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STABILITY ANALYSIS FOR LARGE SCALE TIME DELAY SYSTEMS VIA THE MATRIX LYAPUNOV FUNCTION

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In this paper, we analyze the stability of large-scale systems with multiple time delays in both isolated parts and interconnections via the scalar approach of the matrix Lyapunov function. This approach of the matrix Lyapunov function estimates the stability of a large-scale interconnected system based on a decomposition-aggregation method. The candidate Lyapunov function here takes advantage of a weighted sum of individual Lyapunov function for each free subsystem and every interconnection related to all in pairs isolated subsystems in case of nondelay.

1. INTRODUCTION

The scalar approach of a matrix Lyapunov function method for the stability estimation of a large-scale interconnected system uses a decomposition–aggregation method [1]. Here the candidate Lyapunov function of the approach takes advantage of a weighted sum of individual Lyapunov functions for each free subsystem and every interconnection among isolated subsystems without time delay. By doing this, we can estimate the influence of interconnections between subsystems on the stability.

In this paper, after considering a nonlinear system stability which involves time delays by some perturbations, the scalar approach of the matrix Lyapunov function method has been taken into account for the stability of large-scale nonlinear systems without time delay [2]. Then, this method is applied to a nonlinear system with time delays following the application to the stability analysis of linear systems with time delays. It is noticed that if a weakly coupled nonlinear system is perturbed with time delays, then those delays do not destroy the stability of the system provided that the original system is stable [3, 4]. However, the time delays can be involved in isolated terms as well as interconnections in many cases as pointed out in [3, 5] and these delays really have an effect on the stability of the system.

2. STABILITY OF A LARGE-SCALE NONLINEAR SYSTEM

Consider a large-scale nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\, t] \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{x}(t) = \left[\mathbf{x}_1(t)^T, \mathbf{x}_2(t)^T \ldots \mathbf{x}_m(t)^T\right]^T$ and $\mathbf{f} = \mathbb{R}^n \times T \to \mathbb{R}^n$, T : time interval $(-\infty, \infty)$. The system is assumed to be decomposed into m subsystems,

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}_i\,[\mathbf{x}_i(t),\, t] + \mathbf{g}_i\,[\mathbf{x}(t),\, t] \tag{2}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^{n_i}$, $f_i:\ \mathbb{R}^{n_i} \times T \to \mathbb{R}^{n_i}$,

$$\mathbf{g}_i:\ \mathbb{R}^n \times T \to \mathbb{R}^{n_i}, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} n_i = n.$$

The requirements of $\mathbf{f}_i(\cdot)$, $\mathbf{g}_i(\cdot)$ guarantee the existence and uniqueness of the solution of equation (2). The $\mathbf{x}(t) = 0$ is an equilibrium satisfying

$$\mathbf{f}(0, t) = 0, \qquad \mathbf{f}_i(0, t) = 0, \qquad \mathbf{g}_i(0, t) = 0. \tag{3}$$

Let us first survey stability conditions for the nonlinear interconnected system (1) which is represented as:

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \mathbf{g}(\mathbf{x}, t). \tag{4}$$

The equilibrium state of this system is asymptotically stable (A. S.) if the following conditions are satisfied [2]:

(1) $\mathbf{g}(0, t) = 0$

(2) $\lim\limits_{\|\mathbf{x}(t)\| \to 0} \frac{\|\mathbf{g}(\mathbf{x}, t)\|}{\|\mathbf{x}(t)\|} = 0$

provided that the stability of linear part is A. S.

The system (4) can be perturbed to have time delay in interconnection as:

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \mathbf{g}\,[\mathbf{x}(t-h),\, t] \tag{5}$$

where

$A \in \mathbb{R}^{n \times n}$, $\mathbf{x}(t) \in \mathbb{R}^n$, $0 \le h$ constant delay.

The stability of the system above can be estimated by the following theorem.

Theorem 1. Consider a nonlinear system (5). If the following conditions for vector function $\mathbf{g}[\mathbf{x}(t-h),\, t]$ hold,

(1) $\mathbf{g}(0, t) = 0$

(2) $\lim\limits_{\|\mathbf{x}(t)\| \to 0} \frac{\|\mathbf{g}(\mathbf{x}, t), t\|}{\|\mathbf{x}(t)\|} = 0$

(3) solutions of $\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t)$ are A. S.

then the equilibrium state is A. S.

Proof. For a candidate Lyapunov function for the system (5)

$$\nu(\mathbf{x}, t) = \mathbf{x}(t)^T P\,\mathbf{x}(t),$$

where P is a positive definite matrix

$$\begin{aligned}
\dot{\nu}(\mathbf{x}, t) &= \dot{\mathbf{x}}(t)^T P\,\mathbf{x}(t) + \mathbf{x}(t)^T P\,\dot{\mathbf{x}}(t) \\
&= [A\,\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t-h), t)]^T P\,\mathbf{x}(t) + \mathbf{x}(t)^T P\,[A\,\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t-h), t)] \\
&= \mathbf{x}(t)^T \left[A^T P + P\,A\right] \mathbf{x}(t) + 2\mathbf{g}^T(\cdot)\,P\,\mathbf{x}(t)
\end{aligned}$$

where $A^T P + P\,A$ becomes a negative definite matrix by (3). Further using (2), $2\mathbf{g}^T(\cdot)\,P\,\mathbf{x} = 0$ for small value of $\mathbf{x}(t)$ and $\dot{\nu}(\mathbf{x}, t) < 0$ with $\nu(\mathbf{x}, t) > 0$. This shows the proof of the theorem. □

Let the candidate Lyapunov function for the system (1) be represented as:

$$\nu(\mathbf{x}, t) = \left[\mathbf{d}^T\right] \left[\nu_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t), t)\right] [\mathbf{d}], \qquad i, j = 1, 2, \ldots, m \tag{6}$$

where $\mathbf{d} = [d_1, d_2 \ldots d_m]^T$, $d_i > 0$ are constant scalars and $\nu_{ij}(\cdot)$ here represents the relationship which the jth subsystem affects the ith subsystem and $\nu_{ii}\,[\mathbf{x}_i(t), t] = x_i\,P_{ii}\,x_i$, $\nu_{ij}\,[\mathbf{x}_i(t), \mathbf{x}_j(t), t] = \mathbf{x}_i^T\,P_{ij}\,\mathbf{x}_j$ with appropriate dimensions of positive definite matrices P_{ii} and matrices P_{ij}. To estimate the lower bound of $\nu(\mathbf{x}(t), t)$

$$\nu_{ii}(\mathbf{x}_i, t) \geq \gamma_{ii}\,\xi_i^2(\mathbf{x}_i) \tag{7a}$$

$$\nu_{ij}\,(\mathbf{x}_i(t), \mathbf{x}_j(t), t) \geq \gamma_{ij}\,\xi_i(\mathbf{x}_i)\,\xi_j(\mathbf{x}_j). \tag{7b}$$

By letting

$$\gamma_{ii} = \lambda_m\,(P_{ii}) \tag{8a}$$

$$\gamma_{ij} = \gamma_{ji} = -\text{sign}\,(d_i\,d_j)\,\lambda_M^{\frac{1}{2}}\,\left(P_{ij}\,P_{ij}^T\right), \qquad i \neq j \tag{8b}$$

and

$$\xi_i(\mathbf{x}_i) = \|\mathbf{x}_i\| \tag{8c}$$

where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ represent the minimum and maximum eigenvalues of $(\cdot)$ respectively and $\|\mathbf{x}_i\|$ is a norm of $\mathbf{x}_i$. The lower bound $\nu_\ell(\mathbf{x}(t), t)$ of $\nu(\mathbf{x}(t), t)$ is

$$\nu(\mathbf{x}(t), t) \geq \nu_\ell(\mathbf{x})$$

$$= \xi^T \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & d_m \end{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & d_m \end{bmatrix} \xi \tag{9}$$

or as a compact form

$$\nu_\ell(\mathbf{x}) = \xi^T D^T G D \xi \tag{10}$$

where

$$\xi = [\|\mathbf{x}_1\| \, \|\mathbf{x}_2\| \cdots \|\mathbf{x}_m\|]^T, \qquad D = \text{diag } (d_1 \cdots d_m)$$

and matrix $G = [\gamma_{ij}]$ for $i, j = 1, 2, \ldots, m$. If $G = [\gamma_{ij}] > 0$, then $\nu_\ell(\mathbf{x})$ is positive definite (p. d.). From (6),

$$\nu(\mathbf{x}(t), t) = \sum_{i=1}^{m} \left[d_i^2 \, \nu_{ii}(\mathbf{x}_i, t) + \sum_{\substack{j=1 \\ j \neq i}}^{m} d_i \, d_j \, \nu_{ij}(\mathbf{x}_i(t), \, \mathbf{x}_j(t), \, t) \right], \qquad \nu_{ij}(\cdot) = \nu_{ji}(\cdot). \tag{11}$$

Then the total time derivative of $\nu(\mathbf{x}, t)$ for (2) becomes

$$\dot{\nu}(\mathbf{x}, t) = \sum_{i=1}^{m} \left[d_i^2 \, \dot{\nu}_{ii}(\mathbf{x}_i, t) + \sum_{j \neq i}^{m} d_i \, d_j \, \dot{\nu}_{ij}(\mathbf{x}_i, \, \mathbf{x}_j, \, t) \right] \tag{12}$$

$$\dot{\nu}_{ii}(\cdot) = \frac{\partial \nu_{ii}(\mathbf{x}_i, t)}{\partial t} + \left(\frac{\partial \nu_{ii}(\mathbf{x}_i, t)}{\partial \mathbf{x}_i} \right)^T \dot{\mathbf{x}}_i(t) \tag{13}$$

$$\dot{\nu}_{ij}(\cdot) = \frac{\partial \nu_{ij}(\mathbf{x}_i, \mathbf{x}_j, t)}{\partial t} + \sum_{\substack{k=i,j \\ i \neq j}} \left\{ \left[\frac{\partial \nu_{ij}(\mathbf{x}_i, \mathbf{x}_j, t)}{\partial \mathbf{x}_k} \right]^T [\mathbf{f}_k(\cdot) + \mathbf{g}_k(\cdot)] \right\}. \tag{14}$$

Then the upper bound $\dot{\nu}_M$ of $\dot{\nu}(\mathbf{x}, t)$ with $\dot{\nu}_{ii}(\mathbf{x}_i, t)$ and $\dot{\nu}_{ij}(\mathbf{x}_i, \mathbf{x}_j, t)$ can be obtained by collecting terms of degree of $[q(\cdot)] = 2$ and terms of degree of $[r(\cdot)] > 2$ as

$$\dot{\nu}_M = q\, [\psi_1(\mathbf{x}_1), \ldots, \psi_m(\mathbf{x}_m)] + r\, [\psi_1(\mathbf{x}_1), \ldots, \psi_m(\mathbf{x}_m)] \tag{15}$$

where $\psi_i(\mathbf{x}_i)$, $i = 1, \ldots, m$ is a positive definite continuous function and

$$q(0, \ldots, 0) = 0 \qquad \text{and} \qquad r(0, \ldots, 0) = 0.$$

$q(\cdot)$ could be written as a quadratic form

$$q\, [\psi_1(\mathbf{x}_1), \ldots, \psi_m(\mathbf{x}_m)] = \psi^T \, W \, \psi \tag{16}$$

where W is a constant symmetric matrix

$$\psi = [\psi_1(\mathbf{x}_1), \ldots, \psi_m(\mathbf{x}_m)]^T. \tag{17}$$

From (16), $q(\cdot)$ is negative definite near the origin if and only if $W < 0$ and this gives $\dot{\nu}(\cdot) < 0$ with $\nu(\cdot) > 0$.

Theorem 2. The equilibrium state $\mathbf{x}(t) = 0$ of the system (1) is A. S. near the origin if $G > 0$ in (10) and $-W$ in (16) is an M-matrix [6].

Proof. The proof is clear from Theorem 3. □

3. MATRIX LYAPUNOV FUNCTION FOR THE STABILITY OF A LINEAR TIME DELAY SYSTEMS

Case 1 – Linear System with Time Delay in Two Interconnections

Based on the discussion of previous section for the stability of a nonlinear system through the scalar approach of the matrix Lyapunov function method, the stability of a linear time delay system is investigated as a special case of a nonlinear system. In this case the degree of the system function is one and $r(\cdot)$ in (15) does not appear. The time delay terms in the system however, can be analyzed by using linearity with respect to initial state. Consider a simple interconnected system with constant delays in each subsystem,

$$\dot{\mathbf{x}}_1(t) = A_1\,\mathbf{x}_1(t) + G_{12}\,\mathbf{x}_2(t-h_2) \tag{18a}$$

$$\dot{\mathbf{x}}_2(t) = G_{21}\,\mathbf{x}_1(t-h_1) + A_2\,\mathbf{x}_2(t) \tag{18b}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, h_i is a constant delay for $i = 1,2$ and $G_{ij} \in \mathbb{R}^{n_i \times n_j}$ for $i, j = 1,2$ $(i \neq j)$.

The candidate Lyapunov function for system (18)

$$\nu(\mathbf{x},t) = [d_1\, d_2] \begin{bmatrix} \nu_{11}(\mathbf{x}_1,t) & \nu_{12}(\mathbf{x}_1,\mathbf{x}_2,t) \\ \nu_{21}(\mathbf{x}_2,\mathbf{x}_1,t) & \nu_{22}(\mathbf{x}_2,t) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= d_1^2\,\nu_{11}(\mathbf{x}_1) + 2d_1 d_2 \nu_{12}(\mathbf{x}_1,\mathbf{x}_2) + d_2^2 \nu_2(\mathbf{x}_2), \qquad \nu_{ij} = \nu_{ji} \tag{19}$$

where

$$\nu_{ii}(\mathbf{x}_i) = \mathbf{x}_i^T P_{ii}\,\mathbf{x}_i, \qquad \nu_{ij}(\mathbf{x}_i,\mathbf{x}_j) = \mathbf{x}_i^T P_{ij}\,\mathbf{x}_j. \tag{20}$$

Therefore, $\dot{\nu}(\mathbf{x}_i,t)$ is arranged by substituting $\dot{\mathbf{x}}_i$ given in (18) as follows:

$$\dot{\nu}(\mathbf{x}(t),\,t) = [d_1\, d_2] \begin{bmatrix} \dot{\nu}_{11}(\cdot) & \dot{\nu}_{12}(\cdot) \\ \dot{\nu}_{21}(\cdot) & \dot{\nu}_{22}(\cdot) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{21}$$

$$= \left[\mathbf{x}_1^T\,\mathbf{x}_2^T\right] \begin{bmatrix} d_1^2\left(A_1^T P_{11} + P_{11} A_1\right) & d_1 d_2 \left(A_1^T P_{12} + P_{12} A_2\right) \\ d_2 d_1 \left(A_1^T P_{21} + P_{21} A_1\right) & d_2^2 \left(A_2^T P_{22} + P_{22} A_2\right) \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

$$+ \left[\mathbf{x}_1^T\,\mathbf{x}_2^T\right] \begin{bmatrix} d_1 d_2 P_{12} G_{21} & d_1^2 P_{11} G_{12} \\ d_2^2 P_{22} G_{21} & d_1 d_2 P_{21} G_{12} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1h_1} \\ \mathbf{x}_{2h_2} \end{bmatrix}$$

$$+ \left[\mathbf{x}_{1h_1}^T\,\mathbf{x}_{2h_2}^T\right] \begin{bmatrix} d_1 d_2 G_{21}^T P_{21} & d_2^2 G_{21}^T P_{22} \\ d_1^2 G_{12}^T P_{11} & d_1 d_2 G_{12}^T P_{12} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{22}$$

$$\dot{\nu}_M(\mathbf{x},t) = -\left[\|\mathbf{x}_1^T\|\,\|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix}$$

$$- \left[\|\mathbf{x}_1^T\|\,\|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\beta_{11} & -\beta_{12} \\ -\beta_{21} & -\beta_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_{1h_1}\| \\ \|\mathbf{x}_{2h_2}\| \end{bmatrix} \tag{23}$$

where $\alpha_{ii} = \lambda_{\max}\left\{d_i^2\left(A_i^T P_{ii} + P_{ii} A_i\right)\right\}$ for $i = 1, 2$

$$\alpha_{12} = \left\|d_1\, d_2\left(A_1^T P_{12} + P_{12} A_2\right)\right\|_2, \quad \beta_{11} = \lambda_{\max}\left\{d_1\, d_2\left(P_{12} G_{21} + G_{21}^T P_{21}\right)\right\},$$

$$\beta_{12} = \left\|d_1^2\left(P_{11} G_{12} + G_{12}^T P_{11}\right)\right\|_2, \quad \beta_{21} = \left\|d_2^2\left(P_{22} G_{21} + G_{21}^T P_{22}\right)\right\|_2$$

and

$$\mathbf{x}_{1h_1} = \mathbf{x}_1(t - h_1), \qquad \mathbf{x}_{2h_2} = \mathbf{x}_2(t - h_2), \qquad \alpha_{12} = \alpha_{21}, \qquad \beta_{11} = \beta_{22}.$$

If we can find $d_i > 0$ such that leading principal minors of

$$W = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \quad \text{and} \quad W_h = \begin{bmatrix} -\beta_{11} & -\beta_{12} \\ -\beta_{21} & -\beta_{22} \end{bmatrix}$$

are positive, that is, W and W_h are M-matrices, then $\dot{\nu}_M(\cdot) < 0$. This leads the system to be asymptotically stable according to Theorem 2.

Remark.

(1) Even if both W and W_h are not M-matrices, the system can be stable if the one which is an M-matrix is dominant over the other which is not an M-matrix.

(2) If we can find a scale factor for $\mathbf{x}_i(t - h_i)$ with respect to $\mathbf{x}_i(t)$, then W and W_h with the scaling factor can be added up together as in Section 3.5 of Chapter 3 in [3].

Case 2 – Linear System with Time Delays in Three Interconnections

The stability of a system with time delays in two interconnections can be extended to the system with time delays in three interconnections. Consider a system whose subsystems are represented by

$$\dot{\mathbf{x}}_1 = A_1\,\mathbf{x}_1(t) + G_{12}\,\mathbf{x}_2(t - h_2) + G_{12}\,\mathbf{x}_3(t - h_3) \tag{24a}$$

$$\dot{\mathbf{x}}_2 = G_{21}\,\mathbf{x}_1(t - h_1) + A_2\,\mathbf{x}_2(t) + G_{23}\,\mathbf{x}_3(t - h_3) \tag{24b}$$

$$\dot{\mathbf{x}}_3 = G_{31}\,\mathbf{x}_1(t - h_1) + G_{32}\,\mathbf{x}_2(t - h_2) + A_3\,\mathbf{x}_3(t) \tag{24c}$$

where A_i, G_{ij}, $\mathbf{x}_i$ and h_i are defined similar to Case 1. Thus,

$$\nu_{ii}(\mathbf{x}_i) = \mathbf{x}_i^T P_{ii}\,\mathbf{x}_i, \qquad \nu_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T P_{ij}\,\mathbf{x}_j \quad \text{for } i, j = 1, 2, 3 \text{ and } i \neq j, \tag{25}$$

$$\dot{\nu}(\mathbf{x}(t)) = [d_1\ d_2\ d_3] \begin{bmatrix} \dot{\nu}_{11}(\cdot) & \dot{\nu}_{12}(\cdot) & \dot{\nu}_{13}(\cdot) \\ \dot{\nu}_{21}(\cdot) & \dot{\nu}_{22}(\cdot) & \dot{\nu}_{23}(\cdot) \\ \dot{\nu}_{31}(\cdot) & \dot{\nu}_{32}(\cdot) & \dot{\nu}_{33}(\cdot) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \tag{26}$$

$\nu_{ij}(\cdot) = \nu_{ji}(\cdot)$ for $i \neq j$ and $d_i > 0$.

$$\dot{\nu}(\mathbf{x},t) = \mathbf{x}^T \begin{bmatrix} d_1^2\left(A_1^T p_{11}+p_{11}A_1\right) & d_1 d_2\left(A_1^T p_{12}+p_{12}A_2\right) & d_1 d_3\left(A_1^T p_{13}+p_{13}A_3\right) \\ d_2 d_1\left(A_2^T p_{21}+p_{21}A_1\right) & d_2^2\left(A_2^T p_{22}+p_{22}A_2\right) & d_2 d_3\left(A_2^T p_{23}+p_{23}A_3\right) \\ d_3 d_1\left(A_3^T p_{31}+p_{31}A_1\right) & d_3 d_2\left(A_3^T p_{32}+p_{32}A_2\right) & d_3^2\left(A_3^T p_{33}+p_{33}A_3\right) \end{bmatrix} \mathbf{x}$$

$$+2\mathbf{x}^T \begin{bmatrix} d_1 d_2 p_{12}G_{21}+d_1 d_3 p_{13}G_{31} & d_1^2 p_{11}G_{12}+d_1 d_3 p_{13}G_{32} & d_1^2 p_{11}G_{13}+d_1 d_2 p_{12}G_{23} \\ d_2^2 p_{22}G_{21}+d_2 d_3 p_{23}G_{21} & d_2 d_3 p_{23}G_{32}+d_2 d_1 p_{21}G_{21} & d_2^2 p_{22}G_{23}+d_2 d_1 p_{21}G_{13} \\ d_3^2 p_{33}G_{21}+d_3 d_2 p_{33}G_{32} & d_3^2 p_{32}G_{32}+d_3 d_2 p_{31}G_{12} & d_3 d_1 p_{31}G_{13}+d_3 d_2 p_{32}G_{23} \end{bmatrix} \mathbf{x}_h \quad (27)$$

where

$$\mathbf{x} \equiv \left[\mathbf{x}_1^T\ \mathbf{x}_2^T\ \mathbf{x}_3^T\right]^T, \qquad \mathbf{x}_h \equiv \left[\mathbf{x}_{1h_1}^T\ \mathbf{x}_{2h_2}^T\ \mathbf{x}_{3h_3}^T\right]^T.$$

Furthermore,

$$\dot{\nu}_M(\mathbf{x},t) = -\|\mathbf{x}\|^T \begin{bmatrix} -\alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & -\alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{bmatrix} \|\mathbf{x}\|$$

$$-\|\mathbf{x}\|^T \begin{bmatrix} -(\beta_{11}+\gamma_{11}) & -(\beta_{12}+\delta_{12}) & -(\beta_{13}+\delta_{13}) \\ -(\beta_{21}+\delta_{21}) & -(\beta_{22}+\gamma_{22}) & -(\beta_{23}+\delta_{23}) \\ -(\beta_{31}+\delta_{31}) & -(\beta_{32}+\delta_{32}) & -(\beta_{33}+\gamma_{33}) \end{bmatrix} \|\mathbf{x}_h\| \quad (28)$$

where

$$\|\mathbf{x}\| \equiv \left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\ \|\mathbf{x}_3^T\|\right]^T, \qquad \|\mathbf{x}_h\| \equiv \left[\|\mathbf{x}_{1h_1}^T\|\ \|\mathbf{x}_{2h_2}^T\|\ \|\mathbf{x}_{3h_3}^T\|\right]^T,$$

$$\begin{aligned} \alpha_{ii} &= \lambda_{\max}\left\{d_i^2\left(A_i^T P_{ii} + P_{ii} A_i\right)\right\}, \\ \alpha_{ij} &= \left\|d_i\, d_j\left(A_i^T P_{ij} + P_{ij} A_j\right)\right\|_2 \qquad \text{for } i,\, j = 1,2,3 \text{ and } i \neq j \\ \beta_{ii} &= \lambda_{\max}\left\{d_i\, d_{i+1}\left(P_{ii+1}\, G_{i+1i} + G_{i+1i}^T\, P_{ii+1}^T\right)\right\} \\ \gamma_{ii} &= \lambda_{\max}\left\{d_i\, d_{i+2}\left(P_{ii+2}\, G_{i+2i} + G_{i+2i}^T\, P_{ii+2}^T\right)\right\} \end{aligned}$$

here P_{ii+1}, P_{ii+2}, G_{i+1i}, and G_{i+2i} will cycle from 1 to 3.

$$\begin{aligned} \beta_{ij} &= \left\|d_i^2\left(P_{ii}\, G_{ij} + G_{ij}^T\, P_{ii}\right)\right\|_2 \\ \delta_{12} &= \left\|d_1\, d_3\left(P_{13}\, G_{32} + G_{32}^T\, P_{31}\right)\right\|_2 \\ \delta_{13} &= \left\|d_1\, d_2\left(P_{12}\, G_{23} + G_{23}^T\, P_{21}\right)\right\|_2 \\ \delta_{21} &= \left\|d_2\, d_3\left(P_{23}\, G_{31} + G_{31}^T\, P_{32}\right)\right\|_2 \\ \delta_{23} &= \left\|d_2\, d_1\left(P_{21}\, G_{13} + G_{13}^T\, P_{12}\right)\right\|_2 \\ \delta_{31} &= \left\|d_3\, d_2\left(P_{32}\, G_{21} + G_{21}^T\, P_{23}\right)\right\|_2 \\ \delta_{32} &= \left\|d_3\, d_1\left(P_{31}\, G_{12} + G_{12}^T\, P_{13}\right)\right\|_2. \end{aligned}$$

If we can find α_{ij}, β_{ij}, γ_{ij} for $i,\, j = 1,2,3$ such that the leading principal minors of the matrices

$$W = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & -\alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{bmatrix} \quad (29a)$$

and

$$W_h = \begin{bmatrix} -(\beta_{11}+\gamma_{11}) & -(\beta_{12}+\delta_{12}) & -(\beta_{13}+\delta_{13}) \\ -(\beta_{21}+\delta_{21}) & -(\beta_{22}+\gamma_{22}) & -(\beta_{23}+\delta_{23}) \\ -(\beta_{31}+\delta_{31}) & -(\beta_{32}+\delta_{32}) & -(\beta_{33}+\gamma_{33}) \end{bmatrix} \tag{29b}$$

are positive, then $\dot{\nu}(\mathbf{x}(t)) < 0$ from (28). This gives the system stability and we can find a general rule for the matrix formation.

Case 3 – Linear System with Time Delays in Both Isolation and Interconnection

Consider a linear system with multiple time delays in its ith isolation and interconnection as:

$$\dot{\mathbf{x}}_i(t) = A_i\,\mathbf{x}_i(t) + \sum_{k=1}^{2} F_{ik}\,\mathbf{x}_i(t-h_{ik}) + \sum_{j=1}^{2} B_{ij}\,\mathbf{x}_j(t) + \sum_{j=1}^{2} G_{ij}\,\mathbf{x}_j(t-\tau_{ij}). \tag{30}$$

For $i = k = 2$

$$\dot{\mathbf{x}}_1(t) = A_1\,\mathbf{x}_1(t) + F_{11}\,\mathbf{x}_1(t-h_{11}) + B_{12}\,\mathbf{x}_2(t) + G_{12}\,\mathbf{x}_2(t-\tau_{12}) \tag{31a}$$

$$\dot{\mathbf{x}}_2(t) = A_2\,\mathbf{x}_2(t) + F_{22}\,\mathbf{x}_2(t-h_{22}) + B_{21}\,\mathbf{x}_1(t) + G_{21}\,\mathbf{x}_1(t-\tau_{21}) \tag{31b}$$

where

$$A_i \in \mathbb{R}^{n_i\times n_i}, \qquad F_{ii} \in \mathbb{R}^{n_i\times n_i},$$
$$B_{ij} \in \mathbb{R}^{n_i\times n_j} \quad \text{and} \quad G_{ij} \in \mathbb{R}^{n_i\times n_j} \quad \text{for } i,j = 1,2\ (i \neq j)$$

are all real matrices.

Let the candidate Lyapunov function for the system whose subsystems are represented by (31) be:

$$\nu(\mathbf{x},t) = [d_1\ d_2] \begin{bmatrix} \nu_{11}(\cdot) & \nu_{21}(\cdot) \\ \nu_{21}(\cdot) & \nu_{22}(\cdot) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$= d_1^2\,\nu_{11}(\cdot) + 2d_1\,d_2\,\nu_{12}(\cdot) + d_2^2\,\nu_{22}(\cdot), \qquad \nu_{12}(\cdot) = \nu_{21}(\cdot). \tag{32}$$

Then

$$\dot{\nu}(\mathbf{x},t) = [d_1\ d_3] \begin{bmatrix} \nu_{11}(\cdot) & \nu_{21}(\cdot) \\ \nu_{21}(\cdot) & \nu_{22}(\cdot) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$= d_1^2\,\dot{\nu}_{11}(\cdot) + 2d_1\,d_2\,\dot{\nu}_{12}(\cdot) + d_2^2\,\dot{\nu}_{22}(\cdot) \tag{33}$$

with

$$\nu_{ii}(\mathbf{x}_i,t) = \mathbf{x}_i^T\,P_{ii}\,\mathbf{x}_i, \qquad \nu_{ij}(\mathbf{x}_i,\mathbf{x}_j,t) = \mathbf{x}_i^T\,P_{ij}\,\mathbf{x}_j.$$

From $\nu_{ii}(\mathbf{x}_i,t)$ and $\nu_{ij}(\mathbf{x}_i,\mathbf{x}_j,t)$, we can obtain $\dot{\nu}(\mathbf{x},t)$ along the (31).

$$\dot{\nu}(\cdot) = \mathbf{x}^T \begin{bmatrix} -d_1^2\,Q_1 + d_1\,d_2\,(B_{21}^T\,P_{21} + P_{12}\,B_{21}) & d_1\,d_2\,(P_{11}\,B_{12} + B_{21}^T\,P_{22} + A_1\,P_{12} + P_{12}\,A_2) \\ d_2\,d_1\,(B_{12}^T\,P_{11} + P_{22}\,B_{21} + A_2^T\,P_{21} + P_{21}\,A_1) & -d_2^2\,Q_2 + d_1\,d_2\,(B_{12}^T\,P_{12} + P_{21}\,B_{12}) \end{bmatrix} \mathbf{x}$$

$$+\mathbf{x}^T \begin{bmatrix} d_1^2 P_{11} F_{11} & d_1 d_2 P_{12} F_{22} \\ d_2 d_1 P_{21} F_{11} & d_2^2 P_{22} F_{22} \end{bmatrix} \mathbf{x}_h + \mathbf{x}_h^T \begin{bmatrix} d_1^2 F_{11} P_{11} & d_1 d_2 F_{11}^T P_{12} \\ d_2 d_1 F_{22}^T P_{21} & d_2^2 F_{22}^T P_{22} \end{bmatrix} \mathbf{x}$$

$$+\mathbf{x}^T \begin{bmatrix} d_1 d_2 P_{12} G_{21} & d_1^2 P_{11} G_{12} \\ d_2^2 P_{22} G_{21} & d_2 d_2 P_{21} G_{12} \end{bmatrix} \mathbf{x}_\tau + \mathbf{x}_\tau^T \begin{bmatrix} d_2 d_1 G_{21}^T P_{21} & d_2^2 G_{21}^T P_{22} \\ d_1^2 G_{12}^T P_{11} & d_1 d_2 G_{12}^T P_{12} \end{bmatrix} \mathbf{x} \quad (34)$$

where

$$\mathbf{x} \equiv \left[\mathbf{x}_1^T\ \mathbf{x}_2^T\right]^T, \quad \mathbf{x}_h \equiv \left[\mathbf{x}_{1h}^T\ \mathbf{x}_{2h}^T\right]^T, \quad \mathbf{x}_\tau \equiv \left[\mathbf{x}_{1\tau}^T\ \mathbf{x}_{2\tau}^T\right]^T,$$

$$\text{and} \quad \mathbf{x}_{ih} \equiv \mathbf{x}_i(t - h_{ik}), \quad \mathbf{x}_{i\tau} \equiv \mathbf{x}_i(t - \tau_{ij})$$

$$A_i^T P_{ii} + P_{ii} A_i = -Q_i, \qquad i = 1, 2.$$

If $B_{12} = B_{21}$, then

$$\dot{\nu}(\mathbf{x}) \le -\left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\alpha_{11} & -\alpha_{21} \\ -\alpha_{12} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix}$$

$$-\left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\beta_{11} & -\beta_{21} \\ -\beta_{12} & -\beta_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_{1h}\| \\ \|\mathbf{x}_{2h}\| \end{bmatrix} - \left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\gamma_{11} & -\gamma_{21} \\ -\gamma_{12} & -\gamma_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_{1\tau}\| \\ \|\mathbf{x}_{2\tau}\| \end{bmatrix} \quad (35)$$

where

$$\begin{aligned}
\alpha_{11} &= \lambda_{\max}\left[-d_1^2 Q_1 + d_1 d_2 \left(B_{21}^T P_{21} + P_{12} B_{21}\right)\right] \\
\alpha_{12} &= \alpha_{21} = \left\|d_1 d_2 \left(P_{11} B_{21} + B_{21}^T P_{22} + A_1 P_{12} + P_{12} A_2\right)\right\|_2 \\
\alpha_{22} &= \lambda_{\max}\left[-d_2^2 Q_2 + d_1 d_2 \left(B_{21}^T P_{12} + P_{21} B_{21}\right)\right] \\
\beta_{ii} &= \lambda_{\max}\left[d_i^2 \left(P_{ii} F_{ii} + F_{ii}^T P_{ii}\right)\right] \\
\beta_{12} &= \|d_1 d_2 (P_{12} F_{22} + F_{22} P_{21})\|_2 \\
\beta_{21} &= \left\|d_2 d_1 \left(P_{21} F_{11} + F_{11}^T P_{12}\right)\right\|_2 \\
\gamma_{11} &= \lambda_{\max}\left[d_1 d_2 \left(P_{12} G_{21} + G_{21}^T P_{21}\right)\right] \\
\gamma_{12} &= \left\|d_1^2 \left(P_{11} G_{12} + G_{12}^T P_{11}\right)\right\|_2 \\
\gamma_{21} &= \left\|d_2^2 \left(P_{22} G_{21} + G_{21}^T P_{22}\right)\right\|_2 \\
\gamma_{22} &= \lambda_{\max}\left[d_2 d_1 \left(P_{21} G_{12} + G_{12}^T P_{12}\right)\right]
\end{aligned}$$

and $\mathbf{x}_{ih}$, $\mathbf{x}_{i\tau}$ are $\mathbf{x}_i(t - h_{ij})$ and $\mathbf{x}_i(t - \tau_{ij})$ for $i \neq j$, respectively. If further $\mathbf{x}_{1h} = \mathbf{x}_{1\tau}$, $\mathbf{x}_{2h} = \mathbf{x}_{2\tau}$, and $d_i = 1$, $i = 1, 2$, then

$$\dot{\nu}_M(\mathbf{x}, t) = -\left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix}$$

$$-\left[\|\mathbf{x}_1^T\|\ \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\beta_{11} - \gamma_{11} & -\beta_{12} - \gamma_{12} \\ -\beta_{21} - \gamma_{21} & -\beta_{22} - \gamma_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_{1h}\| \\ \|\mathbf{x}_{2h}\| \end{bmatrix} \quad (36)$$

If the leading principal minors of

$$W = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \quad (37a)$$

and

$$W_h = \begin{bmatrix} -\beta_{11} - \gamma_{11} & -\beta_{12} - \gamma_{12} \\ -\beta_{21} - \gamma_{21} & -\beta_{22} - \gamma_{22} \end{bmatrix} \tag{37b}$$

are positive definite, then the system (31) is stable.

Remark.

(1) If we can scale $\|\mathbf{x}_1(t-h_{11})\|$ and $\|\mathbf{x}_2(t-h_{22})\|$ in (36) to $\|\mathbf{x}_1\|$ and $\|\mathbf{x}_2\|$, respectively, then the stability can be examined by the similar way to the method of Section 3.5 in Chapter 3 in [3].

(2) If $F_{ii} = 0$ for $i = 1, 2$ and $B_{12} = B_{21} = 0$ in (31), then the result is the same as the Case 1, the linear system with time delays in two interconnections. That is,

$$\begin{aligned} \dot{\nu}_M(\mathbf{x}, t) &= -\left[\|\mathbf{x}_1^T\| \; \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix} \\ &\quad - \left[\|\mathbf{x}_1^T\| \; \|\mathbf{x}_2^T\|\right] \begin{bmatrix} -\beta_{11} & -\beta_{12} \\ -\beta_{21} & -\beta_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_{1h}\| \\ \|\mathbf{x}_{2h}\| \end{bmatrix} \end{aligned}$$

equals to equation (23).

4. MATRIX LYAPUNOV FUNCTION APPLICATIONS TO NONLINEAR TIME DELAY SYSTEMS

Case 1 – Nonlinear System with Time Delays in Interconnections

Consider a large-scale system

$$\dot{\mathbf{x}} = \mathbf{f}\left[\mathbf{x}(t),\ \mathbf{x}_1(t-\overline{h}_1), \ldots, \mathbf{x}_m(t-\overline{h}_m),\ t\right], \tag{38}$$

where $\overline{h}_i = [h_{i1}\ h_{i2} \ldots h_{im}]^T$. And its ith subsystem is represented by

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}[\mathbf{x}_i, t] + \mathbf{g}_i\left[\mathbf{x}(t),\ \mathbf{x}_1(t-\overline{h}_1), \ldots, \mathbf{x}_m(t-\overline{h}_m),\ t\right]. \tag{39}$$

The candidate Lyapunov function for the system (38) can be expressed as

$$\nu(\mathbf{x}, t) = \sum_i^m \left[d_i^2\, \nu_{ii}(\mathbf{x}_i, t) + \sum_j^m d_i\, d_j\, \nu_{ij}(\mathbf{x}_i, \mathbf{x}_j, t) \right], \qquad d_i > 0. \tag{40}$$

From this

$$\dot{\nu}(\mathbf{x}, t) = \sum_i^m \left\{ d_i^2\, \dot{\nu}_{ii}(\mathbf{x}_i, t) + \sum_j^m d_i\, d_j\, \dot{\nu}_{ij}(\mathbf{x}_i, \mathbf{x}_j, t) \right\} \tag{41}$$

$$= \sum_i^m \left[d_i^2 \left(\frac{\partial \nu_{ii}(\cdot)}{\partial t} + \left(\frac{\partial \nu_{ii}(\cdot)}{\partial \mathbf{x}_i} \right)^T \dot{\mathbf{x}}_i \right) + \sum_j^m d_i d_j \left\{ \frac{\partial \nu_{ij}(\cdot)}{\partial t} + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_i} \right)^T \dot{\mathbf{x}}_i + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_j} \right)^T \dot{\mathbf{x}}_i \right\} \right]$$

$$= \sum_i^m \left[d_i^2 \left\{ \frac{\partial \nu_{ii}(\cdot)}{\partial t} + \left(\frac{\partial \nu_{ii}(\cdot)}{\partial \mathbf{x}_i}\right)^T \mathbf{f}_i(\cdot) + \left(\frac{\partial \nu_{ii}(\cdot)}{\partial \mathbf{x}_i}\right)^T \mathbf{g}_i(\cdot) \right\} \right.$$
$$+ \sum_i^m d_i d_j \left\{ \frac{\partial \nu_{ij}(\cdot)}{\partial t} + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_i}\right)^T \mathbf{f}_i(\cdot) + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_j}\right)^T \mathbf{f}_j(\cdot) \right.$$
$$\left.\left. + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_i}\right)^T \mathbf{g}_i(\cdot) + \left(\frac{\partial \nu_{ij}(\cdot)}{\partial \mathbf{x}_j}\right)^T \mathbf{g}_j(\cdot) \right\} \right].$$

Separating the terms of degree 2 from the terms of degree greater than 2,

$$\begin{aligned} \dot{\nu}_M(\mathbf{x}, t) &= q_{11}\left[\xi_1(\mathbf{x}_1), \xi_2(\mathbf{x}_2), \ldots, \xi_m(\mathbf{x}_m)\right] \\ &+ q_{12}\left[\xi_1(\mathbf{x}_1(t-h_{1j})), \xi_2(\mathbf{x}_2(t-h_{2j})), \ldots, \xi_m(t-h_{mj}))\right] \\ &+ r_{11}\left[\xi_1(\mathbf{x}_1), \xi_2(\mathbf{x}_2), \ldots, \xi_m(\mathbf{x}_m)\right] \\ &+ r_{12}\left[\xi_1(\mathbf{x}_1(t-h_{1j})), \xi_2(\mathbf{x}_2(t-h_{2j})), \ldots, \xi_m(\mathbf{x}_m(t-h_{mj}))\right] \end{aligned} \tag{42}$$

where q_{11}, q_{12} are negative definite continuous functions of degree 2 and r_{11}, r_{12} are negative definite continuous functions whose degree is greater than 2. Then the stability near the origin can be evaluated from $q_{11}(\cdot)$ and $q_{12}(\cdot)$ by constructing quadratic forms:

$$q_{11}\left[\xi_1(\mathbf{x}_1), \xi_2(\mathbf{x}_2), \ldots, \xi_m(\mathbf{x}_m)\right] = \xi^T W_1 \xi \tag{43a}$$

$$q_{12}\left[\xi_1(\mathbf{x}_1(t-h_1)), \xi_2(\mathbf{x}_2(t-h_2)), \ldots, \xi_m(\mathbf{x}_m(t-h_m))\right] = \xi_h^T W_{1h} \xi_h. \tag{43b}$$

If $W_1 < 0$ and $W_{1h} < 0$, then $\dot{\nu}_M(\mathbf{x}, t) < 0$ near the origin. This says that the system is A. S. near the origin. Similarly we can analyze the stability of a large-scale system with time delays in isolated states and interconnections.

Case 2 – Nonlinear System with Time Delay in Both Isolation and Interconnection

We are considering the following system:

$$\dot{\mathbf{x}} = \mathbf{f}\left[\mathbf{x}(t), \mathbf{x}_1(t-\overline{h}_1), \ldots, \mathbf{x}_m(t-\overline{h}_m), t\right], \tag{44a}$$

where $\overline{h}_i = [h_{i1}\, h_{i2} \ldots h_{im}]^T$. Its initial function is given as

$$\mathbf{x}(\tau) = \phi(\tau), \quad t_0 - \hat{h} \le \tau \le t_0 \tag{44b}$$

with the maximum delay $\hat{h}$ and its initial condition

$$\mathbf{f}(0, 0, \ldots, t) = 0 \tag{44c}$$

and with ith subsystem,

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}_i\left[\mathbf{x}_i, \mathbf{x}_i(t-h_{ik}), t\right] + \mathbf{g}_i\left[\mathbf{x}(t), \mathbf{x}_1(t-\overline{h}_1), \ldots, \mathbf{x}_m(t-\overline{h}_m), t\right]. \tag{44d}$$

The candidate Lyapunov function (40) for the system (44a) is used again:

$$\dot{\nu}(\mathbf{x},t) = \sum_{i}^{m} \left[d_i^2\, \dot{\nu}_{ii}(\mathbf{x}_i,t) + \sum_{j=1}^{m} d_i\, d_j\, \dot{\nu}_{ij}(\mathbf{x}_i,\, \mathbf{x}_j,\, t) \right] \tag{41}$$

The term $\dot{\nu}(\mathbf{x},t)$ using $\dot{\nu}_{ii}(\mathbf{x}_i,t)$ and $\dot{\nu}_{ij}(\mathbf{x}_i,\,\mathbf{x}_j,\,t)$ with $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{x}}_j(t)$ from (44d) is arranged to lead to the followings. We are assuming that the upper bound $\dot{\nu}_M(\mathbf{x},t)$ is obtained from (41) as

$$\dot{\nu}_M(\mathbf{x},t) = q_{21}\,[\xi_1(\mathbf{x}_1),\ldots,\xi_m(\mathbf{x}_m)] + q_{22}\,[\xi_1(\mathbf{x}_1(t-h_{1j})),\ldots,\xi_m(\mathbf{x}_m(t-h_{mj}))] +$$
$$+\, r_{21}\,[\xi_1(\mathbf{x}_1),\ldots,\xi_m(\mathbf{x}_m)] + r_{22}\,[\xi_1(\mathbf{x}_1(t-h_{1j})),\ldots,\xi_m(\mathbf{x}_m(t-h_{mj}))] \tag{45}$$

where $q_{21}(\cdot)$, $q_{22}(\cdot)$ are negative definite continuous functions of degree 2 and r_{21} and r_{22} are negative definite continuous functions of degree over 2. Similarly to the Case 1,

$$q_{21}\,[\xi_1(\mathbf{x}_1),\ldots,\xi_m(\mathbf{x}_m)] = \xi^T\, W_2\, \xi \tag{46a}$$

$$q_{22}\,[\xi_1(\mathbf{x}_1(t-h_1)),\ldots,\xi_m(\mathbf{x}_m(t-h_m))] = \xi_h^T\, W_{2h}\, \xi_h. \tag{46b}$$

Based on Theorem 2 and the Case 2 in this section, the following theorem can be established for the system (44).

Theorem 3. The equilibrium state $\mathbf{x}_e = 0$ of the time delay system (44) is asymptotically stable if $-W_2$ and $-W_{2h}$ in (46) are M-matrices, provided that $G > 0$ in (10).

Proof. From the candidate Lyapunov function (40) for the system (44), it is not difficult to get the upper bound $\dot{\nu}_M(\mathbf{x},t)$ in (45). Since $-W_2$ and $-W_{2h}$ in (46) are M-matrices and the degree of $q_{21}(\cdot)$ and $q_{22}(\cdot)$ are lower than those of $r_{21}(\cdot)$ and $r_{22}(\cdot)$, respectively, the system is guaranteed to be asymptotically stable near the origin. □

Remark. The difference of equation (45) from (42) is that the equation (45) should have time delay terms coming from the isolated part in (44d).

5. CONCLUSIONS

The stability of a large-scale system with time delays is evaluated by the way of scalar approach of a matrix Lyapunov function method and an M-matrix. By using a decomposition-aggregation method, the contribution of the interconnections between subsystems to the stability is well visualized especially in linear systems with time delays.

Sufficient conditions for asymptotic stability of the equilibrium state of the system are obtained by examination of M-matrices consisting of delay and nondelay terms together and the M-matrices here has been considered for the quadratic terms rather than the terms over degree two to watch the stability of the system near the origin.

In nonlinear time delay systems, the method can be applied to the stability analysis of large-scale nonlinear systems whose subsystems with time delays are possibly unstable. If the system could not be well formed like (43) or (46), then we could deal with the problem using a computer program [3].

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