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THE ALGEBRA OF FIRST-ORDER FUZZY LOGIC

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In [13], Novák introduced first-order fuzzy logic and proved, among other things, the semantico-syntactical completeness of this logic. He also demonstrated that the algebra of his logic is a generalized residuated lattice. In this note we specify the algebraic structure of first-order fuzzy logic by proving that the examination of Novák's logic can be reduced to the examination of locally finite MV-algebras.

Ever since 1965 when Zadeh published his study 'Fuzzy Sets' [16] thousands of papers have been written in this topic. Most of these investigations are practical in nature and fuzziness is mostly regarded in them rather as a new technique than a deep mathematical theory. Since the foundations of fuzzy logic are not fully clear, this new approach has not yet gained unreserved approval among mathematicians. A refreshing exception in Novák's study [13], in which fuzzy logic is investigated as a special kind of non-classical mathematical logic. Novák first defines the semantics of fuzzy logic and then shows that this logic is axiomatizable. It is known in literature that the examination of classical logic can be reduced to the examination of Boolean algebras. Therefore the discussion of any new type of non-classical logic raises a question about the corresponding abstract algebra. Novák proves that the algebra of his logic is a generalized residuated lattice. Since this algebraic structure is quite general, it is relevant to ask whether one can specify the structure. In this note we demonstrate that Novák's logic can be reduced to the examinationof locally finite MV-algebras. For that purpose we briefly recollect some definitions and theorems (for details, cf. [13]).

Definition 1. Assume a lattice $\mathsf{L} = \langle \mathsf{L}, \wedge, \vee, \mathbf{1}, \mathbf{0}, \wedge, \vee \rangle$, where $\mathbf{1}$ and $\mathbf{0}$ are the maximal and minimal elements, respectively, be *complete*, that is to say, the infinite joins $\bigvee_{i\in I} a_i$ and meets $\bigwedge_{i\in I} a_i$ ($a_i \in \mathsf{L}$, $i \in I$) always exist and are in L. Let L be *infinitely distributive* in the sense that

$$\bigvee_{i\in I}(a_i \wedge b) = \left(\bigvee_{i\in I} a_i\right) \wedge b. \tag{1}$$

Let L be endowed by binary operations $\odot$ and $\rightarrow$ such that $\odot$ is isotone, associative and commutative and

$$a \odot b \leq c \quad \text{iff} \quad a \leq b \rightarrow c. \tag{2}$$

We say that $\mathsf{L} = \langle \mathsf{L}, \wedge, \vee, \odot, \rightarrow, \mathbf{1}, \mathbf{0}, \bigwedge, \bigvee \rangle$ is a *generalized residuated lattice.* A unary operation $\star$ on L is defined by $a^\star = a \rightarrow \mathbf{0},\ a \in \mathsf{L}$.

In Novák's system L *the truth values set,* is either the Lukasiewicz interval of reals where the binary operations $\odot$ and $\rightarrow$ are defined by

$$a \odot b = \max\{0, a + b - 1\}, \tag{3}$$
$$a \rightarrow b = \min\{1, 1 - a + b\}, \tag{4}$$

or L is a finite Lukasiewicz chain $\mathbf{0} = a_0 < \cdots < a_m = \mathbf{1}$ and

$$a_k \odot a_p = a_{\max\{0, k+p-m\}}, \tag{5}$$
$$a_k \rightarrow a_p = a_{\min\{m, m-k+p\}}, \qquad 0 \leq k,\ p \leq m. \tag{6}$$

The *language* of first-order fuzzy logic is that of classical logic (cf. [13]) endowed with *symbols of truth values* $\mathbf{a},\ a \in \mathsf{L}$. *Terms* and *formulae* are constructed in the usual way. The *set of well formed formulae* is defoned by $\mathbb{F}_J$. The following abbreviations of formulae are introduced $\neg\mathsf{A} = \mathsf{A} \Rightarrow \mathbf{0}$ *(negation)* $\mathsf{A} \cup \mathsf{B} = (\mathsf{A} \Rightarrow \mathsf{B}) \Rightarrow \mathsf{B}$ *(disjunction)*, $\mathsf{A} \cap \mathsf{B} = \neg(((\neg\mathsf{A}) \Rightarrow \neg(\mathsf{B})) \Rightarrow (\mathsf{B}))$ *(conjunction)*, $\mathsf{A} \,\&\, \mathsf{B} = \neg(\mathsf{A} \Rightarrow \neg\mathsf{B})$ *(bold conjunction)*, $\exists (x)\, \mathsf{A} = \neg(\forall x) \neg\mathsf{A}$ *(existential quantifier)*.

A *truth valuation* of formulae is a function $\mathcal{D} : \mathbb{F}_J \rightarrow \mathsf{L}$, which assigns a truth value to every formula as follows

$$\mathcal{D}(\mathbf{a}) = a, \qquad a \in \mathsf{L}, \tag{i}$$

$$\mathcal{D}\left(p(t_1, \ldots, t_n)\right) = p_D\left(\mathcal{D}(t_1), \ldots, (t_n)\right), \tag{ii}$$

where $\mathcal{D}(t_i)$ is an interpretation of the term t_i without variables, $i = 1, \ldots, n$,

$$\mathcal{D}(\mathsf{A} \Rightarrow \mathsf{B}) = \mathcal{D}(\mathsf{A}) \rightarrow \mathcal{D}(\mathsf{B}), \tag{iii}$$

provided that A and B are closed formulae.

$$\mathcal{D}(\forall x)\, \mathsf{A}(x)) = \bigwedge_{d \in D} \mathcal{D}\left(\mathsf{A}_x[\mathbf{d}]\right), \tag{iv}$$

where $\mathbf{d}$ is a name of the element d.

$$\mathcal{D}\left(\mathsf{A}(x_1, \ldots, x_n)\right) = \bigwedge_{d_i \in D} \mathcal{D}\left(\mathsf{A}_{x_1,\ldots,x_n}[\mathbf{d}_1, \ldots, \mathbf{d}_n]\right). \tag{v}$$

The fuzzy set of *semantic consequences* of the fuzzy set T is

$$(\mathcal{C}^{\text{sem}}\mathsf{T})\ \mathsf{A} = \bigwedge \left\{\mathcal{D}(\mathsf{A}) \mid \mathcal{D} \text{ a valuation},\ \mathsf{T}(\mathsf{B}) \leq \mathcal{D}(\mathsf{B}) \text{ for any } \mathsf{B} \text{ in } \mathbb{F}_J\right\}.$$

For $(\mathcal{C}^{\text{sem}}\mathsf{T})\,\mathsf{A} = a$ we write

$$\mathsf{T} \models_a \mathsf{A}.$$

The following are *sound rules of inference* in fuzzy logic

(i) Modus Ponents r_{MP}:

$$\left(\frac{\mathsf{A},\ \mathsf{A} \Rightarrow \mathsf{B}}{\mathsf{B}},\ \frac{a,\ b}{a \odot b}\right),$$

(ii) a-Lifting rule r_{Ra}:

$$\left(\frac{\mathsf{A}}{\mathsf{a} \Rightarrow \mathsf{A}},\ \frac{b}{a \to b}\right),$$

(iii) Generalization r_{G}:

$$\left(\frac{\mathsf{A}}{(\forall x)\,\mathsf{A}},\ \frac{a}{a}\right).$$

Similarly as in classical logic, certain form of formulae A (such that $\mathsf{T} \models_1 \mathsf{A}$) are the *logical axioms of fuzzy logic.* Other axioms are called *special axioms. A proof w* of a formula A in fuzzy logic is (essentially) a finite sequence of formulae such that each formula is an axiom or obtained by means of rules of inference from some previous formulae. The *value of a proof w* $\text{Val}_x(w)$ is the truth value of last formula in the proof. The *syntactic consequence* of the fuzzy set T is

$$(\mathcal{C}^{\text{syn}}\mathsf{T})\ \mathsf{A} = \vee\,\{\text{Val}_x(w)\,|\,w \text{ is a proof of } \mathsf{A} \text{ from } \mathsf{T}\}\,.$$

For $(\mathcal{C}^{\text{syn}}\mathsf{T})\,\mathsf{A} = a$ we write

$$\mathsf{T} \vdash_a \mathsf{A}.$$

Novák proved the following

Theorem 1. Let T be a consistent Henkin theory. Then

(a) $$\mathsf{A} \leq \mathsf{B} \quad \text{if and only if} \quad \mathsf{T} \vdash_1 \mathsf{A} \Rightarrow \mathsf{B}$$

is a preorder on $\mathbb{F}_J$ and a relation $\approx$, defined by

(b) $$\mathsf{A} \approx \mathsf{B} \quad \text{if and only if } \mathsf{A} \leq \mathsf{B} \quad \text{and} \quad \mathsf{B} \leq \mathsf{A}$$

is a congruence on $\mathbb{F}_J$.

(c) Define on the factor algebra $\mathbb{F}_J/\approx$

$$\begin{aligned} |\mathsf{A}| \leq_T |\mathsf{B}| &\quad \text{if and only if} \quad \mathsf{A} \leq \mathsf{B}, \\ |\mathbf{1}| = \mathbf{1}, &\quad |\mathbf{0}| = \mathbf{0}, \\ |\mathsf{A}| \wedge |\mathsf{B}| = |\mathsf{A} \cap \mathsf{B}|, &\quad |\mathsf{A}| \vee |\mathsf{B}| = |\mathsf{A} \cup \mathsf{B}|, \\ |\mathsf{A}| \odot |\mathsf{B}| = |\mathsf{A}\,\&\mathsf{B}|, &\quad |\mathsf{A}| \to |\mathsf{B}| = |\mathsf{A} \Rightarrow \mathsf{B}|, \\ \wedge_{t \in M_V} |\mathsf{A}_x[t]| = |\forall\,(x)\,\mathsf{A}|, &\quad \vee_{t \in M_V} |\mathsf{A}_x[t]| = |\exists\,(x)\,\mathsf{A}|. \end{aligned}$$

Then $\mathsf{L}(T) = \langle \mathbb{F}_J/\approx, \wedge, \vee, \odot, \to, \mathbf{1}, \mathbf{0}, \wedge, \vee \rangle$ is a generalized residuated lattice.

Theorem 2. Let T be a consistent Henkin theory. Then

$$T \vdash_a \mathsf{A} \quad \text{if and only if} \quad \mathsf{T} \models_a \mathsf{A}$$

holds true for every formula $\mathsf{A} \in F_J$, $a \in \mathsf{L}$.

Proposition 1. In Lukasiewicz lattices, it holds, for any a, $b \in \mathbf{L}$

$$(a \to \mathbf{0}) \to \mathbf{0} = a, \qquad \mathbf{1} \to a = a, \tag{7}$$

$$(a \to \mathbf{0}) \to b = (b \to \mathbf{0}) \to a, \tag{8}$$

$$a \to \mathbf{1} = \mathbf{1}, \tag{9}$$

$$\mathbf{0} \to \mathbf{0} = \mathbf{1}, \tag{10}$$

$$(a \odot b) \to \mathbf{0} = a \to (b \to \mathbf{0}), \tag{11}$$

$$(a \to b) \to b = (b \to a) \to a. \tag{12}$$

Moreover, any $a \neq \mathbf{0}$, $\mathbf{1}$ is *nilpotent*, i. e., there exists $n \in \mathbf{N}$ such that

$$a^n = a \odot \cdots \odot a = \mathbf{0} \quad (n \text{ times}). \tag{13}$$

Proof. Direct substitution for (3) and (4) or (5) and (6) in (7)–(13). □

Definition 2. An *MV-algebra* $\mathsf{A} = \langle \mathsf{A}, \oplus, \odot, \star, \mathbf{1}, \mathbf{0} \rangle$ is an abstract algebra such that $\langle \mathsf{A}, \oplus, \mathbf{0} \rangle$ is an Abelian monoid, $x \oplus \mathbf{1} = \mathbf{1}$, $x^{\star\star} = x$, $\mathbf{0}^\star = \mathbf{1}$, $x \odot y = (x^\star \oplus y^\star)$, $(x^\star \oplus y)^\star \oplus y = (y^\star \oplus x)^\star \oplus x$ (cf. [11]), where $\mathbf{0}$ and $\mathbf{1}$ are the minimal and maximal elements, respectively, in A.

MV-algebras has been studied in [1, 2, 3, 4, 7, 8, 9, 10]. Defining $x \vee y = (x \odot y^\star) \oplus y$ and $x \wedge y = (x \oplus y^\star) \odot y$ for any $x, y \in \mathsf{A}$ we have that $\langle \mathsf{A}, \leq, \wedge, \vee \rangle$ is a distributive lattice under the partial order $x \leq y$ iff $x \wedge y = x$. By [4], in any MV-algebra A there holds $x \leq y$ iff $x^\star \oplus y = \mathbf{1}$.

Proposition 2. Let A be a complete MV-algebra. Define $a \to b = a^\star \oplus b$ for a, $b \in \mathsf{A}$. Then $\mathsf{A} = \langle \mathsf{A}, \leq, \wedge, \vee, \odot, \to, \mathbf{1}, \mathbf{0}, \wedge, \vee \rangle$ is a generalized residuated lattice.

Proof. Since the binary operation is $\odot$ isotone, associative and commutative (cf. [4]), the only thing to demonstrate are the conditions (1) and (2). Indeed,

$$a \odot b \leq c \quad \text{iff} \quad (a \odot b)^\star \oplus c = \mathbf{1} \quad \text{iff} \quad a^\star \oplus (b^\star \oplus c) = \mathbf{1} \quad \text{iff} \quad a \leq b^\star \oplus c,$$

so (2) holds.

Because A is distributive and complete, (1) holds. □

Remark 1. The Galois correspondence (2) in Proposition 2 is established also in [6], where it is proved by means of so called Wajsberg algebras.

Remark 2. As Proposition 2 states, every MV-algebra can be regarded as a residuated lattice, but not vice versa. For example any Bouwerian lattice can be viewed as a residuated lattice but not as an MV-algebra, since there the condition $x^{**} = x$ does not always hold.

Definition 3. An MV-algebra A is *locally finite* if, for any $a \in \mathsf{A}$ $(a \neq 0, 1)$, there exists $n \in \mathbf{N}$ such that

$$na = a \oplus \cdots \oplus a = \mathbf{1} \quad \text{or equally,} \quad a^n = a \odot \cdots \odot a = \mathbf{0} \quad (n \text{ times}).$$

Remark 3. As is well known, locally finite MV-algebras are the subalgebras of the Lukasiewicz lattices.

Proposition 3. (Assumptions like in Theorem 1). Define

$$|\mathsf{A}|^* = |\mathsf{A} \Rightarrow \mathbf{0}|, \qquad |\mathsf{A}| \oplus |\mathsf{B}| = |\neg\mathsf{A} \Rightarrow \mathsf{B}|.$$

Then $\mathsf{L}'(\mathsf{T}) = \langle F_J/\approx, \oplus, \odot, \star, \mathbf{1}, \mathbf{0}\rangle$ is a locally finite MV-algebra.

Proof. Let $|\mathsf{A}|$, $|\mathsf{B}| \in F_J/\approx$. We demonstrate the last equation of Definition 2.

$$(|\mathsf{A}|^* \oplus |\mathsf{B}|)^* \oplus |\mathsf{B}| = (|\mathsf{B}|^* \oplus |\mathsf{A}|)^* \oplus |\mathsf{A}| \tag{14}$$

if and only if

$$(|\mathsf{A}|^* \oplus |\mathsf{B}|)^* \oplus |\mathsf{B}| \leq_{\mathsf{T}} (|\mathsf{B}|^* \oplus |\mathsf{A}|)^* \oplus |\mathsf{A}| \tag{15}$$

and

$$(|\mathsf{B}|^* \oplus |\mathsf{A}|)^* \oplus |\mathsf{B}| \leq_{\mathsf{T}} (|\mathsf{A}|^* \oplus |\mathsf{B}|)^* \oplus |\mathsf{A}|. \tag{16}$$

Now, (15) holds if and only if

$$|(\mathsf{A} \Rightarrow \mathsf{B}) \Rightarrow \mathsf{B}| \leq_{\mathsf{T}} |(\mathsf{B} \Rightarrow \mathsf{A}) \Rightarrow \mathsf{A}|$$

if and only if

$$\mathsf{T} \vdash_1 [(\mathsf{A} \Rightarrow \mathsf{B}) \Rightarrow \mathsf{B}] \Rightarrow [(\mathsf{B} \Rightarrow \mathsf{A}) \Rightarrow \mathsf{A}]$$

if and only if

$$\mathsf{T} \models_1 [(\mathsf{A} \Rightarrow \mathsf{B}) \Rightarrow \mathsf{B}] \Rightarrow [(\mathsf{B} \Rightarrow \mathsf{A}) \Rightarrow \mathsf{A}]$$

if and only if

$$\mathcal{D}([(\mathsf{A} \Rightarrow \mathsf{B}) \Rightarrow \mathsf{B}] \Rightarrow [(\mathsf{B} \Rightarrow \mathsf{A}) \Rightarrow]) = \mathbf{1} \qquad \text{for any } \mathcal{D}$$

if and only if

$$(\mathcal{D}(\mathsf{A}) \to \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}) \leq (\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathsf{A})) \to \mathcal{D}(\mathsf{A}) \quad \text{for any } \mathcal{D},$$

which is true by (12). We conclude that (15) holds. In a similar manner one establishes (16). So (14) holds. The rest of Definition 2 can be demonstrated equally. This completes the proof. □

Concluding remark. A very interesting open question concerns the generality of first-order fuzzy logic, i. e., can one substitute a more general algebraic structure than Lukasiewicz interval for the truth-value set Ł. A generalized residuated lattice is too general, since the rule r_{Ra} is not a rule of inference in a pseudo-Boolean algebra. What about substituting complete MV-algebra for the Lukasiewicz structure?

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