spaces:

Theorem 1'. Laminated fuzzy topological spaces (X_1, L_1, τ_1) and (X_2, L_2, τ_2) are homeomorphic (in $FT(\mathcal{L}_1)$) iff their Plotkin semigroups $S_{\mathcal{L}_1}(X_1)$ and $S_{\mathcal{L}_2}(X_2)$ are ω-isomorphic.

Theorem 3'. Laminated L-fuzzy topological spaces [2] (X_1, τ_1) and (X_2, τ_2) are homeomorphic iff their Plotkin semigroups $S_{\mathcal{L}_i}(X_1, L, \tau_1)$ and $S_{\mathcal{L}_i}(X_2, L, \tau_2)$ are toughly isomorphic ($i = 1, 2$).

Example 1. The condition of laminatedness is of essence. Let (X, T) be a topological space such that $C(X, T) = \{\varepsilon_X\} \cup \{c_X : c \in X\}$. Thus the semigroup of endomorphisms of X consists only of constant mappings and the identity. (Such a space can be found e.g. in [6].) Fix two constants $0 < \alpha < \beta < 1$ and two points $a, b \in X$. Let M denote the set of all mappings $\mu : I \to I$ preserving non-empty suprema and finite infima such that $\mu(\alpha) = \alpha$, $\mu(\beta) = \beta$. Define fuzzy sets $U_i : X \to I$, $i = 1, 2$ as follows. Let $U_1(x) = \alpha$ if $x \neq a$ and $U_1(a) = \beta$ and let $U_2(x) = \alpha$ if $x \neq a, b$ and $U_2(a) = U_2(b) = \beta$. Let τ_i, $i = 1, 2$, be the fuzzy topology having $T \cup \{U_i\}$ as its subbase. It is easy to notice that the semigroups $S_{\mathcal{L}_i}(X, I, \tau_1)$ and $S_{\mathcal{L}_i}(X, I, \tau_2)$ are ω-isomorphic (even toughly isomorphic) but nevertheless the spaces (X, I, τ_1) and (X, I, τ_2) are not $\mathcal{L}_i$-homeomorphic, $i = 1, 2$.

Example 2. Inadequacy of semigroups of continuous transformations in fuzzy setting. Let (X,T) be a topological space. For a constant $a \in (0,1]$ let τ_a be a fuzzy topology on X generated by the subbase $\sigma_a = \{aU : U \in T\} \cup \{\alpha_X : \alpha \in I\}$. (Obviously, $\tau_1 = \omega T$ is the set of all lower semicontinuous functions $M : (X,T) \to I$; see [3].) It is easy to notice that $C_{\mathcal{L}_2}(X,I,\tau_a) = C_{\mathcal{L}_2}(X,I,\tau_{a'})$ for any a, $a' \in (0,1]$ and if $a, a' \neq 1$, then $C_{\mathcal{L}_1}(X,I,\tau_a)$ and $C_{\mathcal{L}_1}(X,I,\tau_{a'})$ are isomorphic. On the other hand, if $a \neq a'$, then the fs-sets (X,I,τ_a) and $(X,I,\tau_{a'})$ are neither $\mathcal{L}_2$-homeomorphic, nor $\mathcal{L}_1$-homeomorphic.

REFERENCES

[1] C. L. Chang: Fuzzy topological spaces. J. Math. Anal. Appl. *24* (1968), 182–190.

[2] J. A. Goguen: The fuzzy Tychonoff theorem. J. Math. Anal. Appl. *43* (1973), 734–742.

[3] R. Lowen: Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. *56* (1976), 621–633.

[4] K. D. Magill: A survey of semigroups of continuous selfmaps. Semigroup Forum *11* (1975/76), 189–282.

[5] B. I. Plotkin: Algebra of automata: some problems. Vestnik MGU Ser. Math. Mech. *4* (1980), 96 (in Russian).

[6] A. Pultr and V. Trnková: Combinatorial, algebraic and topological representation of groups, semigroups and categories. Prague 1980.

[7] S. E. Rodabaugh: A categorical accommodation of various notions of fuzzy topology. Fuzzy Sets Syst. *9* (1983), 241–265.

[8] S. E. Rodabaugh: A point-set lattice-theoretic framework **T** for topology which contains LOC as a subcategory of singleton spaces..., Preprint, February 1986, Youngstown State University Youngstown, Ohio.

[9] Ye. M. Vechtomov: Problems of definability of topological spaces by algebraic systems of continuous functions. Itogi Nauki i Techn., Ser. Algebra, Topology, Geometry *28* (1990), 3–46.

Dr. Veniamin Shteinbuk, Department of Applied Mathematics, Riga Technical University, Riga 226355. Latvia.

Dr. Alexander Šostak, Department of Mathematics, Latvian University, Riga 226098. Latvia.