KYBERNETIKA — VOLUME 28 (1992), NUMBER 5, PAGES 337–356

MULTIPLICATION OF FUZZY QUANTITIES

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The addition operation over the class of fuzzy numbers or fuzzy quantities was investigated and discussed e. g. in [1], [8] or [7]. It is easy to define in an analogous way also the operation of multiplication (cf. [1] or in certain sense also [3] and [4]). Moreover, some of the methods and concepts suggested for the addition case in [5] and further used in [6] and [7] can be evidently adapted to the multiplication. In this way the group axioms and some other useful algebraical properties of multiplication can be derived also for fuzzy quantities, at least for some of them and up to certain degree of similarity between them. The specific properties of the multiplication mean that the methods derived for the addition cannot be mechanically transmitted to the multiplicative case, and that rather different approach must be used. The main purpose of this paper is to show these differences and their consequences for the obtained results.

0. INTRODUCTION

Numerous problems concerning e. g. optimal decision-making, network analysis or planning complex activities are connected with uncertain or vague numerical data. These data, often represented by fuzzy numbers or more generally by fuzzy quantities, must be usually arithmetically handled at least on the level of elementary operations. It is well known (cf. [1], [5] or [6]) that some of the useful properties fulfilled for the crisp numbers fail in case of the fuzzy ones. It concerns also the existence of inverse elements and the distributivity rule.

However, it was possible to prove the validity of some of these properties, namely the existence of the additive inverse element; and in a special case also the distributivity of crisp-fuzzy product, up to certain type of equivalence between fuzzy quantities (cf. [5], [6], [7], [8]). It is evident that an analogous way can be used in the case of multiplication if the equivalence is rather modified. The purpose of the presented paper is to describe this multiplication and multiplicative equivalence, and to show their properties.

As the methods and many results described below are closely analogous to those ones presented e. g. in [5] or [6] for the additive case, their presentation here is often abbreviated and focused to the concepts which do essentially differ from the additive version. This approach led to certain variety of subjects explained and discussed in the following sections. The operation of multiplication over fuzzy quantities is rather more complicated than the addition, and the corresponding structures describing its properties are necessarily more complex and also more specialized. This fact was one of the principal arguments for writing the presented article instead of simple referring the analogy with the known results for the additive case.

1. NORMAL FUZZY QUANTITIES

In the whole paper we denote by R the set of all real numbers and by $R_0 = R - \{0\}$ the set of all non-zero real numbers. By *normal fuzzy quantity (n. f. q.)* we call any fuzzy subset a of R with membership function $f_a : R \to [0,1]$ such that

$$\begin{aligned} &\sup\,(f_a(x) : x \in R) = 1, \\ &\exists\, x_1 < x_2 \in R,\ \forall\, x :\ (x > x_2) \text{ or } (x < x_1),\quad f_a(x) \Rightarrow 0. \end{aligned} \tag{1}$$

The set of all normal fuzzy quantities fulfilling (1) is denoted by $\mathbb{R}$. The special position of 0 among real numbers concerning the multiplication has to be respected also if the multiplication of fuzzy quantities is considered. Due to [1] we often assume for an n. f. q. a also

$$f_a(0) = 0. \tag{2}$$

The set of n. f. q. fulfilling (1) and (2) is denoted by $\mathbb{R}_0 \subset \mathbb{R}$.

The first one of conditions (1) is not quite necessary and its absence can be treated analogously to the procedure used in [5] for the additive case. The second condition of (1) will be essentially used in Section 4.1 (in Theorem 6) and in this sense its acceptance is more significant. However, both conditions (1) can be considered for natural and realistic, and moreover they mean an important simplification of the formalism used below. The connection between (2) and the properties of multiplication over real numbers (R is not the multiplicative group, e. g.) is mentioned above as well as in [1].

In the following sections we use the strict equality relation between n. f. q. If $a,\, b \in \mathbb{R}$ then we write $a = b$ iff $f_a(x) = f_b(x)$ for all $x \in R$. This approach does not reflect the naturally vague relations between fuzzy quantities. It is only a simplified notation for certain very strong connection between membership functions. A weaker similarity concept was suggested in [5] as an (additive) equivalence (cf. also [6], [7] and [8]), and its analogy suitable for the representation of multiplicative similarity is suggested below in Section 4. In general, the concept of fuzzy equality relation between fuzzy quantities can be approached in more ways which do not concern the topic of this paper and are not mentioned here at all.

1.1. Multiplication

Definition 1. If $a,\, b \in \mathbb{R}_0$ are normal fuzzy quantities with membership functions f_a, f_b, respectively, then the n. f. q. $a \odot b \in \mathbb{R}$ with membership function $f_{a \odot b}$ defined by

$$f_{a\odot b}(x) = \sup_{y \in R_0}\,(\min\,(f_a(y),\, f_b(x/y))) \tag{3}$$

is called the product of a and b. To distinguish the multiplication over real numbers and over n. f. q. we denote $x \cdot y$ for $x, y \in R$ and $a \odot b$ for $a, b \in \mathbb{R}_0$.

Remark 1. Relation (1) immediately implies that for $a, b \in \mathbb{R}_0$ also $a \odot b \in \mathbb{R}_0$, and that

$$f_{a\odot b}(x) = \sup_{y \in R_0} \left(\min \left(f_a(x/y),\, f_b(y) \right) \right). \tag{4}$$

Lemma 1. If $a, b \in \mathbb{R}_0$ then $a \odot b = b \odot a$.

Proof. The commutativity follows from (3) and (4) immediately. □

Lemma 2. If $a, b, c \in \mathbb{R}_0$ then

$$(a \odot b) \odot c = a \odot (b \odot c).$$

Proof. If $a, b, c \in \mathbb{R}_0$ then

$$\begin{aligned}
f_{(a\odot b)\odot c}(u) &= \sup_{v \neq 0} \left(\min \left(f_{a\odot b}(v),\, f_c(u/v) \right) \right) = \\
&= \sup_{v \neq 0} \left(\min \left(\sup_{x \neq 0} \left(\min \left(f_a(x),\, f_b(v/x) \right) \right),\, f_c(u/v) \right) \right) = \\
&= \sup_{v \neq 0} \left(\sup_{x \neq 0} \left(\min \left(f_a(x),\, f_b(v/x) \right),\, f_c(u/v) \right) \right) = \\
&= \sup_{x \neq 0} \left(\sup_{v \neq 0} \left(\min \left(f_b(v/x),\, f_c(u/v) \right),\, f_a(x) \right) \right) = \\
&= \sup_{x \neq 0} \left(\min \left(f_a(x),\, \sup_{v \neq 0} \left(\min \left(f_b(v/x),\, f_c(u/v) \right) \right) \right) \right) = \\
&= \sup_{x \neq 0} \left(\min \left(f_a(x),\, \sup_{y \neq 0} \left(\min \left(f_b(y),\, f_c(u/(x \cdot y)) \right) \right) \right) \right) = \\
&= \sup_{x \neq 0} \left(\min \left(f_a(x),\, f_{b\odot c}(u/x) \right) \right) = f_{a\odot(b\odot c)}(u).
\end{aligned}$$

If $y \in R$ is a real number, then we denote by $\langle y \rangle$ the n. f. q. with membership function $f_{\langle y \rangle}$ defined by

$$\begin{aligned}
f_{\langle y \rangle} &= 1 \quad \text{for } x = y \\
&= 0 \quad \text{for } x \neq y.
\end{aligned} \tag{5}$$

□

Lemma 3. If $a \in \mathbb{R}_0$ then $\langle 1 \rangle \odot a = a$.

Proof. By (4)

$$f_{\langle 1 \rangle \odot a}(x) = \sup_{y \neq 0} \left(\min \left(f_{\langle 1 \rangle}(y),\, f_a(x/y) \right) \right) = f_a(x/1) = f_a(x).$$

□

Theorem 1. The set $\mathbb{R}_0$ of normal fuzzy quantities fulfilling (1) and (2) is a commutative monoid according to the multiplication relation (3).

Proof. The statement follows from Lemmas 1, 2 and 3 immediately. □

Corollary. The previous theorem implies that $\mathbb{R}_0$ is a commutative semigroup.

If $a \in \mathbb{R}_0$ then we denote by $1/a$ the n. f. q. for which

$$\begin{aligned} f_{1/a}(x) &= f_a(1/x) \quad \text{for all } x \in R,\ x \neq 0, \\ f_{1/a}(0) &= 0. \end{aligned} \tag{6}$$

It is not difficult to verify that generally $a \odot (1/a)$ is not $\langle 1 \rangle$, as shown in the following simple example.

Example 1. Let $a \in \mathbb{R}_0$, $f_a(1) = 1$, $f_a(2) = 1$, $f_a(x) = 0$ for $1 \neq x \neq 2$. Then also $(1/a) \in \mathbb{R}_0$ and

$$f_{1/a}(1) = 1 = f_{1/a}(1/2), \quad f_{1/a}(x) = 0 \quad \text{for} \quad 1/2 \neq x \neq 1.$$

Hence

$$f_{(1/a)\odot a}(1) = f_{(1/a)\odot a}(1/2) = f_{(1/a)\odot a}(2) = 1, \quad f_{(1/a)\odot a}(x) = 0, \quad x \neq \{1/2,\ 1,\ 2\}.$$

bigskip

The previous fact shows that $\mathbb{R}_0$ cannot be a multiplicative group. An analogous problem appeared in the additive case where it could be solved by substituting certain type of equivalence for the equality, as suggested in [5]. The multiplicative case, however more complicated, can be treated in rather similar way, presented and discussed in Section 4.

1.2. Crisp Product

It is useful in numerous practical models of uncertainty to multiply an f. f. q. by crisp (i. e. deterministic) real number.

Definition 2. Let $a \in \mathbb{R}$ and $r \in R$. The normal fuzzy quantity $r \cdot a$ with the membership function

$$\begin{aligned} f_{r\cdot a}(x) &= f_a(x/r) \quad \text{for } r \neq 0, \\ &= f_{\langle 0 \rangle}(x) \quad \text{for } r = 0,\ x \in R, \end{aligned} \tag{7}$$

is called the crisp product of r and a.

Even if we, for practical reasons, distinguish between the product of two fuzzy quantities $\odot$ and the crisp product, both operations coincide.

Remark 2. Comparing Definitions 1 and 2 it is easy to verify that for $r \in R_0$ and $a \in \mathbb{R}_0$

$$r \cdot a = \langle r \rangle \odot a.$$

Remark 3. Evidently for $r \neq 0$ and $a \in \mathbb{R}_0$ also $r \cdot a \in \mathbb{R}_0$.

Remark 4. Definition 2 immediately implies that for $r, r' \in R$, $a \in \mathbb{R}$, the equality $r \cdot (r' \cdot a) = (r \cdot r') \cdot a$ holds.

1.3. Addition

Even if the addition of n. f. q. is investigated in other papers it is worth mentioning it here.

Definition 3. If $a, b \in \mathbb{R}$ are normal fuzzy quantities then the n. f. q. $a \oplus b \in \mathbb{R}$ with membership function defined by

$$f_{a \oplus b}(x) = \sup_y \left(\min \left(f_a(y), f_b(x-y) \right) \right), \quad x \in R, \tag{8}$$

is called the sum of a and b.

The properties of the addition operation $\oplus$ are described e. g. in [1], [5], [6] or [8]. Here we remember its connection with the distributivity of the crisp product.

Lemma 4. If $a, b \in \mathbb{R}$ and $r \in R$ then $r \cdot (a \oplus b) = (r \cdot a) \oplus (r \cdot b)$.

Proof. The statement follows from (7) and (8) immediately. □

The opposite distributivity law, $(r_1 + r_2) \cdot a = r_1 a \oplus r_2 a$ for $r_1, r_2 \in R$, $a \in \mathbb{R}$, does not generally hold, as shown e. g. in [1] or [5]. A way how to guarantee its validity in a weaker form for at least certain class of n. f. q. is suggested in [7]. This class is formed by the symmetric n. f. q. specified in the following subsection, and the weaker form of distributivity means that the equality in the distributivity formula is substituted by an equivalence relation "up to fuzzy zero" (cf. [7]).

However, if we consider an n. f. q. $a \in \mathbb{R}$ then generally

$$a + a \neq 2 \cdot a$$

and this fact provokes some considerations. Loosing the exactness of crisp numbers, we inevitably loose some of their pleasant properties, even such which we used to accept for being selfevident. We may also ask if really the repetitive addition of two (or n) numbers is arithmetically exactly the same like the multiplication of the same number by a coefficient which can be arbitrary (including non-integer values) and which only in this case is equal to 2 (or n), i. e. to the number of the repetitions of the considered quantity in the addition. The coincidence of both operations, selfevident for crisp numbers, can vanish if vague (fuzzy) quantities are considered.[1]

Essential results concerning the interconnection and distributivity between the operations of addition $\oplus$ and multiplication $\odot$ are summarized in [1].

1.4. Symmetry

Symmetric n. f. q. especially dealt in [7] and partly used also in some other papers concerning the addition of n. f. q. can be interesting also for the multiplicative case.

Definition 4. If $y \in R$ and $a \in \mathbb{R}$ then we say that a is y-symmetric iff for all $x \in R$

$$f_a(y + x) = f_a(y - x). \tag{9}$$

The set of all y-symmetric n. f. q. will be denoted by $\mathbb{S}_y$, the union of these sets is denoted by $\mathbb{S}$,

$$\mathbb{S} = \bigcup_{y \in R} \mathbb{S}_y. \tag{10}$$

If we denote for $a \in \mathbb{R}$ the n. f. q. $(-a)$ where

$$f_{-a}(x) = f_a(-x), \quad \text{for all } x \in R, \tag{11}$$

then evidently $a \in \mathbb{S}_0$ iff $a = (-a)$.

Remark 5. It follows from (8) and (9) immediately (cf. [5] or [6]) that for any $a \in \mathbb{R}$

$$a + (-a) \in \mathbb{S}_0. \tag{12}$$

Remark 6. If $r \in R$ and $a \in \mathbb{R}$ then $r \cdot (-a) = (-r) \cdot a = -(r \cdot a)$ as follows from (7), (9) and (11) immediately.

Remark 7. It can be easily seen (cf. [7]) that for any $a \in \mathbb{S}_y$, $y \in R$, there exists $s \in \mathbb{S}_0$ such that $a = \langle y \rangle \oplus s$.

Lemma 4. If $a \in \mathbb{R}_0$ and $s \in \mathbb{S}_0 \cap \mathbb{R}_0$ then $a \odot s \in \mathbb{S}_0 \cap \mathbb{R}_0$.

P r o o f. Preserving the notation used in the statement,

$$\begin{aligned} f_{a \odot s}(x) &= \sup_{y \neq 0} (\min (f_a(y), f_s(x/y))) = \\ &= \sup_{y \neq 0} (\min (f_a(y), f_s(-x/y))) = f_{a \odot s}(-x) \end{aligned}$$

for all $x \in R$. □

[1] The author thanks Dr. Kamila Bendová from the Institute of Mathematics in Prague for this idea which in its essence offers new view on some traditional certainties of numerical calculations.

Lemma 5. If $a \in \mathbb{R}_0$ then $a \odot a = (-a \odot (-a))$.

Proof. For all $x \in R$,

$$\begin{aligned} f_{a\odot(-a)}(x) &= \sup_{y\neq 0}(\min(f_a(y), f_{-a}(x/y))) = \\ &= \sup_{y\neq 0}(\min(f_a(y), f_a(-x/y))) = f_{a\odot a}(-x). \end{aligned}$$

□

Remark 8. If $a \in \mathbb{S}_0$ then also $(1/a) \in \mathbb{S}_0$ as follows from (6) and (9).

2. SIGNED NORMAL FUZZY QUANTITIES

In the case of multiplication over n. f. q. the fact if their supports belong to exactly one (positive or negative) semiaxis plays a significant role.

Definition 5. Let $a \in \mathbb{R}_0$ be an f. f. q. We say that a is positive iff $f_a(x) = 0$ for all $x \leq 0$, and that a is negative iff $f_a(x) = 0$ for all $x \geq 0$. The sets of all positive or negative n. f. q. will be denoted by $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively. Fuzzy quantities from $R^* = \mathbb{R}^+ \cup \mathbb{R}^- \subset \mathbb{R}_0$ will be called signed.

Lemma 6. If $a, b \in \mathbb{R}^+$, $r_1, r_2 \in R$, $r_1 < 0 < r_2$, then $a \odot b \in \mathbb{R}^+$, $r_1 \cdot a \in \mathbb{R}^-$, $r_2 \cdot a \in \mathbb{R}^+$, $a \oplus b \in \mathbb{R}^+$, $(1/a) \in \mathbb{R}^+$.

Proof. Relations $r_1 \cdot a \in \mathbb{R}^-$, $r_2 \cdot a \in \mathbb{R}^+$ and $(1/a) \in \mathbb{R}^+$ follow from (7), (6) and Definition 5 immediately. Let us choose an arbitrary $x \leq 0$. Then by (3) $f_{a\odot b}(x) > 0$ iff both, $f_a(y)$ and $f_b(x/y)$, are positive for some $y \in R$, $y \neq 0$. It is impossible as either $y > 0$ and $x/y < 0$ or vice versa for any such y and $a, b \in \mathbb{R}^+$. Analogously $f_{a\oplus b}(x) > 0$ iff both, $f_a(y)$ and $f_b(x - y)$, are positive for some $y \in R$, as follows from (8). □

Lemma 7. If $a, b \in \mathbb{R}^-$, $r_1, r_2 \in R$, $r_1 < 0 < r_2$, then $a \odot b \in \mathbb{R}^+$, $r_1 \cdot a \in \mathbb{R}^+$, $r_2 \cdot a \in \mathbb{R}^-$, $a \oplus b \in \mathbb{R}^-$, $(1/a) \in \mathbb{R}^-$.

Proof. The proof of this statement is completely analogous to the one of Lemma 6. □

Lemma 8. Let $a \in \mathbb{R}^+$, $b \in \mathbb{R}^-$ then $a \odot b \in \mathbb{R}^-$.

Proof. Also this statement can be proved analogously to the proof of the corresponding statement of Lemma 6. If $x \geq 0$ and $y > 0$ then x/y cannot be negative which means that $f_{a\odot b}(x)$ cannot be positive as follows from (3). □

Remark 9. If $a \in \mathbb{R}_0 - \mathbb{R}^*$ then obviously $(-a) \in \mathbb{R}_0 - \mathbb{R}^*$, $(1/a) \in \mathbb{R}_0 - \mathbb{R}^*$ and $r \cdot a \in \mathbb{R}_0 - \mathbb{R}^*$ for any $r \in R_0$.

3. TRANSVERSIBILITY

The concept of symmetry (9) useful for the investigation of addition over $\mathbb{R}$ has its multiplicative analogy. Similarly to the procedure suggested in [5] to guarantee the additive group properties of $\mathbb{R}$ at least up to certain equivalence based on the 0-symmetry, we will use in the next sections the notion of transversibility and especially 1-transversibility to derive a weaker form of multiplicative group properties for $\mathbb{R}_0$. Here we present some auxiliary results concerning this concept.

Definition 6. If $y \in R_0$ and $a \in \mathbb{R}_0$ then we say that a is y-transversible iff

$$f_a(y \cdot x) = f_a(y/x) \quad \text{for } x > 0, \qquad f_a(y \cdot x) = 0 \quad \text{for } x \leq 0. \tag{13}$$

The set of all y-transversible n. f. q. is denoted by $\mathbb{T}_y$. By $\mathbb{T}$ we denote the union

$$\mathbb{T} = \bigcup_{y \in R_0} \mathbb{T}_y. \tag{14}$$

Remark 10. If $y \in R_0$ then $\langle y \rangle \in \mathbb{T}_y$. Hence $\mathbb{T}_y \cap \mathbb{R}^* \neq \emptyset$.

Remark 11. Equality (13) immediately implies that $\mathbb{T}_y \subset \mathbb{R}^+$ for $y > 0$ and $\mathbb{T}_y \subset \mathbb{R}^-$ for $y < 0$.

Remark 12. The second one of conditions (1) immediately implies that for every n. f. q. $t \in \mathbb{T}$ there exists $\varepsilon > 0$ and an ε-neighborhood of 0, $\mathcal{U}(0, \varepsilon) < R$, such that $f_t(x) = 0$ for $x \in \mathcal{U}(0, \varepsilon)$.

Lemma 9. Let $y \in R_0$, $a \in \mathbb{R}_0$. Then $a \in \mathbb{T}_y$ iff there exists $t \in \mathbb{T}_1$ such that $a = t \odot \langle y \rangle$.

Proof. If $a = t \odot \langle y \rangle$ for $t \in \mathbb{T}_1$ then for $x \in R$

$$\begin{aligned} f_a(y \cdot x) &= f_{t \odot \langle y \rangle}(y \cdot x) = \sup_{z \neq 0} \left(\min\left(f_t(y \cdot x/z),\, f_{\langle y \rangle}(z)\right)\right) = f_t(x), \\ f_a(y/x) &= f_{t \odot \langle y \rangle}(y/x) = \sup_{z \neq 0} \left(\min\left(f_t(y/(x \cdot z)),\, f_{\langle y \rangle}(z)\right)\right) = f_t(1/x) = f_t(x), \end{aligned}$$

and $a \in \mathbb{T}_y$. Let, on the other hand, $a \in \mathbb{T}_y$. Then

$$a = a \odot \langle 1 \rangle = a \odot (\langle y \rangle \odot \langle 1/y \rangle) = (a \odot \langle 1/y \rangle) \odot \langle y \rangle$$

as by (5) and (4) $\langle y \rangle \odot \langle 1/y \rangle = \langle 1 \rangle$, $y \neq 0$. It is sufficient, now, to show that $a \odot \langle 1/y \rangle \in \mathbb{T}_1$. For $x \in R$

$$\begin{aligned} f_{a \odot \langle 1/y \rangle}(x) &= \sup_{z \neq 0} \left(\min\left(f_a(x/z),\, f_{\langle 1/y \rangle}(z)\right)\right) = f_a(x \cdot y), \\ f_{a \odot \langle 1/y \rangle}(1/x) &= \sup_{z \neq 0} \left(\min\left(f_a(1/(x \cdot z)),\, f_{\langle 1/y \rangle}(z)\right)\right) = f_a(y/x) = f_a(y \cdot x), \end{aligned}$$

and $t = a \odot \langle 1/y \rangle \in \mathbb{T}_1$. □

Lemma 10. If $a,\ b \in \mathbb{T}_1$ then $a \odot b \in \mathbb{T}_1$.

Proof. For $x \in R,\ x > 0$

$$\begin{aligned} f_{a\odot b}(x) &= \sup_{y\neq 0}(\min(f_a(y),\ f_b(x/y))) = \sup_{y\neq 0}(\min(f_a(1/y),\ f_b(y/x))) = \\ &= \sup_{z\neq 0}(\min(f_a(z),\ f_b(1/(x\cdot z))) = f_{a\odot b}(1/x), \end{aligned}$$

where $z = 1/y$ was substituted. For $x < 0$ $f_{a\odot b}(x) = f_{a\odot b}(1/x) = 0$ by Lemma 6 and Remark 10. □

Theorem 2. If $x,\ y \in R_0$, and if $a \in \mathbb{T}_x,\ b \in \mathbb{T}_y$, then $a \odot b \in \mathbb{T}_{x\cdot y}$.

Proof. Let $a \in \mathbb{T}_x,\ b \in \mathbb{T}_y$. Then by Lemma 9 $a = t_1 \odot \langle x\rangle$, $b \in t_2 \odot \langle y\rangle$ for $t_1,\ t_2 \in \mathbb{T}_1$, and

$$a \odot b = t_1 \odot t_2 \odot \langle x\rangle \odot \langle y\rangle = t \odot \langle x \cdot y\rangle \in T_{x\cdot y}$$

as follows from Lemma 10, Lemma 9 and from the fact that by (4) and (5) $\langle x\rangle \odot \langle y\rangle = \langle x \cdot y\rangle$. □

Corollary. The class $\mathbb{T}$ of transversible n. f. q. is a closed set regarding the multiplication $\odot$.

The preceding statements imply a few relations concerning the algebraic structure of the set $\mathbb{T}$ namely if the multiplication operation over $\mathbb{T}$ is considered.

Remark 13. Let $x,\ y,\ z \in R_0,\ \mathbb{T}_x,\ \mathbb{T}_y,\ \mathbb{T}_z \subset \mathbb{T}$. If $a,\ b,\ c,\ e$ are arbitrary n. f. q. from $\mathbb{R}_0$ such that $a \in \mathbb{T}_x,\ b \in \mathbb{T}_y,\ c \in \mathbb{T}_z,\ e \in \mathbb{T}_1$, then

$$a \odot b = b \odot a \in \mathbb{T}_{x\cdot y}, \tag{15}$$

$$(a \odot b) \odot c = a \odot (b \odot c) \in \mathbb{T}_{x\cdot y\cdot z}, \tag{16}$$

$$a \odot e \in \mathbb{T}_a, \tag{17}$$

$$a \odot (1/a) \in \mathbb{T}_1, \tag{18}$$

and if $s \in \mathbb{R}_0 \cap \mathbb{S}_0$ then also $a \odot s \in \mathbb{R}_0 \cap \mathbb{S}_0$.

The relations summarized in Remark 13 are remarkably similar to the commutative group properties which fact can be rather generalized and used to introduce a weaker form of group including as many n. f. q. from $\mathbb{R}_0$ as possible.

4. MULTIPLICATIVE GROUP

We have already introduced the auxiliary concepts and results which enable us to formulate the weaker form of multiplicative group properties valid for $\mathbb{R}^*$, analogously to the procedure used in [5] for the additive case.

4.1. Multiplicative equivalence

In this section we suggest certain concept of similarity between n. f. q. up to "multiplicative fuzzy 1".

Definition 7. Let a, $b \in \mathbb{R}^*$ be signed normal fuzzy quantities. We say that a is equivalent (or multiplicative-equivalent) to b and write $a \sim_{\odot} b$ iff there exist 1-transversible n. f. q. r, $t \in \mathbb{T}_1$ such that

$$r \odot a = t \odot b. \tag{19}$$

Theorem 3. Relation $\sim_{\odot}$ defined above is reflexive, symmetrical and transitive.

Proof. If $a \in \mathbb{R}^*$ then for every $t \in \mathbb{T}_1$ $a \odot t = a \odot t$, and the symmetry of (19) evidently implies the symmetry of $\sim_{\odot}$. Let a, b, $c \in \mathbb{R}^*$ and t_1, t_2, t_3, $t_4 \in \mathbb{T}_1$ be such that

$$a \odot t_1 = b \odot t_2, \qquad b \odot t_3 = c \odot t_4.$$

Then by Theorem 2 also $t_1 \odot t_3 \in \mathbb{T}_1$, $t_2 \odot t_3 \in \mathbb{T}_1$, $t_2 \odot t_4 \in \mathbb{T}_1$, and

$$a \odot t_1 \odot t_3 = b \odot t_2 \odot t_3 = c \odot t_2 \odot t_4$$

which means that $a \sim_{\odot} c$ if $a \sim_{\odot} b$ and $b \sim_{\odot} c$. □

Remark 14. If a, $b \in \mathbb{R}^*$, $t \in \mathbb{T}_1$, and if $a \odot t = b$ then $a \sim_{\odot} b$ as $b = b \odot \langle 1 \rangle$ and $\langle 1 \rangle \in \mathbb{T}_1$.

Lemma 11. If $a \in \mathbb{R}^*$ then $a \odot (1/a) \in \mathbb{T}_1$.

Proof. Lemmas 6 and 7 imply that $a \odot (1/a) \in \mathbb{R}^+$ for any $a \in \mathbb{R}^*$. For $x > 0$

$$\begin{aligned} f_{a \odot (1/a)}(x) &= \sup_{y \neq 0} \left(\min\left(f_a(y), f_{(1/a)}(x/y)\right)\right) = \\ &= \sup_{y \neq 0} \left(\min\left(f_{(1/a)}(1/y), f_a(y/x)\right)\right) = \\ &= \sup_{z \neq 0} \left(\min\left(f_{(1/a)}(z), f_a(1/(x \cdot z))\right)\right) = f_{a \odot (1/a)}(1/x). \end{aligned}$$ □

Theorem 4. If a, $b \in \mathbb{R}^*$ and $a \odot (1/b) \in \mathbb{T}_1$, then $a \sim_{\odot} b$.

Proof. If $a \odot (1/b) = t \in \mathbb{T}_1$ then also

$$a \odot (1/b) \odot b = t \odot b$$

and by Lemma 11 $(1/b) \odot b = r \in \mathbb{T}_1$. It means that $a \odot r = b \odot t$ and $a \sim_{\odot} b$. □

Theorem 5. Let $a, b, c, d \in \mathbb{R}^*$ be signed n. f. q. If $a \sim_\odot b$ and $c \sim_\odot d$ then also $a \odot c \sim_\odot b \odot d$.

Proof. Equality (3) means that that for n. f. q. $a_1, a_2, a_0 \in \mathbb{R}$ the equality $a_1 = a_2$ implies $a_1 \odot a_0 = a_2 \odot a_0$. If $a \sim_\odot b$ and $c \sim_\odot d$ then for some $t_1, t_2, z_1, z_2 \in \mathbb{T}_1$ $a \odot t_1 = b \odot z_1$, $c \odot t_2 = d \odot z_2$. Then

$$a \odot c \odot t_1 \odot t_2 = b \odot d \odot z_1 \odot z_2$$

and by Lemma 10

$$a \odot c \odot t = b \odot d \odot z$$

for $t = t_1 \odot t_2 \in \mathbb{T}_1$, $z = z_1 \odot z_2 \in \mathbb{T}_1$. Hence $a \odot c \sim_\odot b \odot d$. □

Remark 15. If $a \sim_\odot b$ for $a, b \in \mathbb{R}^*$ then (19) and Lemmas 6 and 8 imply that either both a and b are positive or both of them are negative.

4.2. Equivalence of transversible n. f. q.

As $\sim_\odot$ is a correct equivalence relation, which follows from Theorem 3, it parts the set $\mathbb{R}^*$ into disjoint equivalence classes. Each of them is either a subset of $\mathbb{R}^+$ or of $\mathbb{R}^-$, as mentioned in Remark 15. For the special case of transversive n. f. q. from $\mathbb{T}$ the equivalence classes can be specified as follows.

Theorem 6. The equivalence relation $\sim_\odot$ parts the set $\mathbb{T}$ of transversive n. f. q. into disjoint equivalence classes $\mathbb{T}_y$, $y \in R_0$.

Proof. If $a \in \mathbb{T}_y$ for $y \neq 0$ then by Lemma 9 $a = t \odot \langle y \rangle$ for some $t \in \mathbb{T}_1$ and, by Remark 14, $a \sim_\odot \langle y \rangle$. It means that for $a, b \in \mathbb{T}_y$ the transitivity of $\sim_\odot$ implies $a \sim_\odot b$. Let $a \in \mathbb{T}_x$, $b \in \mathbb{T}_y$, $x \neq y$, and let $a \sim_\odot b$. Then $a = t_1 \odot \langle x \rangle$, $b = r_1 \odot \langle y \rangle$, for some $t_1, r_1 \in \mathbb{T}_1$, and there exist $t_2, r_2 \in \mathbb{T}_1$ such that

$$\langle x \rangle \odot t_1 \odot t_2 = a \odot t_2 = b \odot r_2 = \langle y \rangle \odot r_1 \odot r_2,$$

or

$$\langle x \rangle \odot t = \langle y \rangle \odot r \tag{20}$$

if we denote $t = t_1 \odot t_2 \in \mathbb{T}_1$, $r = r_1 \odot r_2 \in \mathbb{T}_1$ (cf. Lemma 10). If it is so then either both $x > 0$, $y > 0$ or both $x < 0$ and $y < 0$. Let us suppose, now, that $x > y > 0$. Then by (13) for any $z \in R_0$

$$f_{\langle x \rangle \odot t}(x^2/z) = f_{\langle x \rangle \odot t}(z) = f_{\langle y \rangle \odot r}(z) = f_{\langle y \rangle \odot r}(y^2/z). \tag{21}$$

Condition (1) implies that there exists $z_0 > 0$ and $\varepsilon \in R$ such that $1 > \varepsilon > y^2/x^2 > 0$, and

$$f_{\langle x \rangle \odot t}(z_0) = f_{\langle y \rangle \odot r}(z_0) > 0, \qquad f_{\langle x \rangle \odot t}(z) = f_{\langle y \rangle \odot r}(z) = 0$$

for all $z \in (0, \varepsilon \cdot z_0)$. Then also, using (21),

$$\begin{aligned} f_{\langle x\rangle \odot t}(x^2/z_0) = f_{\langle y\rangle \odot r}(y^2/z) > 0 \\ f_{\langle x\rangle \odot t}(x^2/z) = f_{\langle y\rangle \odot r}(y^2/z) = 0 \text{ for } z \in (0, \varepsilon \cdot z_0), \end{aligned} \tag{22}$$

i. e.

$$f_{\langle x\rangle \odot t}(u) = 0 = f_{\langle y\rangle \odot r}(u) \text{ for all } u > y^2/(\varepsilon \cdot z_0). \tag{23}$$

But

$$x^2/z_0 = (x^2 \cdot y^2) \,/\, (z_0 \cdot y^2) > y^2 \,/\, (\varepsilon \cdot z_0)$$

and then by (22) $f_{\langle x\rangle \odot t}(x^2/z_0) > 0$ and by (23) $f_{\langle x\rangle \odot t}(x^2/z_0) = 0$. This contradiction means that equality (20) cannot be fulfilled. □

4.3. Group up to equivalence

The multiplication $\odot$ over $\mathbb{R}^*$ does not fulfill all group properties in the usual equality form. However, $\mathbb{R}^*$ is a group up to the equivalence $\sim_\odot$ as shown below.

Theorem 7. The set $\mathbb{R}^*$ of signed normal fuzzy quantities and the product operation $\odot$ defined by (3) form a multiplicative commutative group up to the equivalence relation $\sim_\odot$, i. e.

$$a \odot b \sim_\odot b \odot a, \tag{24}$$

$$a \odot (b \odot c) \sim_\odot (a \odot b) \odot c, \tag{25}$$

$$a \odot \langle 1 \rangle \sim_\odot a, \tag{26}$$

$$a \odot (1/a) \sim_\odot \langle 1 \rangle, \tag{27}$$

for any a, b, $c \in \mathbb{R}^*$.

Proof. If a, $b \in \mathbb{R}^*$ then also $a \odot b \in \mathbb{R}^*$ as follows from Lemmas 6, 7 and 8. Relations (24) and (25) follow from Lemmas 1 and 2, respectively, as by Definition 7 $a = b$ implies $a \sim_\odot b$. Analogously (26) follows from Lemma 3. Finally Lemma 11 shows that $a \odot (1/a) \in \mathbb{T}_1$ which means, by Theorem 6, the validity of (27). □

4.4. Why $\mathbb{R}^*$ only?

The limitation of our preceding considerations to the signed n. f. q. only is rather strong as doing so we omit a wide class of possible fuzzy quantities. The necessity to proceed in such way follows from serious difficulties appearing when we apply the procedures described above to more general n. f. q. The most essential of them are briefly discussed in this section.

Let us call an n. f. q. $a \in \mathbb{R}_0$ *bisigned* iff there exist $x_1 < 0$ and $x_2 > 0$ such that $f_a(x_1) > 0$, $f_a(x_2) > 0$.

First, it is very easy to verify that, for bisigned n. f. q. a, $a \odot (1/a) \notin \mathbb{T}_1$ but it is also bisigned even if also in this case $a \odot (1/a) \in \mathbb{R}_0$ and

$$f_{a\odot(1/a)}(x) = f_{a\odot(1/a)}(1/x) \qquad \text{for all } x \in R_0. \tag{28}$$

It means that there are no t_1, $t_2 \in \mathbb{T}_1$ such that

$$a \odot (1/a) \odot t_1 = \langle 1 \rangle \odot t_2$$

and the group property (27) cannot be fulfilled for bisigned n. f. q.

If we suggest a wider concept of the y-transversibility, e. g. if we say that $a \in \mathbb{R}_0$ is y^*-transversible, $y \in R_0$, iff

$$f_a(y \cdot x) = f_a(y/x) \qquad \text{for all } x \neq 0, \tag{29}$$

and if we denote by $\mathbb{T}^*_y$ the set of all y^*-transversible n.f.q. then evidently $\langle -1 \rangle \in \mathbb{T}^*_1$, $\langle 1 \rangle \in \mathbb{T}^*_{-1}$, $\mathbb{T}^*_1 \cap \mathbb{S}_0 \neq \emptyset$, $\mathbb{T}^*_1 \cap \mathbb{T}^*_{-1} \neq \emptyset$, etc.

If we proceed in this way then evidently $a \odot (1/a) \in \mathbb{T}^*_1$ for any $a \in \mathbb{R}_0$, but it is necessary to modify the equivalence concept in the way resulting to the equivalence between $\langle 1 \rangle$ and $a \odot \langle 1/a \rangle$ for general n. f. q. $a \in \mathbb{R}_0$. Let us define the $*$-equivalence $\sim^*_\odot$ as follows. If a, $b \in \mathbb{R}_0$ then $a \sim^*_\odot b$ iff there exist t_1, $t_2 \in \mathbb{T}^*_1$ such that

$$a \odot t_1 = b \odot t_2.$$

Then always $a \odot (1/a) \sim^*_\odot \langle 1 \rangle$ but also $\langle -y \rangle \sim^*_\odot \langle y \rangle$ for arbitrary $y \in R_0$ as also the n. f. q. $\langle -1, 1 \rangle$ such that

$$\begin{aligned} f_{\langle -1,1 \rangle}(x) &= 1 \qquad \text{for } x = 1 \text{ or } x = -1, \\ &= 0 \qquad \text{for } -1 \neq x \neq 1, \end{aligned}$$

belongs to $\mathbb{T}^*_1$. By Lemma 4 $a \odot \langle -1, 1 \rangle \in \mathbb{S}_0$ and it is easy to prove that generally

$$a \odot \langle -1, 1 \rangle = (-a) \odot \langle -1, 1 \rangle$$

for $(-a)$ fulfilling (11), which implies that $a \sim^*_\odot (-a)$ for any $a \in \mathbb{R}_0$.

These and other possible paradoxes essentially complicate the application of the method suggested for signed n. f. q. to their more general types.

5. POWERS

Some of the concepts and results derived in the preceding sections can be used also for the fuzzy version of powers. A brief note about the crisp exponents over fuzzy quantities can be found in [1]. In fact, the precision of definitions and statements in the general case of n. f. q. and also in the case of signed n. f. q. from $\mathbb{R}^*$ demands rather complex and sophisticated formalism. As the powers represent a marginal topic as regards the main subject of this paper we limit our interest to the very simple case of positive normal fuzzy quantities from $\mathbb{R}^+$ and their crisp or fuzzy exponents. More detailed and more general approach to this topic would demand the complexity of presentation fairly exceeding the expected extent of this paper.

5.1. Crisp exponent

In some applications it is useful to manage at least the basic algebraical operations concerning the crisp (often integer or natural) exponents over (positive) n. f. q.

Definition 8. Let $a \in \mathbb{R}^+$ be positive n. f. q. and let $r \in R_0$ be non-zero real number. Then we denote by a^r and call the rth power of a the n. f. q. a^r with membership function

$$\begin{aligned} f_{a^r}(x) &= f_a\left(x^{1/r}\right) \qquad \text{for all } x > 0, \\ &= 0 \qquad \text{for all } \ x \leq 0. \end{aligned}$$

Remark 14. If $a \in \mathbb{R}^+$ then $a^r \in \mathbb{R}^+$ as well.

Lemma 12. If $a \in \mathbb{R}^+$, $r \in R_0$ then $a^{-r} = 1/a^r = (1/a)^r$.

Proof. For any $x > 0$

$$f_{a^{-r}}(x) = f_a\left(x^{-1/r}\right) = f_a\left(x^{-1})^r\right) = f_{a^r}(1/x) = f_{1/a^r}(x),$$

and

$$f_{a^{-r}}(x) = f_a\left(x^{-1/r}\right) = f_a\left((x^r)^{-1}\right) = f_{1/a}(x^r) = f_{(1/a)^r}(x). \qquad \square$$

Remark 15. The preceding lemma implies that $(a^r)^{-1} = a^{-r}$ for $a \in \mathbb{R}^+$, $r \in R_0$.

Theorem 8. For any $a \in \mathbb{R}^+$ and $r \in R_0$ the relation $a^r \odot a^{-r} \sim_{\odot} \langle 1 \rangle$, i.e. $a^r \odot r^{-r} \in \mathbb{T}_1$, holds.

Proof. The statement follows from Lemma 12, Lemma 11 and Theorem 7 immediately. $\square$

Corollary. For any $a \in \mathbb{R}^+$ the relation $a \odot a^{-1} \in \mathbb{T}_1$ holds.

Theorem 9. If $a \in \mathbb{R}^+$ and $r, s \in R_0$ then $(a^r)^s = a^{r \cdot s}$.

Proof. For all $x > 0$

$$\begin{aligned} f_{(a^r)^s}(x) &= f_{a^r}\left(a^{1/s}\right) = f_a\left(\left(x^{1/s}\right)^{1/r}\right) = \\ &= f_a\left(x^{1/(r \cdot s)}\right) = f_{a^{r \cdot s}}(x). \end{aligned}$$

Theorem 10. If $r \in R_0$ and $a,\, b \in \mathbb{R}^+$ then $a^r \odot b^r = (a \odot b)^r$.

Proof. For any $x > 0$

$$\begin{aligned} f_{a^r \odot b^r}(x) &= \sup_{y \neq 0} \left(\min \left(f_a\left(y^{1/r}\right),\, f_b\left(x^{1/r}/y^{1/r}\right)\right)\right) = \\ &= \sup_{z \neq 0} \left(\min \left(f_a(z),\, f_b(x^{1/r}/z)\right)\right) = \\ &= f_{a \odot b}\left(x^{1/r}\right) = f_{(a \odot b)^r}(x), \end{aligned}$$

where the substitution $y^{1/r} = z$ was used. □

Some of the useful properties of deterministic powers are not true in case of fuzzy variables. Namely the equality $a^r \odot a^s = a^{r+s}$ is not generally fulfilled, as shown by the following example.

Example 2. Let us consider an n. f. q. $a \in \mathbb{R}^+$ as follows.

$$f_a(1) = f_a(2) = 1, \qquad f_a(x) = 0 \qquad \text{for} \qquad 1 \neq x \neq 2.$$

Then

$$\begin{aligned} f_{a \odot a}(x) &= 1 \qquad \text{for } x = 1,\, 2,\, 4, \\ &= 0 \qquad \text{for other } x, \end{aligned}$$

and

$$\begin{aligned} f_{a^2}(x) &= 1 \qquad \text{for } x = 1,\, 4, \\ &= 0 \qquad \text{for other } x. \end{aligned}$$

It means that $f_{a^2}(2) \neq f_{a \odot a}(2)$, and

$$a \odot a \neq a^2.$$

Remark 16. If $y > 0$ then (29) implies that $\langle y \rangle^r = \langle y^r \rangle$ and consequently $\langle y \rangle^r \odot \langle y \rangle^s = \langle y \rangle^{r+s}$ for $r,\, s \in R_0$.

Some other results can be derived for (positive) transversible n. f. q.

Lemma 13. If $a \in \mathbb{T}_1$ and $r \in R_0$ then $a^r \in \mathbb{T}_1$.

Proof. For any $x > 0$

$$\begin{aligned} f_{a^r}(x) &= f_a\left(x^{1/r}\right) = f_a\left(x^{-1/r}\right) = f_a\left(\left(x^{-1}\right)^{1/r}\right) = \\ &= f_{a^r}\left(x^{-1}\right) = f_{a^r}(1/x). \end{aligned}$$

□

Theorem 11. If $y > 0$, $a \in \mathbb{T}_y$ and $r \neq 0$ then $a^r \in \mathbb{T}_{y^r}$.

Proof. Lemma 9 implies that there exists $t \in \mathbb{T}_1$ such that $a = \langle y \rangle \cdot t$. Then, using Theorem 10 and Remark 16,

$$a^r = \langle y \rangle^r \odot t^r = \langle y^r \rangle \odot t^r,$$

where $t^r \in \mathbb{T}_1$ by Lemma 13. Hence, $a^r \in \mathbb{T}_{y^r}$. □

Theorem 12. If $y > 0$, $a \in \mathbb{T}_y$ and $r, s \in R_0$, $r + s \neq 0$, then $a^r \odot a^s \sim_\odot a^{r+s}$.

Proof. By Lemma 9 and Theorem 11 for $r \neq -s$

$$a^r \odot a^s = \langle y \rangle^r \odot t_1 \odot \langle y \rangle^s \odot t_2 = \langle y^r \rangle \odot \langle y^s \rangle \odot t_1 \odot t_2$$

for some $t_1, t_2 \in \mathbb{T}_1$. Relations (3) and (5) immediately imply that

$$\langle y^r \rangle \odot \langle y^s \rangle = \langle y^r \cdot y^s \rangle = \langle y^{r+s} \rangle$$

and by Lemma 10 $t_1 \odot t_2 \in \mathbb{T}_1$. It means that (cf. Remark 16)

$$\langle y^r \rangle \odot \langle y^s \rangle \odot t_1 \odot t_2 = \langle y^{r+s} \rangle \odot t \in \mathbb{T}_{y^{r+s}}.$$

On the other hand, Theorem 11 implies that also $a^{r+s} \in \mathbb{T}_{y^{r+s}}$ and, due to Theorem 6,

$$a^r \odot a^s \sim_\odot a^{r+s}.$$ □

Remark 17. The power a^0 is not defined. However the modification of the previous theorem for $r = -s$ is given in Theorem 8.

5.2. Fuzzy exponent

The concept of the power with crisp exponent over a positive n. f. q. can be generalized to the case of fuzzy exponent.

Definition 9. Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}_0$ be normal fuzzy quantities, positive or signed, respectively. Then the n. f. q. a^b with the membership function

$$\begin{aligned} f_{a^b}(x) &= \sup_{y \neq 0} \left(\min \left(f_a(x^{1/y}), f_b(y) \right) \right), \qquad x > 0, \\ &= 0 \qquad \text{for } x \leq 0 \end{aligned} \tag{30}$$

is called the bth power of a.

As any crisp number $r \in \mathbb{R}_0$ is a special case of (signed) n. f. q. (namely $\langle r \rangle \in \mathbb{R}^*$), the contraexample 2 keeps generally significant, and for $a \in \mathbb{R}^+$, $b, c \in \mathbb{R}_0$ the equality $a^b \odot a^c = a^{b \oplus c}$ cannot be generally fulfilled.

Lemma 14. If $a \in \mathbb{R}^+$ and $r \in R_0$ then the crisp power a^r by (29) is identical with the fuzzy power $a^{\langle r \rangle}$ by (30).

Proof. If $x > 0$ then

$$\begin{aligned} f_{a^{\langle r \rangle}} &= \sup_y \left(\min\left(f_a(x^{1/y}),\, f_{\langle r \rangle}(y)\right)\right) = \\ &= f_a(x^{1/r}) = f_{a^r}(x). \end{aligned}$$

□

Theorem 13. If $a \in \mathbb{R}^+$ and $b \in \mathbb{R}_0$ then $a^{-b} = (1/a)^b = 1/a^b$.

Proof. For any $x > 0$

$$\begin{aligned} f_{a^{-b}}(x) &= \sup_{y \neq 0} \left(\min\left(f_a(x^{1/y}),\, f_{-b}(y)\right)\right) = \sup_{y \neq 0} \left(\min\left(f_a(x^{1/y}),\, f_b(-y)\right)\right) = \\ &= \sup_{z \neq 0} \left(\min\left(f_a(x^{-1/z}),\, f_b(z)\right)\right) = \sup_{z \neq 0} \left(\min\left(f_a((x^{-1})^{1/2}),\, f_b(z)\right)\right) = \\ &= f_{a^b}(x^{-1}) = f_{1/a^b}(x), \end{aligned}$$

and

$$\begin{aligned} f_{a^{-b}}(x) &= \sup_{y \neq 0} \left(\min\left(f_a\left(x^{1/y}\right) f_b(-y)\right)\right) = \sup_{z \neq 0} \left(\min\left(f_a\left(x^{1/z}\right)^{-1}\right),\, f_b(z)\right)\right) = \\ &= \sup_{z \neq 0} \left(f_{1/a}\left(x^{1/z}\right),\, f_b(z)\right) = f_{(1/a)^b}(x). \end{aligned}$$

□

Lemma 15. If $y > 0$, $y \neq 1$, and if $a \in \mathbb{R}_0$ then the membership function of $\langle y \rangle^a$ is given by

$$f_{\langle y \rangle^a}(x) = f_a\left(\frac{\ln x}{\ln y}\right) \qquad \text{for all } x > 0. \tag{31}$$

Proof. For all $x > 0$

$$f_{\langle y \rangle^a}(x) = \sup_{z \neq 0} \left(\min\left(f_{\langle y \rangle}(x^{1/z}),\, f_a(z)\right)\right) = f_a(z_x)$$

for the $z_x \in R$ for which $f_{\langle y \rangle}(z_x) = 1$. It means that by (5) $z_x = y$. It is valid for the z for which $x^{1/z} = y$, which means $(1/z) \cdot \ln x = \ln y$. Consequently

$$z = \ln x \,/\, \ln y.$$

□

Theorem 14. If $a \in \mathbb{R}^+$, $b \in \mathbb{R}_0$ then

$$a^b \odot a^{\langle -b \rangle} \in \mathbb{T}_1, \qquad \text{i.e.} \qquad a^b \odot a^{-b} \sim_{\odot} \langle 1 \rangle.$$

Proof. The statement immediately follows from Theorem 13, as $a^b \odot a^{\langle -b \rangle} = a^b \odot (1/a^b)$, and Theorem 7 holds. □

Theorem 15. If $a \in \mathbb{R}^+$, $b, c \in \mathbb{R}_0$ then $(a^b)^c = a^{b \odot c}$.

Proof. For any $x > 0$

$$\begin{aligned}
f_{(a^b)^c}(x) &= \sup_{u \neq 0} \left(\min \left(f_{a^b}(x^{1/u}),\, f_c(u)\right)\right) = \\
&= \sup_{u \neq 0} \left[\min \left(\sup_{v \neq 0} \left[\min \left(f_a \left((x^{1/u})^{1/v}\right),\, f_b(v)\right),\, f_c(u)\right)\right]\right] = \\
&= \sup_{u \neq 0} \left[\min \left(\sup_{v \neq 0} \left[\min \left(f_a \left(x^{1/(u \cdot v)}\right),\, f_b(v)\right),\, f_c(u)\right)\right]\right] = \\
&= \sup_{w \neq 0} \left[\min \left(\sup_{v \neq 0} \left[\min \left(f_a \left(x^{1/w}\right),\, f_b(v)\right),\, f_c(w/v)\right)\right]\right] = \\
&= \sup_{w \neq 0} \left[\min \left(f_a \left(x^{1/w}\right),\, \sup_{v \neq 0} \left[\min \left(f_b(v),\, f_c(w/v)\right)\right)\right]\right] = \\
&= \sup_{w \neq 0} \left[\min \left(f_a(x^{1/w}),\, f_{b \odot c}(w)\right)\right] = f_{a^{b \odot c}}(x),
\end{aligned}$$

where the substitution $w = u \cdot v$ was used. □

Theorem 16. If $a \in \mathbb{R}^+$ and $s \in \mathbb{S}_0 \cap \mathbb{R}_0$ then $a^s \in \mathbb{T}_1$.

Proof. For any $x > 0$

$$\begin{aligned}
f_{a^s}(x) &= \sup_{y \neq 0} \left(\min \left(f_a(x^{1/y}),\, f_s(y)\right)\right) = \sup_{y \neq 0} \left(\min \left(f_a(x^{1/y}),\, f_s(-y)\right)\right) = \\
&= \sup_{z \neq 0} \left(\min \left(f_a \left((x^{-1})\right)^{1/z},\, f_s(z)\right)\right) = f_{a^s}(x^{-1}).
\end{aligned}$$
□

Corollary. If $a \in \mathbb{R}^+$ and $s \in \mathbb{S}_0 \cap \mathbb{R}_0$ then $a^s \sim_{\odot} \langle 1 \rangle$.

Theorem 17. Let $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}_0$. Then $a^c \odot b^c = (a \odot b)^c$.

Proof. Let us remember relations (30) and (3). Using them we obtain for any $x > 0$

$$\begin{aligned}
f_{a^c \odot b^c} &= \sup_{y \neq 0} \left(\min \left(f_{a^c}(x/y),\, f_{b^c}(y)\right)\right) = \\
&= \sup_{y \neq 0} \left(\min \left[\sup_{i \neq 0} \left(\min \left(f_a \left(x^{1/i}/y^{1/i}\right),\, f_c(i)\right)\right),\, \sup_{i \neq 0} \left(\min \left(f_b(y^{1/i}),\, f_c(i)\right)\right)\right]\right) = \\
&= \sup_{y \neq 0} \left[\sup_{i \neq 0} \left(\min \left[f_a \left(x^{1/i}/y^{1(i)}\right),\, f_b(y^{1/i}),\, f_c(i)\right]\right)\right] = \\
&= \sup_{i \neq 0} \left[\sup_{y \neq 0} \left(\min \left[f_a \left(x^{1/i}/y^{i/i}\right),\, f_b(y^{1/i}),\, f_c(i)\right]\right)\right] = \\
&= \sup_{i \neq 0} \left(\min \left[f_c(i),\, \sup_{y \neq 0} \left(\min \left(f_a \left(x^{1/i}/y^{1/i}\right),\, f_b(y^{1/i})\right)\right)\right]\right) = \\
&= \sup_{i \neq 0} \left(\min \left(f_c(i),\, f_{a \odot b}(x^{1/i})\right)\right) = f_{(a \odot b)^c}(x).
\end{aligned}$$
□

Some further results can be derived for transversible n. f. q.

Lemma 16. Let $a \in \mathbb{R}_0$, $t \in \mathbb{T}_1$. Then $t^a \in \mathbb{T}_1$ and consequently $t^a \sim_{\odot} t$.

Proof. For any $x > 0$

$$\begin{aligned} f_{t^a}(x) &= \sup_{y \neq 0} \left(\min \left(f_t(x^{1/y}),\, f_a(y)\right)\right) = \\ &= \sup_{y \neq 0} \left(\min \left(f_t\left((1/x)^{1/y}\right),\, f_a(y)\right)\right) = f_{t^a}(1/x). \end{aligned}$$ □

Some analogies between the classical deterministic and fuzzy powers are attractive, but the analogy is not universal. So, it seems natural to expect for $y > 0$, $a \in \mathbb{T}_y$ and $t \in \mathbb{T}_1$ the validity of

$$a^t \in \mathbb{T}_y, \qquad \text{or at least,} \qquad \langle y \rangle^t \in \mathbb{T}_y.$$

The following example shows that for $y \neq 1$ this is not generally true.

Example 3. Let us choose $y = 4$, $t \in \mathbb{T}_1$, $f_t(1/2) = f_t(2) = 1$, $f_t(x) = 0$ for $1/2 \neq x \neq 2$. Then by (30)

$$f_{\langle y \rangle^t}(2) = f_{\langle y \rangle^t}(16) = 1, \qquad f_{\langle y \rangle^t}(x) = 0 \qquad \text{for } 2 \neq x \neq 16.$$

Then evidently $\langle y \rangle^t \notin \mathbb{T}_y$, in our case $\langle 4 \rangle^t \notin \mathbb{T}_4$.

6. CONCLUSIVE REMARKS

Formulating and discussing some multiplicative analogies to the methods developed for the addition over real-valued fuzzy quantities we can see that the multiplication (and power) forms rather more sophisticated structure. The procedures used in the additive case without any practical limitations can be transformed to the multiplicative operation very carefully with consequent checking of the range of their validity.

Nevertheless, even the results and methods presented above offer interesting tools for the application of (mainly linear) algebraic methods to the n. f. q. Having developed both, additive and multiplicative, formal apparates regarding the arithmetics of fuzzy quantities we can also manage at least the fundamental elaboration of the additive or multiplicative fuzzy noise acting in realistic data processing.

The mutual connection between addition and multiplication, represented in the crisp case by the distributivity, is not so easy in the fuzzy case. Its validity in some special cases (cf. also [1] or [7]) does not cover the general set $\mathbb{R}$ of n. f. q. It is not clear yet if e. g. some type of equivalence (derived from $\sim_{\odot}$ and the additive equivalence [5], for example) could guarantee at least some weaker form of the distributivity, analogously to the weaker form of group properties shown in [5] and in the above sections.

(Received November 11, 1991.)

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