SUPPLEMENT TO KYBERNETIKA VOLUME 28 (1992), PAGES 25–27

FUZZY CONCEPTS DEFINED VIA RESIDUATED MAPS

ACHILLE ACHACHE AND ARTURO A. L. SANGALLI[1]

We show how some concepts such as "fuzzy subset" or "fuzzy closed set of a topological space" may be identified with certain maps between complete lattices. Underlying this representation is the fact that the category of closure spaces contains the category of complete lattices and residuated maps as a reflective subcategory. This approach suggests a uniform method for fuzzifying concepts such as "ideals", "subgroups" and other collections of subsets having a complete lattice structure.

1. INTRODUCTION

In this note we present a uniform procedure to fuzzify concepts such as "subgroup", "ideal", "closed set", as well as others having the property that the (crisp) objects form a complete lattice. The fuzzification is achieved by representing the fuzzy objects as residuated maps between complete lattices. The underlying result making this representation possible is the existence of a left adjoint to the functor which embeds the category of residuated maps between complete lattices into the category of closure spaces (defined in 2. below).

We begin by establishing this adjunction (in 3.). In 4., we give several examples of the representation. In each case it is seen that the fuzzy objects (fuzzy subgroups, fuzzy closed sets, etc.) "are" the residuated maps between the complete lattice of crisps objects (lattice of subgroups, lattice of closed sets, etc.) and the complete valuation lattice. We also show how some familiar definitions and results are special cases of our construction.

2. PRELIMINARY DEFINITIONS

A *closure space* is a pair $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a family of subsets of S closed with respect to arbitrary intersections. The subsets in $\mathcal{F}$ are called *closed*, and for each $X \subseteq S$, the *closure* $\overline{X}$ of X is the smallest closed set containing X.

The category $\underline{\mathbf{F}}$ has closure spaces as objects and its morphisms are the maps $S_1 \longrightarrow S_2$ such that inverse images of closed sets are closed.

$\underline{\mathbf{L}}$ is the category of complete lattices with morphisms the residuated maps, a map between complete lattices being *residuated* if it preserves suprema, which is equivalent to requiring that the inverse image of every principal ideal be a principal ideal (a *principal ideal* of the ordered set $(L, \leq)$ is a subset of the form $\{x \in L : x \leq p\}$, for some $p \in L$). $P(L)$ denotes the set of all principal ideals of L.

[1]The authors's contribution was supported by Quebec's Fund FCAR.

Every complete lattice L may be seen as a closure space $(L,\ P(L))$, where the closed sets are precisely the principal ideals of L. The correspondence $L \longmapsto (L,\ P(L))$ defines a functor $I : \underline{\mathbf{L}} \longrightarrow \underline{\mathbf{F}}$. Since every morphism $f : L_1 \longrightarrow L_2$ is also a morphism from $I(L_1)$ to $I(L_2)$ and conversely, the functor I embeds $\underline{\mathbf{L}}$ as a full subcategory of $\underline{\mathbf{F}}$.

3. THE ADJUNCTION

The functor $F : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{L}}$ is defined by $F((S, \mathcal{F})) = \mathcal{F}$ on objects. For a morphism $h : (S_1, \mathcal{F}_1) \longrightarrow (S_2, \mathcal{F}_2)$ de $\underline{\mathbf{F}}$, $F(h) : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is, by definition, the closure of the image, i. e., for all $X \in \mathcal{F}_1$, $F(h)(X) = \overline{h(X)}$.

Theorem 1. The functor $F : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{L}}$ is left adjoint to the embedding functor I. Moreover, the lattices $\hom_{\underline{\mathbf{F}}}((S,\ \mathcal{F}),\ (L,\ P(L))$ and $\hom_{\underline{\mathbf{L}}}(\mathcal{F}, L)$ are isomorphic.

A proof of this theorem appears in [3].

4. CONSEQUENCES OF THE ADJUNCTION

4.1. Fuzzy closed sets of a topological space

Every topological space T is a closure space $(T, \mathcal{F})$, where $\mathcal{F}$ is the complete lattice of closed (in the topology) sets. More precisely, the category **Top** of topological spaces may be viewed as a full subcategory of $\underline{\mathbf{F}}$, for the morphisms $(T_1,\ \mathcal{F}_1) \longrightarrow (T_2,\ \mathcal{F}_2)$ of $\underline{\mathbf{F}}$ are precisely the continuous functions $T_1 \longrightarrow T_2$. If L is a complete lattice, a fuzzy closed set of T is, by definition, a residuated map from $\mathcal{F}$ to L.

We can use the adjunction to give a characterization of the fuzzy closed sets. By the bijection of Theorem 1,

$$\hom_{\underline{\mathbf{F}}}((T,\ \mathcal{F})),\ (L, P(L)) \ \cong\ \hom_{\underline{\mathbf{L}}}(\mathcal{F}, L).$$

Now let a topology in L be defined by taking the principal ideals as a basis for the closed sets (i. e. this is the smallest topology such that every principal ideal is closed), then

$$\hom_{\underline{\mathbf{F}}}((T, \mathcal{F}),\ (L, P(L)) = \ \mathbf{Top}\,(T, L),$$

and thus *the fuzzy closed sets of a topological space T are (in one-to-one correspondence with) the continuous functions from T to L*, where L is endowed with the topology defined above. Using this characterization it is easy to check that if $L = \mathbf{2}$ $(= \{0, 1\})$, then the fuzzy closed sets are precisely the (characteristic functions of the) elements of $\mathcal{F}$.

4.2. Fuzzy subsets

The category **Set** of sets may be identified with the full subcategory of **Top** whose objects are the discrete spaces, i. e. those of the form $(S, \mathbf{2}^S)$. By the results of 4.1, we have $L^S\ \cong\ \hom_{\underline{\mathbf{L}}}(\mathbf{2}^S, L)$.

Thus, an L-fuzzy subset of a set S may be represented as a residuated map from $\mathbf{2}^S$ to L.

The dual of the complete lattice L is denoted L^*. It is known that the residuated maps $Q \longrightarrow L$, where Q is another complete lattice, are precisely the Galois connections from Q to L^* (see for example [4]). We then have, using the above bijection,

$$\mathrm{Gal}\,(\mathbf{2}^S, L) = \hom_{\underline{\mathbf{L}}}(\mathbf{2}^S, L^*) \;\cong\; (L^*)^S = L^S.$$

Moreover, this bijection is an isomorphism of complete lattices. It is also the main result of [1] (Proposition 3 and Corollary 4): *Every L-fuzzy subset of a set S "is" a Galois connection between the lattice of all subsets of S and the complete lattice L.*

4.3. Fuzzy subgroups

Let G be a groupoid (i.e. a set equipped with a binary operation) and let $S(G)$ be the complete lattice of all subgroupoids of G. Then $(G, S(G))$ is a closure space. The correspondence $G \longmapsto (G, S(G))$, defines a functor from the category $\underline{\mathbf{G}}$ of groupoids into $\underline{\mathbf{F}}$ (every groupoid homomorphism is a morphism in $\underline{\mathbf{F}}$, for inverse images of subgroupoids are subgroupoids). If L is a complete lattice, the L-fuzzy subgroupoids of G are, by definition, the residuated maps from $S(G)$ to L. By Theorem 1, these are in one-to-one correspondence with certain maps $\varphi : G \longrightarrow L$ (i.e. the morphisms in $\underline{\mathbf{F}}$). It can be shown ([2], Theorem 4) that these are precisely those φ which satisfy $\varphi(x) \wedge \varphi(y) \leq \varphi(xy)$, for all $x, y \in G$.

In a similar fashion we can define the fuzzy subgroups of a group, and we then find the definition of [5].

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Dr. Achille Achache, Département de mathématiques, Université Claude Bernard (Lyon I), 696 22 Villeurbanne Cedex. France.

Dr. Arturo A. L. Sangalli, Department of Mathematics, Champlain Regional College, Lennoxville J1M 2A1 (Quebec). Canada.