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MODIFIED MODUS PONENS AND MODAL LOGIC

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This paper discusses an inference rule called by *modified modus ponens*, which is used in the logical system $LPC + Ch$ which is first order (or lower) predicate calculus equipped with additional axiomatization of *modifier operators*. This basic forms a system like generalized modal system with several pairs of modal operators.

The main properties of the system $LPC + Ch$ necessary for introducing this topic are considered. It suffices well a propositional system $PC + Ch$ for these purposes. The modal version of *modified modus ponens* is proved to hold in standard modal systems.

1. INTRODUCTION

First we give a short description of the *Ch-extension* of classical propositional logic PC. As an *alphabet* of our *Ch*-language we adopt the alphabet of classical propositional calculus choosing the connectives $\neg$ standing for negation, $\rightarrow$ standing for implication, as primitives, and connectives $\vee$ standing for disjunction, $\wedge$ standing for conjunction, and $\leftrightarrow$ standing for equivalence are derived from those in the known way. We adopt the set of proposition letters $PR = \{p_i \mid i = 0, 1, \ldots, n, \ldots\}$ straight from PC. So, for $PC + Ch$ we get the logical alphabet from PC. We further need some added characters for formalizing a set of *characteristic operators*. The symbolic alphabet consists of a set of *modifier symbols* $\mathbf{0} = \{\Im, \mathcal{F}_1, \mathcal{F}_2, \ldots\}$ where the operators $\mathcal{F}_1$, $\mathcal{F}_2, \ldots$ are *substantiating* (abbreviated by $\Im \precsim \mathcal{F}_i$) and $\Im$ is an *identity operator*. We can denote these *modifier operators* by metavariables $\mathcal{H}$, $\mathcal{F}$, $\mathcal{V}, \ldots$ (with or without numerical subscripts). For any modifier $\mathcal{F} \in \mathbf{0}$, we can form its *dual modifier* $\mathcal{F}^* = \neg\mathcal{F}\neg$, and the set of duals we symbolize by $\mathbf{0}^*$. For the identity operator $\Im$ it holds $\Im^* = \Im$. Modifiers belonging to $\mathbf{0}^*$ are called *weakening operators* (abbreviated by $\mathcal{F}^* \precsim \Im$).

The formation of well-formed formulae (wffs) is similar to that of PC. We give the definition of the set $\mathbf{W}$ of wffs of $PC + Ch$ as follows:

Definition 1.1. $\mathbf{W}$ is the set of wffs of $PC + Ch$ if

(1^0) the set W of wffs of PC is a subset of $\mathbf{W}$;

(2^0) if $\alpha \in \mathbf{W}$ and $\mathcal{F} \in \mathbf{0}$ then $\mathcal{F}(\alpha) \in \mathbf{W}$;

(3^0) if $\alpha \in \mathbf{W}$ then $\neg\alpha \in \mathbf{W}$;

(4^0) if α, $\beta \in \mathbf{W}$ then $(\alpha \rightarrow \beta) \in \mathbf{W}$;

(5^0) All the wffs are generated by the steps (1^0) – (4^0).

The formal semantics of $PC + Ch$ is given in Mattila [7], and we do not consider it here. Instead, we go straight to the axiomatization. In addition to the axiomatization of PC we need in our proof-theoretical system a *characteristic axiom schemata* governing the logical properties of the modifier operators.

Our axiomatization for our system $PC + Ch$ are as follows:

Axiom schemata of Ch.

(i) All the *tautologies of PC* are axioms.

(ii) If $\mathcal{H}$, $\mathcal{F} \in \mathbf{0} \cup \mathbf{0}^*$, and $\mathcal{H} \overset{<}{\sim} \mathcal{F}$ ($\mathcal{H}$ is at most as strong as $\mathcal{F}$), then for all $\alpha \in \mathbf{W}$

$$\mathcal{F}(\alpha) \to \mathcal{H}(\alpha) \tag{AxCh}$$

is an axiom.

(iii) For all wffs $\alpha \in \mathbf{W}$ and for the identity operator $\Im \in \mathbf{0}$

$$\Im(\alpha) \leftrightarrow \alpha \tag{AxId}$$

is an axiom.

We also adopt the following **inference rules:**

Modus ponens:

$$\alpha \to \beta,\ \alpha \vdash \beta, \tag{MP}$$

Modified modus ponens:

$$\alpha \to \beta,\ \mathcal{F}(\alpha) \vdash \mathcal{F}(\beta) \tag{MMP}$$

where $\mathcal{F} \in \mathbf{0}$ is an arbitrary operator.

Rule of Substantiation. For wffs $\alpha \in \mathbf{W}$ and all substantiating operators $\mathcal{F} \in \mathbf{0}$

$$\vdash \alpha \Rightarrow \vdash \mathcal{F}(\alpha) \tag{RS}$$

So, a *Ch-system* is any non-empty set X, such that the tautologies of PC, (AxCh) and (AxId) are included in X, and X is closed under (MP), (MMP), and (RS).

In the sequel we need the following properties of $PC + Ch$, which are proved in Mattila [6], [7]. For any $P \in \mathbf{W}$,

$$\vdash \mathcal{F}(P) \to P \qquad (\Im \overset{<}{\sim} \mathcal{F}) \tag{1.1}$$

$$\vdash P \to \mathcal{H}(P) \qquad (\mathcal{H} \overset{<}{\sim} \Im) \tag{1.2}$$

Then we consider situations in which operators are associated with connected wffs. We have the following result:

If $\mathcal{F}$ is a substantiating operator and $\mathcal{H} = \neg\mathcal{F}\neg$, and $P, Q \in \mathbf{W}$, then

$$\mathcal{F}(P \to Q) \vdash \mathcal{F}(P) \to \mathcal{F}(Q). \tag{1.3}$$

2. SOME SYNTACTICAL SIMILARITIES BETWEEN *Ch*- AND MODAL SYSTEMS

We consider first the system **T** giving its axioms and other rules and definitions we need (for details, see [1]). One useful way for axiomatizing modal systems is to built the system over PC, as usually is done. Thus the system **T** consists of the axioms for propositional logic and of the axioms basing on *necessity*,

$$\Box P \to P \tag{TA1}$$

$$\Box(P \to Q) \to (\Box P \to \Box Q) \tag{TA2}$$

T contains also inference rule MP and so-called **Rule of Necessitation**:

$$\vdash P \Longrightarrow \vdash \Box P \tag{N}$$

The modal concept of *possibility*, is defined by the condition

$$\Diamond P =_{\mathrm{df}} \neg\Box\neg P \tag{Def. $\Diamond$}$$

for any wff P, i.e. possibility is the dual of necessity.

If we interpret the operators $\mathcal{F}$ and $\mathcal{H}$ to be the modal operators $\Box$ standing for *necessity*, and $\Diamond$ standing for *possibility* of aletic modal logic, respectively, and the system has only this dual pair of operators, we get a modal system which contains the modal system **T**. In the modal interpretation of modifier operators the identity operator $\Im$ corresponds to modal operator *'actuality'* (abbreviated often by $\bigcirc$). The formal evidence for that is e.g. the equation $\bigcirc(P) \equiv \neg \bigcirc (\neg P)$ for all $P \in \mathbf{W}$.

The modal counterparts of (AxCh) in the system **T** is

$$\vdash \Box P \to \Diamond P \tag{AxCh''}$$

from which it follows the reflexivity laws by means of the actuality operator.

$$\vdash \Box P \to P \tag{2.1}$$

$$\vdash \Diamond P \to \Diamond P \tag{2.2}$$

Thus in this modal interpretation (TA1) is equivalent to (1.1) operator $\Box$ being substantiating. (TA2) follows directly from the modal counterpart of (1.3). Because MP belongs to the both systems and in $PC + Ch$ substantiating operators have the same formal property than $\Box$ in **T**, namely (N), we have showed that **T** belongs to the modal version $PC + Ch$. Clearly (2.1) implies both (2.2), and (AxCh"), and also (2.2) implies both (2.1) and (AxCh").

3. MODAL VERSION OF MODIFIED MODUS PONENS

There are also other standard modal systems like **S 1**, **S 2**, **S 3**, **S 4**, and **S 5**, which are the most usual ones. The subsystem relations between these are **S 1** ⊂ **S 2** ⊂ **S 3** ⊂ **S 4** ⊂ **S 5** and **S 1** ⊂ **S 2** ⊂ **T** ⊂ **S 4** ⊂ **S 5**, Thus **S 3** is in a way alternative to the system **T**. It suffices to restrict our considerations to **S 1**. The rule of Necessitation does not hold without restrictions in standard systems **S 1**, **S 2** and **S 3** (see [1], p. 225, 230, and 235). The restricted form is

$$\vdash_{PC} \alpha \;\Rightarrow\; \vdash_{S1} \Box\alpha \tag{3.1}$$

For any wff α of PC. Axiom (TA2) of **T** is a theorem in **S 1**. It is proved in Hughes and Cresswell [1] p. 225 and numbered by TS1.21. We need this result below. We now prove the following

Proposition 3.1. The modal version of the rule MMP holds in **S 1**, i. e.

$$\vdash P \to Q,\ \vdash \mathcal{M}(P) \Rightarrow \vdash \mathcal{M}(Q) \tag{3.2}$$

where $\mathcal{M}$ is a modal operator of **S 1**.

Proof. Suppose $\vdash P \to Q$, and $\vdash \mathcal{M}(P)$ hold. It is remarkable that especially $\vdash_{PC} P \to Q$. For $\mathcal{M} = \Box$ we have the deduction.

1.	$P \to Q$	given
2.	$\Box P$	given
3.	$\Box(P \to Q)$	appl. (3.1) to 1
4.	$\Box(P \to Q) \to (\Box P \to \Box Q)$	TS1.21 in H. & C.
5.	$\Box P \to \Box Q$	MP, 3,4
6.	$\Box Q$	MP, 2,5

For $\mathcal{M} = \Diamond$ we have the deduction

1.	$P \to Q$	given
2.	$\Diamond P$	given
3.	$\neg\Diamond Q$	premise
4.	$\Box\neg Q$	$\Diamond Q = \neg\Box\neg Q$
5.	$(P \to Q) \to (\neg Q \to \neg P)$	LA3
6.	$\neg Q \to \neg P$	MP, 1,5
7.	$\Box(\neg Q \to \neg P)$	appl. (3.1) to 6
8.	$\Box(\neg Q \to \neg P) \to (\Box\neg Q \to \Box\neg P)$	TS2.21 in H. & C.
9.	$\Box\neg G \to \Box\neg P$	MP, 7,8
10.	$\Box\neg P$	MP, 4,9
11.	$\neg\Diamond P$	$\Diamond P = \neg\Box\neg P$
12.	$\neg\Diamond P \wedge \Diamond P$	A, 2,11
13.	$\Diamond Q$	PC, 3,12.

□

From this result and from the subsystem relations it follows directly

Proposition 3.2. The modal version of the rule MMP holds in **T**, i.e.

$$\vdash P \rightarrow Q, \vdash \mathcal{M}(P) \Rightarrow \vdash \mathcal{M}(Q) \tag{3.3}$$

Where $\mathcal{M}$ is a modal operator of **T**.

This can be also proved very easily without the knowledge of Proposition 3.1. Because **T** is also a subsystem of Brouwerian system, the modal version of MMP holds also in it.

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