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ON EXTENSIONS OF FUZZY TOPOLOGIES

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In this paper we introduce the concept of extension of fuzzy topologies. If (X,T) is a fuzzy topological space having the property 'P' we find conditions under which the extension of T will also have the same property 'P'.

1. INTRODUCTION

The concept of fuzzy topological spaces was introduced in [2] and ever since this introduction much research has been done in this area. In this note we introduce the concept of extension of fuzzy topological spaces as done for topological spaces in [1, 7].

Let (X,T) be a fuzzy topological space and $T \subset T^*$. Then T^* will be called a simple extension of T iff there exists a $\delta \notin T$ such that $T^* = \{\lambda \vee (\mu \wedge \delta) \mid \lambda, \mu \in T\}$. In this case we write $T^* = T(\delta)$. In this note we attempt to answer the general question:
If (X,T) has property 'P' under what conditions will $(X,(T(\delta))$ also have property 'P', where 'P' is some fuzzy topological property.

2. PRELIMINARIES

For the definition of fuzzy topological space we refer to [2]; but we follow the definition given by Lowen in [5] for Section 5 only. A fuzzy topological space (X,T) is said to be fuzzy connected [3] if it has no proper fuzzy closed set (a fuzzy set λ in X is proper if $\lambda \neq 0$ and $\lambda \neq 1$). It is said to superconnected [4] if it has no proper fuzzy regular open set. (A fuzzy open set λ is called a fuzzy regular open set [2] if $\mathrm{Int}(\lambda) = \lambda$.) It is said to be fuzzy strongly connected [2] if it has no non-zero fuzzy closed sets λ and μ such that $\lambda + \mu \leq 1$. A fuzzy set λ is fuzzy compact [5, 6] (countably compact, Lindelőf) iff for all (countable, all) family $\beta \subset T$ such that $\sup_{\mu\in\beta} \mu \geq \lambda$ and for all $\varepsilon > 0$, there exists a finite (finite, countable) sub-family $\beta_0 \subset \beta$ such that $\sup\limits_{\mu\in\beta_0} \mu > \lambda - \varepsilon$. (X,T) is said to be weakly fuzzy compact [5, 6] iff for each family $\beta \subset T$ such that $\sup\limits_{\mu\in\beta} \mu = 1$ and for each $\varepsilon > 0$, there exists a finite sub-family $\beta_0 \subset \beta$ such that $\sup\limits_{\mu\in\beta_0} \mu > 1 - \varepsilon$.

Fuzzy separation axioms are defined as follows:

T_0: For every pair of distinct points x and y in X there exists a fuzzy open set λ such that $\lambda(x) \neq \lambda(y)$ (cf. [4]),

T_1: $\Delta = \{(x,x) \in X \times X\}$ is fuzzy closed in $(X \times X,\ T \times \delta)$ where δ is the discrete fuzzy topology on X, $T \times \delta$ is the product fuzzy topology on $X \times X$ (cf. [9]),

T_2: $\Delta = \{(x,x) \in X \times X\}$ is fuzzy closed in $X \times X$ (cf. [8]).

Regularity: Every fuzzy open set λ is the union of fuzzy open sets whose closures are contained in λ (cf. [4]).

Normality: For every fuzzy closed set λ and fuzzy open set μ such that $\lambda < \mu$, there exists a set γ such that $\lambda < \operatorname{Int}(\gamma) < \operatorname{cl}(\gamma) < \mu$ (cf. [4]).

If $A \subset X$, μ_A stands for membership function associated with A.

3. BASIC PROPERTIES

(1) Let (X,T) be an fts and $T^* = T(\delta)$. Let Δ be any fuzzy subset of X. Define $T \wedge \Delta = \{\lambda \wedge \Delta \mid \lambda \in T\} = T_\Delta$. Clearly T_Δ is a fuzzy topology on X. Then

$$\operatorname{Int}_{T^*}(\Delta) = \operatorname{Int}_T(\Delta) \vee \operatorname{Int}_{T_\delta}(\Delta \wedge \delta).$$

Proof.

$$\begin{aligned}
\operatorname{Int}_{T^*}(\Delta) &= \vee\{\theta \mid \theta \in T^*,\ \theta \leq \Delta\} \\
&= \vee\{\lambda^\theta \vee (\mu^\theta \wedge \delta) \mid \lambda^\theta, \mu^\theta \in T,\ \lambda^\theta \vee (\mu^\theta \wedge \delta) \leq \Delta\} \\
&= \vee\{\lambda^\theta \mid \lambda^\theta \vee (\mu^\theta \wedge \delta) \leq \Delta\} \vee \{\mu^\theta \wedge \delta \mid \lambda^\theta \vee (\mu^\theta \wedge \delta) \leq \Delta\} \\
&= \vee\{\lambda^\theta \mid \lambda^\theta \leq \Delta,\ \lambda^\theta \in T\} \vee \{\mu^\theta \wedge \delta \mid \mu^\theta \wedge \delta \leq \Delta \wedge \delta,\ \mu^\theta \wedge \delta \in T_\delta\} \\
&= \operatorname{Int}_T(\Delta) \vee \operatorname{Int}_{T_\delta}(\Delta \wedge \delta).
\end{aligned}$$

□

(2) (a) $T^* \wedge \delta = T \wedge \delta$ (b) $1 - \delta$ is T^* closed.

(3) $\operatorname{cl}_{T^*}(\Delta) = \operatorname{cl}_T(\Delta) \wedge \{(1-\delta) \vee \operatorname{cl}_T(\Delta \vee (1-\delta))\}$.

Proof. Follows by (1), De Morgan laws and the definition of

$$\operatorname{cl}_T(\Delta) = 1 - \operatorname{Int}_T(1-\Delta).$$ □

(4) Let (X,T) be an fts, $A \subset X$, $T^* = T(\chi_A)$, $\chi_A \notin T$. Then
(i) $(A, T/A) = (A, T^*/A)$,
(ii) $(X - A, T/X - A) = (X - A, T^*/X - A)$.

(5) $\operatorname{cl}_{T^*}(\Delta \wedge \delta) = \operatorname{cl}_T(\Delta \wedge \delta)$.

Proof.

$$\mathrm{cl}_{T^*}(\Delta \wedge \delta) = \mathrm{cl}_T(\Delta \wedge \delta) \wedge \{(1-\delta) \vee \mathrm{cl}_T[(\Delta \wedge \delta) \vee (1-\delta)]\} = \mathrm{cl}_T(\Delta \wedge \delta)$$

since

$$(1-\delta) \vee \mathrm{cl}_T[(\Delta \wedge \delta) \vee (1-\delta)] \geq \mathrm{cl}_T(\Delta \wedge \delta).$$

□

Δ is T^* is closed $\Longleftrightarrow$ Δ is T closed.

4. SEPARATION AXIOMS

Regarding the fuzzy separation axioms the following results are easy to prove:

(1) $(X, T(\delta))$ is fuzzy T_0, T_1, T_2 whenever (X,T) is fuzzy T_0, T_1, T_2 respectively. This follows from the fact that $T \subset T(\delta)$.

(2) $(X, T(\delta))$ is fuzzy regular if (X,T) is fuzzy regular and δ is such that $\delta \wedge \lambda \neq 0$ for all $0 \neq \lambda \in T$.

(3) Let (X,T) be fuzzy normal and $A \subset X$ such that $\mu_A \notin T$, $1-\mu_A \in T$. Then $(X, T(\mu_A))$ is normal if $(X-A, T/X-A)$ is normal.

5. COVERING AXIOMS

Regarding the covering axioms of fuzzy topological spaces [5] we shall establish the following:

Theorem 1. Let (X,T) be fuzzy countably compact (fuzzy compact or fuzzy Lindelof) and $\delta \notin T$. Then $(X, T(\delta))$ is fuzzy countably compact (fuzzy compact, fuzzy Lindelof) iff $1-\delta$ is fuzzy countably compact (fuzzy compact or fuzzy Lindelof) in (X,T).

Proof. We prove the theorem only for countably compact case. Suppose $(X, T(\delta))$ is countably compact. Since $1-\delta$ is closed in $(X, T(\delta))$, it is countably compact in $(X, T(\delta))$. Then $1-\delta$ is fuzzy countably compact in (X,T) since $T \subset T(\delta)$. Conversely suppose $\sup_{1\leq i\leq\infty} \{[\lambda_i \vee (\mu_i \wedge \delta)]\} = 1$. Then $\sup_{1\leq i\leq\infty} \{\lambda_i\} \geq 1-\delta$ and since $1-\delta$ is fuzzy countably compact there exists a natural number N such that $\sup_{1\leq i\leq N} \{\lambda_i\} \geq (1-\delta)-\varepsilon$, where $\varepsilon > 0$ given arbitrarily. But $\sup_{1\leq i\leq\infty} \{\lambda_i \vee \mu_i\} = 1$ and thus there exists a natural number M such that $\sup_{1\leq i\leq M} \{\lambda_i \vee \mu_i\} = 1$. Then $\delta \leq \sup_{1\leq i\leq M} \{\lambda_i \vee (\mu_i \wedge \delta)\}$ and it follows that $(1-\delta)-\varepsilon+\delta < \sup_{1\leq i\leq M+N} \{\lambda_i \wedge (\mu_i \wedge \delta)\}$. As ε is arbitrary the result follows. □

6. FUZZY CONNECTIVITY AND SOME APPLICATIONS

Firstly we observe that whenever $A \subset X$ is such that $\mu_A \neq 1$ is a T-fuzzy closed set, then $(X,\, T(\mu_A))$ need not be fuzzy connected.

Example. Let X be any non-empty set. Define $T = \{\delta_1,\, \delta_2,\, \delta_3\}$ where $\delta_1 = 1$, $\delta_2 = 0$, $\delta_3 = 1/4$. Clearly T is a fuzzy topology on X and (X, T) is fuzzy connected. But if we take $\delta = 3/4$, then $T(\delta) = \{\delta_1,\, \delta_2,\, \delta_3,\, \delta\}$ and $(X,\, T(\delta))$ is clearly fuzzy disconnected. However in the following results we give a sufficient condition under which $T(\delta)$ is fuzzy connected.

Theorem 2. Let (X, T) be an fts and $A \subset X$ be such that

(1) $\mu_A \notin T$

(2) μ_A is not T-fuzzy closed

(3) $\mu_A \wedge \lambda \neq 0$ for all $0 \neq \lambda \in T$

(4) $(A,\, T/A)$ is a fuzzy connected subspace of X. Then $(X,\, T(\mu_A))$ is fuzzy connected.

Theorem 3. Let (X, T) be an fts and $A \subset X$ be such that

(1) $\mu_A \notin T$

(2) μ_A is not T-fuzzy closed

(3) $(A,\, T/A)$ and $(X - A,\, T/X - A)$ are fuzzy connected subspaces.

Then $(X,\, T(\mu_A))$ is fuzzy connected.

Proof. Suppose $\lambda_1 \vee (\mu_1 \wedge \mu_A)$ and $\lambda_2 \vee (\mu_2 \wedge \mu_A)$ are non-zero fuzzy open set of $T\,(\mu_A)$ such that

$$\lambda_1 \vee (\mu_1 \wedge \mu_A) + \lambda_2 \vee (\mu_2 \wedge \mu_A) = 1, \qquad \lambda_1,\, \lambda_2,\, \mu_1,\, \mu_2 \in T.$$

i.e. $M + N = 1$, where $M = \lambda_1 \vee (\mu_1 \wedge \mu_A) : N = \lambda_2 \vee (\mu_2 \wedge \mu_A)$.
By Corollary 1, $M \neq \mu_A$, $N \neq \mu_A$. Therefore

$$M \wedge \mu_A \neq 0 \quad \text{and} \quad N \wedge \mu_A \neq 0 \tag{1}$$

$$M - \mu_A \neq 0 \quad \text{and} \quad N - \mu_A \neq 0. \tag{2}$$

Case (1): Now $M/A = M \wedge \mu_A = \lambda_1 \vee \mu_1/A \neq 0$. Similarly $N/A = N \wedge \mu_A = \lambda_2 \vee \mu_2/A \neq 0$. And $\lambda_1 \vee \mu_1/A + \lambda_2 \vee \mu_2/A = 1$. This contradicts the fact that $(A,\, T/A)$ is fuzzy connected.

Case (2): This gives rise to a contradiction to the fact that $(X - A,\, T/X - A)$ is fuzzy connected.

Hence the theorem. □

7. APPLICATIONS

Corollary 2. Let (X,T) be an fts and $A \subset X$ such that μ_A is not fuzzy closed. If A and $X - A$ are fuzzy connected subspaces of (X,T), then (X,T) is fuzzy connected.

Proof. By Theorem 3, $(X,\ T(\mu_A))$ is fuzzy connected and since $T \subset T(\mu_A)$ it follows that (X,T) is fuzzy connected. □

The following results can be proved similarly.

Theorem 4. Let (X,T) be an fts and $A \subset X$ be such that μ_A is not fuzzy closed. If $(A,\ T/A)$ and $(X - A,\ T/X - A)$ are fuzzy superconnected subspaces, then $(X,\ T(\mu_A))$ is fuzzy superconnected.

Corollary 3. Let (X,T) be an fts and $A \subset X$ be such that μ_A is not fuzzy closed. If A and $X - A$ are fuzzy superconnected subspaces of (X,T), then (X,T) is fuzzy superconnected.

Theorem 5. Let (X,T) be an fts and $A \subset X$ be such that

(i) $\mu_A \notin T$,

(ii) $(A,\ T/A)$ is fuzzy superconnected,

(iii) $\mu_A \wedge \lambda \neq 0$ for all $\lambda \in T$.

Then $(X,\ T(\mu_A))$ is fuzzy superconnected.

Theorem 6. Let (X,T) be an fts and $A \subset X$ such that

(1) $\mu_A \notin T$,

(2) μ_A is not fuzzy closed.

If $(A,\ T/A)$ and $(X-A,\ T/X-A)$ are fuzzy strongly connected subspaces, then $(X,\ T(\mu_A))$ is strongly connected.

Corollary 4. Let (X,T) be an fts and $A \subset X$ be such that μ_A is not fuzzy closed. If A and $X - A$ are fuzzy strongly connected subspaces of (X,T), then (X,T) is fuzzy strongly connected.

Remarks. Let (X,T) be a fuzzy topological space and $\Im = \{T(\delta)\}_{\delta \in I^X - T}$ be a family of simple extensions of T. Then Δ is the $\Im$-extension of T if Δ is the smallest fuzzy topology on X which contains $T(\delta)$ for all $\delta \in I^X - T$. This is an extension of the concept introduced for topological spaces in [1]. From [1] one can infer that some of the properties of (X,T) can be carried over to (X,Δ) under certain conditions whenever Δ is a countably infinite extension of T.

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REFERENCES

[1] C. J. R. Borges: On extensions of topologies. Canad. J. Math. *19* (1967), 148–151.

[2] C. L. Chang: Fuzzy topological spaces. J. Math. Anal. Appl. *24* (1968), 182–201.

[3] U. V. Fatteh and D. S. Bassan: Fuzzy connectedness and its stronger forms. J. Math. Anal. Appl. *111* (1985), 449–464.

[4] B. W. Hutton and I. L. Reilly: Separation axioms in fuzzy topological spaces. Fuzzy Sets and Systems *3* (1980), 99–104.

[5] R. Lowen: Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. *56* (1976), 621–633.

[6] R. Lowen: A comparison of different compactness notions in fuzzy topological spaces. J. Math. Appl. *64* (1978), 446–454.

[7] N. Levine: Simple extensions of topologies. Amer. Math. Monthly *71* (1964), 22–25.

[8] R. Srivastava, S. N. Lal and A. K. Srivastava: Fuzzy Hausdorff topological spaces. J. Math. Anal. Appl. *81* (1981), 497–506.

[9] R. Srivastava, S. N. Lal and A. K. Srivastava: On fuzzy T_1-topological spaces. J. Math. Anal. Appl. *136* (1988), 124–130.

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