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NONNEGATIVE MULTIVARIATE AR(1) PROCESSES

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Conditions for nonnegativity of a p-dimensional AR(1) process $\mathbf{X}_t = \mathbf{U}\,\mathbf{X}_{t-1} + \mathbf{e}_t$ are investigated in the paper. If all the elements of the matrix $\mathbf{U}$ are nonnegative, a new method for estimating $\mathbf{U}$ is proposed. It is proved that the estimators are strongly consistent. Small-sample properties of the estimators are illustrated in a simulation study.

1. INTRODUCTION

A one-dimensional AR(1) process is given by $X_t = b\,X_{t-1} + e_t$ where e_t is a white noise and $b \in (-1, 1)$. Assume that $b \in [0, 1)$ and that e_t are nonnegative independent identically distributed random variables with a distribution function F. Then, of course, $X_t \geq 0$ for all t. Let a realization $X_1, \ldots, X_n$ be given. Then Bell and Smith [9] proved that

$$b^* = \min_{2 \leq t \leq n} X_t \,/\, X_{t-1}$$

is a strongly consistent estimator for b if and only if

$$F(d) - F(c) < 1$$

holds for all $0 < c < d < \infty$. Anděl [2] derived the distribution of b^* when e_t have an exponential distribution. Some moments of b^* in this case were calculated by Anděl and Zvára [8]. Turkman [11] presents a Bayesian analysis of the model. A generalization to the autoregressive processes of a higher order can be found in [5]. This method was also applied to nonlinear AR process (see [4] and [6]).

In the present paper we deal with multivariate AR(1) processes. First, we derive conditions under which the process is nonnegative. Second, we propose a method for estimating parameters of a nonnegative AR(1) process. It is proved that the estimators are strongly consistent.

2. PRELIMINARIES

Let $\mathbf{X}_t = (X_{t1}, \ldots, X_{tp})'$ be a p-dimensional process given by

$$\mathbf{X}_t = \mathbf{U}\,\mathbf{X}_{t-1} + \mathbf{e}_t \tag{2.1}$$

where $\mathbf{U} = (u_{ij})$ is a $p \times p$ matrix and $\mathbf{e}_t = (e_{t1}, \dots, e_{tp})'$ are random vectors. We make the following assumptions.

A1. All the roots of the matrix $\mathbf{U}$ lie inside the unit circle.

A2. The random vectors $\mathbf{e}_t$ are independent identically distributed with a distribution function F.

A3. The random vectors $\mathbf{e}_t$ have finite second moments.

Our assumptions ensure that there exists a stationary solution $\mathbf{X}_t$ of the equation (2.1) and that it can be written in the form

$$\mathbf{X}_t = \mathbf{e}_t + \mathbf{U}\,\mathbf{e}_{t-1} + \mathbf{U}^2\,\mathbf{e}_{t-2} + \dots \tag{2.2}$$

where the series converges in the quadratic mean. If we denote $\mathbf{U}^k = \left(u_{ij}^{(k)}\right)$, then (2.2) can be also expressed as

$$X_{ti} = e_{ti} + \sum_{k=1}^{\infty} \sum_{j=1}^{p} u_{ij}^{(k)}\, e_{t-k,j} \qquad (i = 1, \dots, p). \tag{2.3}$$

Let us remark that under A1 – A3 we have

$$\sum_k \sum_i \sum_j \left| u_{ij}^{(k)} \right| < \infty. \tag{2.4}$$

We denote $\mu_i = \mathsf{E}\, X_{ti},\ i = 1, \dots, p$.

3. CONDITIONS FOR NONNEGATIVITY

If all the elements u_{ij} of the matrix $\mathbf{U}$ are nonnegative and all the components e_{ti} are also nonnegative, then it is clear that $X_{ti} \geq 0$ for all t and i. On the other hand, these sufficient conditions for nonnegativity of X_{ti} are not necessary (cf. Remarks 3.3 and 3.4). It is possible to generalize the results concerning one-dimensional case introduced in Lemma 10.2 in [3] to multidimensional models.

Theorem 3.1. Assume that the distribution of $\mathbf{e}_t$ has the property that

$$\mathsf{P}\left(\sum_i c_i\, e_{ti} < \varepsilon\right) > 0 \tag{3.1}$$

holds for every $\varepsilon > 0$ and for every reals $c_1, \dots, c_p$. If there exist numbers $q \geq 1$ and $c > 0$ such that

$$\mathsf{P}\left(\sum_j u_{ij}^{(q)}\, e_{tj} < -c\right) > 0 \tag{3.2}$$

for an $i \in \{1, \dots, p\}$, then with probability 1 there exist infinitely many subscripts t such that $X_{ti} < 0$.

Proof. For $m > q$ introduce the events

$$Q_{tm1} = \left\{ \omega : \; e_{ti} + \sum_{k=1}^{m} \sum_{j=1}^{p} u_{ij}^{(k)} \, e_{t-k,j} < -\frac{c}{2} \right\},$$

$$Q_{tm2} = \left\{ \omega : \; \sum_{k=m+1}^{\infty} \sum_{j=1}^{p} u_{ij}^{(k)} \, e_{t-k,j} < \frac{c}{2} \right\}.$$

From A3 and (2.4) we get that $\mathsf{P}\,(Q_{tm2}) \longrightarrow 1$ as $m \to \infty$. Moreover, $\mathsf{P}\,(Q_{tm2})$ does not depend on t. Denote $M_{mq} = \{1, 2, \ldots, q-1, q+1, \ldots, m\}$ for $m > q$. We have

$$\mathsf{P}\,(Q_{tm1}) \geq \pi_m$$

where

$$\begin{aligned} \pi_m &= \mathsf{P}\left(e_{ti} < \frac{c}{2m}, \sum_j u_{ij}^{(k)} \, e_{t-k,j} < \frac{c}{2m} \text{ for } k \in M_{mq}, \sum_j u_{ij}^{(q)} \, e_{t-q,j} < -c \right) = \\ &= \mathsf{P}\left(e_{ti} < \frac{c}{2m} \right) \prod_{k \in M_{mq}} \mathsf{P}\left(\sum_j u_{ij}^{(k)} \, e_{t-k,j} < \frac{c}{2m} \right) \cdot \\ &\quad \cdot \mathsf{P}\left(\sum_j u_{ij}^{(q)} \, e_{t-q,j} < -c \right) > 0. \end{aligned}$$

Let w_m be the smallest integer such that $w_m \, \pi_m \geq 1$. Introduce the subsets $S_{q+2}, S_{q+3}, \ldots$ of positive integers in the following way. Let S_{q+2} contain the elements of w_{q+1} $(q+2)$-tuples $(1, \ldots, q{+}2)$, $(q{+}3, \ldots, 2q{+}4), \ldots, (1 + (w_{q+1} - 1)\,(q+2), \ldots, 2 + q + (w_{q+1} - 1)$ $(q+2))$. Let S_{q+3} contain the elements of w_{q+2} $(q+3)$-tuples starting with

$$(3 + q + (w_{q+1} - 1)\,(q+2), \ldots, 5 + 2q + (w_{q+1} - 1)\,(q+2))$$

and so on. The last terms of $(q+2)$-tuples, $(q+3)$-tuples etc. denote $t_1, t_2, \ldots$. If $t_r \in S_m$, then we use the decomposition

$$X_{t_r,i} = U_{t_r} + Z_{t_r}$$

where

$$U_{t_r} = e_{t_r,i} + \sum_{k=1}^{m-1} \sum_{j=1}^{p} u_{ij}^{(k)} \, e_{t_r-k,j},$$

$$Z_{t_r} = \sum_{k=m}^{\infty} \sum_{j=1}^{p} u_{ij}^{(k)} \, e_{t_r-k,j}.$$

Denote

$$A_r = Q_{t_r,m-1,1}, \qquad B_r = Q_{t_r,m-1,2}.$$

The events $A_1, A_2, \ldots$ are independent,

$$\sum \mathsf{P}(A_r) \geq \sum_{m=q+1}^{\infty} w_m \, \pi_m = \infty,$$

$\mathsf{P}(B_r) \to 1$ as $r \to \infty$ and the events A_r, B_r are independent. Theorem 8.1 yields that with probability 1 infinitely many events $A_r \cap B_r$ occur and thus also infinitely many events $\{X_{ti} < 0\}$. □

The roots of $\mathbf{U}$ are $\lambda_1 = 0,\ \lambda_2 = 2c$. Both of them lie inside the unit circle. Since

$$\mathbf{U}^n = 2^{n-1}\, c^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad n \geq 1,$$

we get

$$\mathbf{U}^n\, \mathbf{e}_{t-n} = 2^{n-2}\, c^n \begin{pmatrix} e_{t-n,1} \\ e_{t-n,1} \end{pmatrix}, \qquad n \geq 1,$$

From (2.2) we have

$$\begin{aligned} X_{t1} &= e_{t1} + \sum_{n=1}^{\infty} 2^{n-2}\, c^n\, e_{t-n,1}, \\ X_{t2} &= -\frac{1}{2}\, e_{t1} + \sum_{n=1}^{\infty} 2^{n-2}\, c^n\, e_{t-n,1}. \end{aligned}$$

It is clear that $X_{t1} > 0$. If we take $c = 0.4,\ a = 1,\ b = 2$, then

$$X_{t2} > -\frac{1}{2}\, b + \sum_{n=1}^{\infty} 2^{n-2}\, c^n\, a = 0.$$

4. AUXILIARY RESULTS FOR ESTIMATION

Till the end of this paper we assume that not only A1 – A3, but also the following assumptions B1 – B4 are satisfied.

B1. All the elements u_{ij} of the matrix $\mathbf{U}$ are nonnegative.

B2. Random vectors $\mathbf{e}_t$ have only nonnegative components.

B3. $\mathsf{P}\,(e_{t1} < z, \ldots, e_{tp} < z) > 0$ for all $z > 0$.

B4. There exists a number $\gamma > 0$ such that for every $\eta > 0$ and for each $i \in \{1, \ldots, p\}$

$$\mathsf{P}\,(e_{t1} < \eta, \ldots, e_{t,i-1} < \eta,\ e_{ti} > \gamma,\ e_{t,i+1} < \eta, \ldots, e_{tp} < \eta) > 0.$$

It was already pointed out that A1 – A3, B1, B2 ensure nonnegativity of all variables X_{ti}.

Remark 4.1. Let $p = 2$. If B1 holds, then $\mathbf{U}$ has only real roots. Really, an easy calculation gives

$$|\mathbf{U} - \lambda\, \mathbf{I}| = \lambda^2 - (u_{11} + u_{22})\, \lambda + u_{11}\, u_{22} - u_{12}\, u_{21}$$

and thus the roots are

$$\lambda_{12} = \frac{1}{2} \left\{ u_{11} + u_{22} \pm \left[(u_{11} - u_{22})^2 + 4 u_{12}\, u_{21} \right]^{\frac{1}{2}} \right\}.$$

Remark 4.2. The assumptions B3 and B4 are independent. This can be shown in an example with $p = 2$. If $\mathsf{P}\,(e_{t1} = 0,\, e_{t2} = 0) = 1$, then B3 is fulfilled but B4 does not hold. If $\mathsf{P}\,(e_{t1} = 0,\, e_{t2} = 5) = \mathsf{P}\,(e_{t1} = 5,\, e_{t2} = 0) = \frac{1}{2}$, then B3 is not fulfilled but B4 holds.

Remark 4.3. Consider the case $p = 2$. Let ξ_{ti} be i.i.d. random variables with exponential distribution $Ex(\lambda)$ where $i = 1,2,3$ and $t = \dots,-1,0,1,\dots$. If $e_{t1} = \xi_{t1} + \xi_{t3}$, $e_{t2} = \xi_{t2} + \xi_{t3}$, then the condition B4 is fulfilled, since

$$\begin{aligned} &\mathsf{P}\,(\xi_{t1} + \xi_{t3} < \eta,\, \xi_{t2} + \xi_{t3} > \gamma) \geq \\ \geq\ &\mathsf{P}\left(\xi_{t1} < \frac{\eta}{2},\, \xi_{t3} < \frac{\eta}{2},\, \xi_{t2} > \gamma\right) = \\ =\ &\mathsf{P}\left(\xi_{t1} < \frac{\eta}{2}\right)\, \mathsf{P}\left(\xi_{t3} < \frac{\eta}{2}\right)\, \mathsf{P}\,(\xi_{t2} > \gamma) > 0 \end{aligned}$$

for every $\eta > 0,\ \gamma > 0$. If $e_{t1} = \xi_{t1} + \xi_{t2}$, $e_{t2} = \xi_{t1}$, then $\mathsf{P}\,(\xi_{t1} + \xi_{t2} < \eta,\, \xi_{t1} > \gamma) = 0$ for every $0 < \eta < \gamma$, and thus B4 is not fulfilled.

Theorem 4.4. Define

$$u_{ij}^0 = \min_{2 \leq t \leq n} (X_{ti} \,/\, X_{t-1,j})$$

for $i, j = 1,\dots,p$. Then $u_{ij}^0 \longrightarrow u_{ij}$ a.s. as $n \to \infty$ for each $i, j \in \{1,\dots,p\}$.

Proof. First, consider the case $i = j = 1$. Since

$$X_{t1} = \sum_{\beta=1}^{p} u_{1\beta}\, X_{t-1,\beta} + e_{t1},$$

we obtain

$$u_{11}^0 = u_{11} + \min_{2 \leq t \leq n} \left(\sum_{\beta=2}^{p} u_{1\beta}\, X_{t-1,\beta} + e_{t1} \right) \Big/ X_{t-1,1}.$$

Since $X_{t-1,1} \geq e_{t-1,1}$, it is sufficient to prove that

$$\min_{2 \leq t \leq n} \left(\sum_{\beta=2}^{p} u_{1\beta}\, X_{t-1,\beta} + e_{t1} \right) \Big/ e_{t-1,1} \longrightarrow 0 \qquad \text{a.s.}$$

Let $\varepsilon > 0$ be a given number. Consider the events

$$Q_t = \left\{ \omega : \left(\sum_{\beta=2}^{p} u_{1\beta}\, X_{t-1,\beta} + e_{t1} \right) \Big/ e_{t-1,1} < \varepsilon \right\}.$$

Using (2.3) we can write

$$\begin{aligned} Q_t \ = \ &\left\{ \omega : \ e_{t1} + \sum_{\beta=2}^{p} u_{1\beta} \left(e_{t-1,\beta} + \sum_{k=2}^{m} \sum_{r=1}^{p} u_{\beta r}^{(k)}\, e_{t-k,r} \right) + \right. \\ &\left. + \sum_{\beta=2}^{p} u_{1\beta} \sum_{k=m+1}^{\infty} \sum_{r=1}^{p} u_{\beta r}^{(k)}\, e_{t-k,r} < \varepsilon\, e_{t-1,1} \right\}. \end{aligned}$$

Denote $A = 2p[1 + (p-1)(m-1)]$. It is clear that $Q_t \supset Q_{tm1} \cap Q_{tm2}$ where

$$\begin{aligned} Q_{tm1} &= \{\omega :\ e_{t-1,1} > \gamma,\ e_{t1} < \varepsilon\gamma/A,\ u_{1\beta}\, e_{t-1,\beta} < \varepsilon\gamma/A \text{ for } \beta = 2,\ldots,p; \\ &\qquad u_{1\beta}\, u_{\beta r}^{(k)}\, e_{t-k,r} < \varepsilon\gamma/A \text{ for } \beta = 2,\ldots,p,\ k = 2,\ldots,m,\ r = 1,\ldots,p\Big\}. \\ Q_{tm2} &= \{\omega :\ Z_{tm} < \varepsilon\gamma/2\} \end{aligned}$$

with

$$Z_{tm} = \sum_{\beta=2}^{p} u_{1\beta} \sum_{k=m+1}^{\infty} \sum_{r=1}^{p} u_{\beta r}^{(k)}\, e_{t-k,r}.$$

From (2.4) we can see that there exists $\Lambda > 0$ such that

$$0 \le u_{ij}^{(k)} < \Lambda \qquad \text{for all } i,\, j,\, k.$$

Therefore $\mathsf{P}(Q_{tm1}) \ge \pi_m$ where

$$\begin{aligned} \pi_m &= \mathsf{P}(e_{t1} < \varepsilon\gamma/A)\ \mathsf{P}\Big(e_{t-1,1} > \gamma,\ e_{t-1,\beta} < \frac{\varepsilon\gamma}{A\Lambda} \text{ for } \beta = 2,\ldots,p\Big) \cdot \\ &\quad \cdot \Big[\mathsf{P}\Big(e_{t-2,r} < \frac{\varepsilon\gamma}{A\Lambda^2} \text{ for } r = 1,\ldots,p\Big)\Big]^{m-1} \end{aligned}$$

Our assumptions imply that neither $\mathsf{P}(Q_{tm1})$ nor π_m depend on t. The value of γ can be chosen in such a way that $\pi_m > 0$.

It is easy to show that $\mathsf{E}\, Z_{tm} \to 0$ and $\mathsf{var}\ Z_{tm} \to 0$ as $m \to \infty$ for every fixed t. Thus $\mathsf{P}(Q_{tm2}) \to 1$. Moreover, $\mathsf{P}(Q_{tm2})$ also does not depend on t.

Let w_m be the smallest integer such that $w_m\, \pi_m \ge 1$ $(m = 2,3,\ldots)$. Let the set S_2 contain elements of j_2 triples

$$(1,\, 2,\, 3),\ \ldots,\ (3j_2 - 2,\ 3j_2 - 1,\ 3j_2),$$

let S_3 contain elements of j_3 four-tuples

$$(3j_2 + 1,\ 3j_2 + 2,\ 3j_2 + 3,\ 3j_2 + 4),\ \ldots,\ (3j_2 + 4j_3 - 3,\ \ldots,\ 3j_2 + 4j_3)$$

and so on. The last numbers of the triples, four-tuples etc. denote $t_1, t_2, \ldots$. If $t_i \in S_m$, then we define

$$A_i = Q_{t_i m 1}, \qquad\qquad B_i = Q_{t_i m 2}.$$

The events $A_1, A_2, \ldots$ are independent,

$$\sum_{i=1}^{\infty} \mathsf{P}(A_i) \ge \sum_{m=2}^{\infty} w_m\, \pi_m = \infty,$$

events A_i and B_i are independent for each i, and $\mathsf{P}(B_i) \to 1$ as $i \to \infty$. It follows from Theorem 8.1 that with probability 1 infinitely many events $A_i \cap B_i$ occur, and thus also infinitely many events Q_t.

The proof for other estimators u_{ij}^0 is quite similar. □

Although u_{ij}^0 are strongly consistent estimators for u_{ij}, our experience from similar models (see [5]) leads to the suspicion that the convergence $u_{ij}^0 \to u_{ij}$ a. s. as $n \to \infty$ is too slow and u_{ij}^0 cannot be used in practical situations as reasonable estimators. Simulations really confirmed this fact. In the next section we propose other estimators, which are also strongly consistent, but which are good for moderate values of n.

5. ESTIMATING PARAMETERS

To simplify the notation and the proofs, we describe the estimating procedure in this section only in the case $p = 2$. First, we introduce a motivation for our estimators. Let e_{t1}, e_{t2} be independent random variables such that $e_{t1} \sim Ex(\lambda_1)$, $e_{t2} \sim Ex(\lambda_2)$, where $Ex(\lambda)$ denotes the exponential distribution with the density $f(x) = \lambda^{-1} e^{-x/\lambda}$ for $x > 0$. Then the conditional likelihood of $\mathbf{X}_2, \dots, \mathbf{X}_n$, given $\mathbf{X}_1$, is

$$\lambda_1^{-n+1} \exp\left\{-\sum_{t=2}^{n} (X_{t1} - u_{11} X_{t-1,1} - u_{12} X_{t-1,2}) \,/\, \lambda_1\right\} \cdot$$

$$\cdot \ \lambda_2^{-n+1} \exp\left\{-\sum_{t=2}^{n} (X_{t2} - u_{21} X_{t-1,1} - u_{22} X_{t-1,2}) \,/\, \lambda_2\right\}$$

for

$$X_{t1} - u_{11} X_{t-1,1} - u_{12} X_{t-1,2} \geq 0, \tag{5.1}$$

$$X_{t2} - u_{21} X_{t-1,1} - u_{22} X_{t-1,2} \geq 0 \tag{5.2}$$

$(t = 2, \dots, n)$. The conditional likelihood reaches its maximum for such u_{11} and u_{12} which maximize

$$u_{11} \sum_{t=2}^{n} X_{t-1,1} + u_{12} \sum_{t=2}^{n} X_{t-1,2} \tag{5.3}$$

under the conditions (5.1) with $u_{11} \geq 0$, $u_{12} \geq 0$, and for such u_{21} and u_{22} which maximize

$$u_{21} \sum_{t=2}^{n} X_{t-1,1} + u_{22} \sum_{t=2}^{n} X_{t-1,2} \tag{5.4}$$

under the conditions (5.2) with $u_{21} \geq 0$, $u_{22} \geq 0$. Define

$$X_{n1}^0 = n^{-1} \sum_{t=1}^{n} X_{t1}, \qquad X_{n2}^0 = n^{-1} \sum_{t=1}^{n} X_{t2}.$$

If n is large then one can expect that the maximization of (5.3) and (5.4) is nearly the same as the maximization of $X_{n1}^0 u_{11} + X_{n2}^0 u_{12}$ and $X_{n1}^0 u_{21} + X_{n2}^0 u_{22}$, respectively.

Theorem 5.1. Let $u_{i1}^\star$, $u_{i2}^\star$ be a solution of the linear program LP(n)

$$\max\left(X_{n1}^0\, v_{i1} + X_{n2}^0\, v_{i2}\right) \tag{5.5}$$

under conditions

$$X_{ti} - v_{i1}\, X_{t-1,1} - v_{i2}\, X_{t-1,2} \geq 0 \qquad (t = 2, \ldots, n)$$

with $v_{i1} \geq 0$, $v_{i2} \geq 0$, for $i = 1, 2$. Then $u_{ij}^\star \to u_{ij}$ a. s. for all i, $j = 1, 2$ as $n \to \infty$.

Proof. Let $i = 1$. Assume that $u_{11} > 0$, $u_{12} > 0$. Define

$$M_n = \{(v_{11}, v_{12}) :\ v_{11} \geq 0,\ v_{12} \geq 0,\ X_{t1} - v_{11}\, X_{t-1,1} - v_{12}\, X_{t-1,2} \geq 0 \text{ for } t = 2, \ldots, n\}.$$

Let M be the oblong with vertices $(0,0)$, $(u_{11}, 0)$, (u_{11}, u_{12}), $(0, u_{12})$. It is clear that $M_2 \supset M_3 \supset \ldots$. First we prove that $M_n \to M$ a. s. We have

$$\frac{X_{t1}}{X_{t-1,1}} = u_{11} + \frac{X_{t-1,2}}{X_{t-1,1}}\, u_{12} + \frac{e_{t1}}{X_{t-1,1}}. \tag{5.6}$$

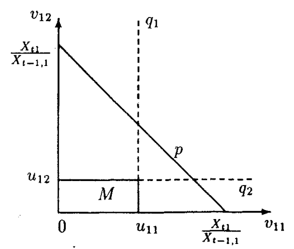

Fig. 1.

Theorem 4.4 implies that there exists a sequence t_r such that

$$X_{t_r 1} \,/\, X_{t_r - 1,1} \longrightarrow u_{11} \qquad \text{a. s.}$$

In view of (5.6) we can see that

$$X_{t_r - 1,2} \,/\, X_{t_r - 1,1} \longrightarrow u_{11} \qquad \text{a. s.} \tag{5.7}$$

Since

$$\frac{X_{t1}}{X_{t-1,2}} = u_{12} + \frac{X_{t-1,1}}{X_{t-1,2}}\, u_{11} + \frac{e_{t1}}{X_{t-1,2}},$$

using (5.7) we obtain

$$X_{t_r 1} \,/\, X_{t_r - 1,2} \longrightarrow \infty \qquad \text{a. s.}$$

In this case the straight line p in Figure 1 approaches the straight line q_1. Similarly we can prove that with probability 1 there exists a sequence of straight lines p converging to q_2.

An elementary calculation gives that p intersects q_1 at the point

$$\left(u_{11},\, u_{12} + \frac{e_{t1}}{X_{t-1,2}}\right)$$

and thus no straight line p intersects M.

Consider the linear program LP(n) (5.5) for $i = 1$. It concerns the problem

$$\max\left(X_{n1}^0\, v_{11} + X_{n2}^0\, v_{12}\right)$$

on M_n. Since $M_n \to M$ and $X_{n1}^0 \to \mu_1$, $X_{n2}^0 \to \mu_2$ a. s. (see [10], Chap. IV.2), the solutions $(u_{11}^\star, u_{12}^\star)$ of LP(n) converge a. s. to a solution of the linear program LP

$$\max\left(\mu_1\, v_{11} + \mu_2\, v_{12}\right) \tag{5.8}$$

on M. It is clear that the maximum (5.8) on M is reached at the point (u_{11}, u_{12}). Thus we have proved that $u_{11}^\star \to u_{11}$, $u_{12}^\star \to u_{12}$ a. s.

If $u_{11} = 0$ and/or $u_{12} = 0$, the proof is similar. The case $i = 2$ is quite analogous. □

6. A SIMULATION STUDY

We simulated the two-dimensional AR(1) process

$$\mathbf{X}_t = \mathbf{U}\,\mathbf{X}_{t-1} + \mathbf{e}_t$$

with

$$\mathbf{U} = \begin{pmatrix} 0.7 & 0.3 \\ 0.1 & 0.5 \end{pmatrix}.$$

The roots of $\mathbf{U}$ are $\lambda_1 = 0.8$, $\lambda_2 = 0.4$. The white noise $\mathbf{e}_t = (e_{t1},\, e_{t2})'$ was constructed in such a way that

$$e_{t1} = \ell_1\,\xi_{t1} + \ell_3\,\xi_{t3}, \qquad e_{t2} = \ell_2\,\xi_{t2} + \ell_3\,\xi_{t3}$$

where ℓ_1, ℓ_2, ℓ_3 were nonnegative constants and ξ_{ti} were nonnegative i. i. d. variables. Three distributions of ξ_{ti} were examined:

(i) exponential distribution $Ex\,(1)$;

(ii) absolutely normal distribution $AN\,(0,1)$; i. e. $\xi_{ti} = |U_{ti}|$, where $U_{ti} \sim N(0,1)$;

(iii) rectangular distribution $R(0,1)$ with the density $f(x) = 1$ for $x \in (0,1)$.

The results of simulations are summarized in Tables 1–5. In each case 100 simulations were performed. The tables contain averages of estimates of the elements of the matrix $\mathbf{U}$. The empirical standard deviations are introduced in parentheses.

Table 1

$n = 20,\ \ell_1 = \ell_2 = \ell_3 = 1,\ \xi_{ti} \sim Ex\,(1)$

$$\begin{bmatrix} 0.70 & 0.37 \\ (0.10) & (0.22) \\ 0.13 & 0.50 \\ (0.10) & (0.20) \end{bmatrix}$$

Table 2

$n = 20,\ \ell_1 = \ell_2 = \ell_3 = 1,\ \xi_{ti} \sim AN\,(0.1)$

$$\begin{bmatrix} 0.71 & 0.37 \\ (0.15) & (0.30) \\ 0.17 & 0.46 \\ (0.13) & (0.26) \end{bmatrix}$$

Table 3

$n = 20,\ \ell_1 = \ell_2 = \ell_3 = 1,\ \xi_{ti} \sim R(0,1)$

$$\begin{bmatrix} 0.68 & 0.44 \\ (0.20) & (0.41) \\ 0.19 & 0.43 \\ (0.15) & (0.29) \end{bmatrix}$$

Table 4

$n = 20,\ \ell_1 = \ell_2 = 1,\ \ell_3 = 0,\ \xi_{ti} \sim Ex\,(1)$

$$\begin{bmatrix} 0.70 & 0.33 \\ (0.05) & (0.10) \\ 0.11 & 0.51 \\ (0.06) & (0.12) \end{bmatrix}$$

Table 5

$n = 50,\ \ell_1 = \ell_2 = \ell_3 = 1,\ \xi_{ti} \sim Ex\,(1)$

$$\begin{bmatrix} 0.71 & 0.32 \\ (0.07) & (0.14) \\ 0.11 & 0.51 \\ (0.06) & (0.12) \end{bmatrix}$$

A simulation of length $n = 50$ with $\ell_1 = \ell_2 = \ell_3 = 1$ and $\xi_{ti} \sim Ex(1)$ is depicted in Figure 2.

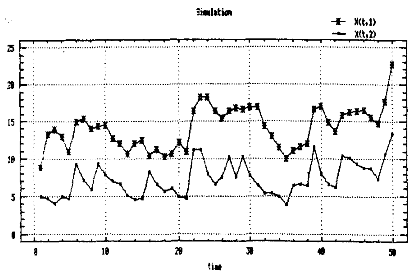

Fig. 2.

The estimate of the matrix $\mathbf{U}$ for this simulation is

$$\begin{pmatrix} 0.74 & 0.26 \\ 0.11 & 0.53 \end{pmatrix}.$$

The experience from our simulations can be briefly summarized as follows. Tables 1 – 4 show that the estimates are better when the distribution of residuals is nearer to the exponential one. This is not surprising, since our method was motivated by the maximum likelihood estimators for exponential distribution. The best results among Tables 1 – 4 are contained in Table 4. The same quality in the case $\ell_1 = \ell_2 = \ell_3 = 1$, $\xi_{ti} \sim Ex(1)$, is reached only when the length of simulation is enlarged from $n = 20$ to $n = 50$ (see Table 5).

Let us remark that the least squares estimates of the elements of the matrix $\mathbf{U}$ for the simulation depicted in Figure 2 are

$$\begin{pmatrix} 0.53 & 0.41 \\ 0.05 & 0.48 \end{pmatrix}.$$

(Of course, first of all the average of the both components of the series were substracted.) The corresponding asymptotic standard deviations are

$$\begin{pmatrix} 0.14 & 0.17 \\ 0.16 & 0.19 \end{pmatrix}.$$

In this case the estimates obtained by the new method are better. Also the empirical standard deviations introduced in Table 5 are smaller than the asymptotic standard deviations of the least squares estimates.

7. ANALYSIS OF REAL DATA

Anděl [1] presents some hydrological data about the small river Volyňka in Czechoslovakia. The mean hourly discharges of the Volyňka river (in m^2/s) and hourly rainfall in the Volyňka basin were measured for three days. The data are graphically presented in Figure 3.

Denote X_{t1} the discharges and X_{t2} the rainfall ($t = 1, \ldots, 72$). The averages are

$$\overline{x}_1 = (1/72) \sum_{t=1}^{72} X_{t1} = 31.78, \qquad \overline{x}_2 = (1/72) \sum_{t=1}^{72} X_{t2} = 0.36$$

and the empirical variances of the components are

$$s_1^2 = 207.59, \qquad s_2^2 = 0.53.$$

The least squares estimates of the autoregressive parameters are

$$\begin{pmatrix} 0.97 & 1.08 \\ 0.00 & 0.76 \end{pmatrix}$$

and their asymptotic standard deviations are

$$\begin{pmatrix} 0.025 & 0.498 \\ 0.004 & 0.075 \end{pmatrix}.$$

The residual variance matrix is

$$\begin{pmatrix} 9.37 & 0.02 \\ 0.02 & 0.21 \end{pmatrix}.$$

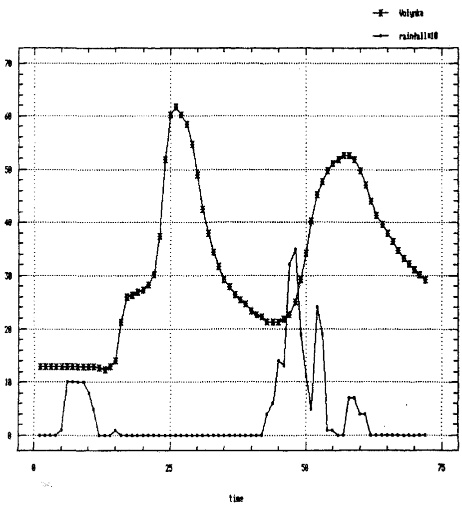

Fig. 3.

Applying our new method we get the estimate of the matrix $\mathbf{U}$

$$\begin{pmatrix} 0.87 & 1.68 \\ 0.00 & 0.00 \end{pmatrix}.$$

The residual variance of the first component is in this case 11.80.

APPENDIX

Theorem 8.1. Let two sequences of events $A_1, A_2, \dots$ and $B_1, B_2, \dots$ satisfy the following conditions:

(i) The events $A_1, A_2, \dots$ are independent.

(ii) The events A_i and B_i are independent for every $i = 1, 2, \dots$.

(iii) $\sum \mathsf{P}(A_i) = \infty$.

(iv) $\mathsf{P}(B_i) \to 1$ as $i \to \infty$.

Then with probability one infinitely many events $C_i = A_i \cap B_i$ occur.

Proof. See [7]. □

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