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HALF A CENTURY OF FUZZY SETS

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Fuzzy sets conceived as mappings from X to [0,1], i. e. as a generalized characteristic functions with operations max, min, and subtraction-from-1 form a structure of De Morgan algebra, and as a such they have been introduced for the first time in 1940 by H. Weyl and independently, in much more elaborated form, by A. Kaplan and H. Schott in 1951. Recently the fuzzy sets were rediscovered by L. Zadeh and they have been developed in many directions by many researchers. An abstract De Morgan algebras have been introduced in 1935 by G. C. Moisil, generalized characteristic functions introduced in 1936 by E. Szpilrajn have been also intensively studied in fifties by H. Rasiowa and R. Sikorski.

So called fuzzy sets are treated as a conceptualization of vagueness, or in other words as a formal tool to treat precisely the phenomenon of vagueness.

The problem of vagueness is a very old one. A discussions of it one may trace back to Greek antiquity. It is enough to remember the controversy concerning the definition of "man" given in Plato's Academy, and the well-known paradox called "falakros".

W. G. Leibniz, the last man who knew all things in the world, was probably the first one who has discussed the problems of vagueness and impreciseness in greater details, considering them as a properties of language (cf. [8]).

L. E. J. Brouwer has also argued that any language is vague and prone to misunderstanding. Even if not explicitly he has connected vagueness phenomenon with ancient principle of "tertium non datur". As early as 1908 he criticized the "long belief in the universal validity of the principle of excluded third", he has argued that in general one cannot expect a subset B of A to be decidable, i. e. for all $x \in A$ whether $x \in B$ or $x \notin B$ (see [3]).

The principle of excluded middle has been also criticized by J. Lukasiewicz, who argued that the principle of bivalence cannot be proved, one can only believe it, and he is alone who considers it self-evident to believe it (see [10]). His own attitude towards this principle was expressed as follows: "I am entitled not to recognize it, and to accept the view that besides truth and falsehood there exist other truth-values, including at least one more, the third truth-value" (cf. [10]). And in 1918 he communicated about the success in developing a new logical system. The famous about 50-lines article on three-valued logic was published in 1920 (see [12]).

In the much cited article of B. Russell [15] we find once again that vagueness and its opposite – precision – are "characteristics which can only belong to representation, of which language is an example" (cf. [15]).

Russell's definition of vagueness by means of one-many relation between symbolizing and symbolized systems has been criticized by M. Black, who following the Peirce's attitude prefers a pragmatical definition, arguing that indeterminacy in meaning of notion is due to usage but not its extension (see [1]). Undoubtedly his "consistency profile" must be considered as a first attempt to define what we know today as a membership function.

Although B. Russell and M. Black differ in their philosophical points of view, they both, more or less explicitly, have grounded their considerations on questioned principle of excluded third. Like L. Brouwer and J. Lukasiewicz they do not want to exclude this third. For M. Black for example, the third was a region of doubtful application of a term to objects. This region has been called "fringe".

The problem of vagueness has been also investigated within the framework of mathematics, where the central role plays the notion of "set", and which is used for definition of meaning (extension) of terms. The notion of "set" has been used (defined) for the first time by W. G. Leibniz, and the most widespread is a Cantorian conception to treat a set as a collection of distinct objects. Followers of G. Frege equate a set with a property (set = Begriffsumfang). In the simplest case when properties are considered as a bivalent predicates, the family of all sentences of the type "x has a property P" is isomorphic with an algebra of all subsets $\{x|P(x)\}$ of a given set X. The both forms the so called Boolean lattice, and furthermore, they are isomorphic with a family of all characteristic functions $V : X \to \{0,1\}$ introduced in 1936 by De la Vallee-Poussin with the following operations:

$$V_A(x) \cup V_B(x) = \max(V_A(x), V_B(x))$$
$$V_A(x) \cap V_B(x) = \min(V_A(x), V_B(x)), \qquad (*)$$
$$V_{\bar{A}}(x) = 1 - V_A(x).$$

As early as 1940 H. Weyl has suggested to consider the grades of an extent to which an object x possesses a property under the study. These grades are interpreted as values of generalized predicate $W : X \to [0,1]$ defined on a set X. On a family of all such predicates H. Weyl has defined the operations by formulas $(*)$, generalizing in this way a notion of characteristic function. Such functions are known today as a fuzzy sets, and this seems to justify, at least partially, the title of this note. Besides this calculus (De Morgan algebra), H. Weyl has discussed some other formal structures, particularly this one of quantum logic originated by G. Birhoff and J. von Neumann in 1936, and intensively studied in present days also within the framework of fuzzy sets theory.

It is well known that Cantorian concept of the set was badly shaken by the antinomies, there are also well known three approaches to the problem of rebuilding the foundations of set theory proposed by L. E. J. Brouwer, B. Russell and E. Zermelo. Unfortunately less known is a system proposed by an eminent Polish logician S. Leśniewski. This system is called mereology, which has already been developed in 1916 (see [19]). It is not a logical theory because its terms and axioms cannot be deduced from the principles of logic, it is an adequate base not only for construction of contemporary mathematics but also for applications in other branches of sciences. This is mainly because in contrast to classical theory of sets, the term class in the mereology considered in collective sense "says absolutely nothing about whether or not certain objects exist in the universe. One can employ it to relate different object, physical or not, provided that one believes in the existence of these objects" (cf. [16]).

The problem of developing a formal system which might provide a more adequate explication of classes in their scientific use, i. e. as an actual or empirical classes has been resoled to a certain extent by A. Kaplan and H. Schott. In 1951 they published an article [4] presenting a calculus for empirical classes (CEC). Essentially, this calculus is identical with calculus of fuzzy sets thought of as mappings from a universe X to the unit interval. Membership function (indicator in CEC terminology) is defined as a composition of two mappings: one, from a given universe X into a set of properties, and the second, from properties space to the unit interval. Values of indicator determining an empirical class are called nominal probabilities, since they are supposed to be fixed only by convention. It is worth noticing that beside the basic operations (*) and the basic relations for empirical classes (fuzzy sets in today terminology) some "recent" results obtained in the theory of fuzzy sets are also considered in paper [4], for example a definition of quasi-null and quasi-universal classes, idealization of a given class, sharpening relations, and similarity or sameness relations among meaning of terms.

It is rather common opinion that vagueness is a property of language. Attempts to formalize this property surprisingly pertain chiefly not to language but to some domains beyond it.

Seemingly the only exception is an interesting work of T. Kubiński (cf. [5, 6]). T. Kubiński has discussed in greater details three semiotical points of view on vagueness: pragmatical, syntactical, and semantical. He has developed an axiomatic system called quasi-ontology as a deductive theory of imprecise names. Very roughly speaking this system is a modification of Leśniewski's ontology, i. e. this is theory of expressions "x is A", which admit also a vague terms.

Fuzzy sets are usually identified with generalized characteristic functions. It is worth noticing that generalized functions have been introduced by E. Szpilrajn as early as 1936, chiefly for enabling logical comparisons between certain questions of point sets and questions belonging to the general theory of sets [17]. Various generalizations have been considered also by H. Rasiowa and R. Sikorski mainly as a tools for interpretations of formal theories (see [13, 14]).

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