KYBERNETIKA — VOLUME 28 (1992), NUMBER 2, PAGES 90–99

POLE PLACEMENT AND RELATED PROBLEMS[1]

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We survey recent work in pole placement and related problems which are notably matrix completion, placement of the zeroes of a triple and feedback simulation. For each of these points we exam the existing and open field, and we point out the connection with pole placement by the use of matrix pencil formulation.

1. INTRODUCTION

A linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1}$$

is called reachable when one can assign free the poles of the closed-loop system

$$\dot{x} = (A + BF)x \tag{2}$$

obtained by applying a static state feedback $u = Fx$. Even in this case, the dynamics of the system cannot be completely modified. The basic result of Rosenbrock (cf. [11], Chap. 5, Thms 4.1 and 4.2) is that the freedom in assigning the dynamics of $A + BF$ by choosing F depends of some inequalities linking the controllability indices of (A, B) and the degrees of the desired invariant factors.

Inequalities similar to Rosenbrock's ones appear in other parts of system theory. In this way, Heymann [3] defined and solved the so-called feedback simulation problem in terms of the controllability indices of the system and of the model. This comes to a characterisation of the freedom in assigning the controllability indices of $(A + BF, BG)$ by choosing F and G. Descusse, Lafay and Malabre [1] (see also [9]) used in the context of Morgan's problem inequalities which link the infinite zero orders and the Morse's list I_2 (Morse [12]) of a triple (C, A, B) to the infinite zero orders of $(C, A + BF, BG)$ where G can be non-invertible.

Other inequalities which generalize Rosenbrock's ones appear in recent results in the area of matrix completion. Zaballa [14], [15] describes the possible values of the invariant factors of a matrix having some unknown rows.

[1]Presented at the IFAC Workshop on System Structure and Control held in Prague during 25 – 27 September 1989.

The aim of the present work is to exam these results within a unifying matrix pencil approach. This permits us to show the exact relationships linking them, given an overview on the subject and offer some open problems.

The following notations will be used. If α and β are polynomials with real coefficients, $\alpha \geq \beta$ stands for "α is divided by β". The least common multiple of α and β is denoted lcm (α, β). The degree of α is denoted $d(\alpha)$. Any finite list of m integers $n_1, n_2, \ldots, n_m$ is identified to the list of p $(p > m)$ integers $n_1, n_2, \ldots, n_m, 0, 0, \ldots, 0$. This permits us to identify the following two sums

$$\sum_{j=1}^{m} n_j \qquad \text{and} \qquad \sum_{j=1}^{p} n_j$$

especially when m is unknown. card $\{\cdot\}$ stands for the number of elements in the set $\{\cdot\}$.

We assume that the reader is familiar with the notions of poles and zeroes of a system [11], Kronecker indices of a matrix pencil [2], Morse's feedback invariants [12] and familiar with the relationships linking these concepts [7].

2. POLE PLACEMENT

We consider system (1) where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and a list $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ of monic polynomials. Assume that (A, B) is controllable and have controllability indices $\sigma_1, \sigma_2, \ldots, \sigma_m$. The basic result of Rosenbrock [11] is the following.

Theorem 1 (cf. [11], [4] for an alternative proof). There exists a feedback F such that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the invariant factors of $A + BF$ if and only if

$$\sum_{j=1}^{m} \sigma_j = \sum_{j=1}^{n} d(\alpha_j) \tag{3}$$

and

$$\sum_{j=1}^{i} \sigma_j \leq \sum_{j=1}^{i} d(\alpha_j), \qquad \text{for } i = 1, 2, \ldots . \tag{4}$$

In order to generalize this result, one can give various formulations of it. Let N be a left anihilator of B, that is any maximal rank matrix so that $NB = 0$, and $A' \in \mathbb{R}^{q \times q}$.

Proposition 1. The following claims are equivalent.

(i) There exist matrices F and T, T invertible, such that

$$T(A + BF)T^{-1} = A'. \tag{5}$$

(ii) There exist matrices P epic and Q invertible such that

$$N(s1 - A) = P(s1 - A')Q. \tag{6}$$

(iii) There exist matrices B', F, T, T invertible, such that

$$T(A+BF)T^{-1} = A' \qquad \text{and} \qquad TB = B'. \tag{7}$$

Proof. Clearly (iii) $\Longrightarrow$ (i). Assuming that (i) hold, one obtains (ii) multiplying both members of (5) by N, and defining $P = NT^{-1}$ and $Q = T$. Finally, if (ii) hold, then $N = PQ$. Thus P is a left anihilator of $B' = QB$. In addition $NA = NQ^{-1}A'Q$, so $A + BF = Q^{-1}A'Q$ for some F. The conclusion follows taking $T = Q$. □

Claim (ii) gives us a matrix pencil formulation of Theorem 1. Claim (iii) shows the reversibility of conditions (4), when one wish to assign the controllability indices of (A', B') by choosing B'. To be perfectly clear, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the invariant factors of A'.

Corollary 1. There exist matrices P epic and Q invertible satisfying (6) if and only if conditions (3) and (4) hold.

Corollary 2. There exists a matrix B' so that (A', B') is controllable and $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the controllability indices of (A', B') if and only if conditions (3) and (4) hold.

In the same way, inequalities (4) can receive various interpretations. In particular, because of the rank condition (3), (4) can be replaced by

$$\sum_{j=i}^{n} \sigma_j \geq \sum_{j=i}^{n} d(\alpha_j), \qquad \text{for } i = 1, 2, \ldots . \tag{8}$$

One should think that one of the two conditions (4) or (8) is preferable to the other one. It is not the case, the two generalizations of Theorem 1 which follow, where the rank condition (3) fall down, are pleasant illustrations of this point. A' is now any matrix having $\alpha_1, \alpha_2, \ldots, \alpha_q$ for invariant factors.

Theorem 2 (cf. [13]). There exists a state feedback F such that $A + BF$ is similar to

$$\begin{bmatrix} A' & \star \\ 0 & \star \end{bmatrix}$$

where $\star$ stands for unspecified values, if and only if conditions (8) hold.

Theorem 3. There exists a matrix B' such that $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the controllability indices of (A', B') if and only if conditions (4) hold.

Proof. Theorem 3 can be proved by direct adaptation of the proof of Corollary 2, that is the proof of Theorem 1 proposed in [11], [4]. In this proof, condition (3) only traduces the hypothesis that (A', B') is controllable. In addition, Theorem 3 can be seen as a consequence of Theorem 5. □

Theorem 1 can also be generalized to more general systems. The last result of this chapter concerns the square singular system

$$E\dot{x} = Ax + Bu, \qquad \text{where } E,\, A \in \mathbb{R}^{n\times m},\ B \in \mathbb{R}^{n\times m}. \tag{9}$$

Let us first introduce some vocabulary (see for example [5], [10]). System (9) is called

- *regular* if $sE - A$ has no column minimal indices nor row minimal indices, that is $sE - A$ is invertible.
- *proper* if $sE - A$ only has infinite elementary divisors with order 1 and finite elementary divisors. In other words, the system is regular and the inverse of $sE - A$ is proper.
- *polynomial* if $s - A$ only has infinite elementary divisors, the system is then regular and the inverse of $sE - A$ is polynomial.
- *reachable* if $N(sE - A)$ has only column minimal indices.

Assume now that (9) is reachable and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ be the column minimal indices of $N(sE - A)$.

Theorem 4. There exists a proportional and derivative feedback (F, K) such that $s(E + BK) - (A + BF)$ is regular and have finite invariant factors $\alpha_1, \alpha_2, \ldots, \alpha_n$ and infinite elementary divisors with degrees $p_1, p_2, \ldots, p_n$ if and only if

$$\sum_{j=1}^{m}(\varepsilon_j + 1) = \sum_{j=1}^{n}(d(\alpha_j) + p_j) \tag{10}$$

and

$$\sum_{j=1}^{i}(\varepsilon_j + 1) \leq \sum_{j=1}^{i}(d(\alpha_j) + p_j), \qquad \text{for } i = 1, 2, \ldots. \tag{11}$$

The proof of Theorem 4 is removed in the Appendix. This proof is based on the variable change $s = (1 + kp)/p$ which brings all the infinite zeros in finite positions: one can then apply Theorem 1.

Note that Kučera and Zagalak [5], [6] solved the problem of pole placement for singular systems by only proportional feedback, that is with $K = 0$. The remarkable fact is that in this case necessary and sufficient conditions are also provided by (10) and (11), together with the unique additional assumption that the number of infinite zeroes of $(sE - (A + BF))$ have to be equal to the number of infinite zeroes of $(sE - A)$ and so do not depend on F.

The case where $sE - A - BF$ is not assumed to be regular is studied in [16]. In this case the situation is more complicated, especially because the quantity

$$\sum_{j=1}^{n}(d(\alpha_j) + p_j)$$

cannot be assigned freely. The complete description of the solution in this case is still an open problem.

Theorems 1, 2, and 4 are only available for reachable systems, Theorem 3 gives an incomplete answer to the problem of pole placement in the non-controllable case: nothing is stated about the invariant factors of (A, B). We will see a more complete answer in the following chapter.

3. MATRIX COMPLETION

Let $A \in \mathbb{R}^{n \times n}$, with invariant factors $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ be positive integers and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ be n monic polynomials. The following was recently stated by Zaballa [14], [15].

Theorem 5. There exists a matrix B such that $\sigma_1, \sigma_2, \ldots, \sigma_m$ and $\beta_1, \beta_2, \ldots, \beta_n$ are the controllability indices and the invariant factors of (A, B) if and only if

$$\alpha_i \geq \beta_i \geq \alpha_{i+m} \qquad \text{for } i = 1, 2, \ldots \tag{12}$$

$$\sum_{j=1}^{m} \sigma_j = \sum_{j=1}^{n} d(\theta_j) \tag{13}$$

and

$$\sum_{j=1}^{i} \sigma_j \leq \sum_{j=1}^{i} d(\theta_j), \qquad \text{for } i = 1, 2, \ldots \tag{14}$$

where

$$\begin{aligned} \theta_j &= \eta^j \,/\, \eta^{j+1}, \\ \eta^j &= \eta_1^j \cdot \eta_2^j \cdot \cdots \cdot \eta_n^j, \\ \eta_i^j &= \operatorname{lcm}(\beta_j, \alpha_{i+j-1}) \qquad \text{for } i, j = 1, 2, \ldots. \end{aligned}$$

Inequalities (14) are quite different from those given in [15]. This is only due to a different ordering of the invariant factors.

Proposition 1 and Corollary 2 can convince anyone that Theorem 5 is a generalization of Theorem 1 and of Theorem 3. In fact Theorem 5 is the most achieved result in this area. An open challenge is to supress the rank condition (13) in the spirit of Theorem 2 and Theorem 3.

4. CONNECTED PROBLEMS

We will rapidly recall two results which have direct connections with the previous work. Let (A, B) and (A', B') be two controllable systems with controllability indices $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ and $\sigma'_1 \geq \sigma'_2 \geq \cdots \geq \sigma'_p$. N and N' are left anihilators of B and B'.

Theorem 6. The following claims are equivalent.

(i) There exist a matrix F, an isomorphism T and G monic such that

$$T(A + BF)T^{-1} = A'$$

and

$$TBG = B'.$$

(ii) There exist matrices P epic and Q invertible such that

$$N(s1 - A) = P\,N'(s1 - A')\,Q.$$

(iii)

$$\sum_{j=1}^{m} \sigma_j = \sum_{j=1}^{p} \sigma'_j \tag{15}$$

and

$$\sum_{j \in \mathcal{J}(i)} \sigma_j \geq \sum_{j \in \mathcal{J}'(i)} \sigma'_j, \qquad \text{for } i = 1, 2, \ldots \tag{16}$$

where

$$\mathcal{J}(i) = \text{card}\,\{j \mid \sigma_j \leq i\}, \qquad \text{for } i = 1, 2, \ldots$$

and

$$\mathcal{J}'(i) = \text{card}\,\{j \mid \sigma'_j \leq i\}, \qquad \text{for } i = 1, 2, \ldots .$$

The equivalence of (i) and (iii) is a result of Heymann [3]. Equivalence between (i) and (ii) is shown as in Proposition 1.

Let us remark that (16) specify all the possible lists of controllability indices of systems $(A + BF,\ BG)$, when (F, G) vary, even in the non-controllable case. (15) only traduces the controllability of $(A + BF,\ BG)$.

Consider now a (C, A, B) triple and let $n_1 \geq n_2 \geq \cdots \geq n_p$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ respectively be its Morse's lists I_4 (in other words the ordered list of the infinite zero orders of our triple) and Morse's list I_2 [12], and let $n'_1 \geq n'_2 \geq \cdots \geq n'_q$ be a non-increasing list of positive integers.

Theorem 7. There exist a feedback F and G monic such that $n'_1, n'_2, \ldots, n'_q$ are the infinite zero orders of $(C, A + BF,\ BG)$ if and only if

$$p - p_i \geq q - q'_i \qquad \text{for } i = 1, 2, \ldots$$

and

$$\sum_{j=1}^{i} \gamma_j \geq \sum_{j=1}^{i} \Delta_j \qquad \text{for } i = 1, 2, \ldots \tag{17}$$

where the list $\Delta_1, \Delta_2, \dots$ is obtained by reordering in a non-increasing order the sequence of differencies $p'_i - p_i$, and where

$$\begin{aligned} p_i &= \text{card}\,\{j \mid n_j \geq i\} \qquad \text{for } i = 1, 2, \dots \\ p'_i &= \text{card}\,\{j \mid n'_j \geq i\} \qquad \text{for } i = 1, 2, \dots \end{aligned}$$

and

$$\gamma_i = \text{card}\,\{j \mid \sigma_j \geq i\} \qquad \text{for } i = 1, 2, \dots .$$

Note that $p_1 = p$ and that $p'_1 = q$.

This last result was stated in [9]. Note that Theorem 5 generalized Theorem 7 if the rank of the system is not modified - in other words if $q = p$.

In that case, Zaballa's result generalizes Theorem 7 to the simultaneous placement of finite and infinite zeroes of $(C,\, A + BF,\, BG)$. This is given by Proposition 1 and the following Proposition 2. Be given two linear triples (C, A, B) and (C', A', B'), with $A,\, A' \in \mathbb{R}^{n \times n}$, let N (respectively N') be a left anihilator of B (B') and K (K') be a matrix basis of C (C').

Proposition 2 (cf. [6]). There exist matrices P epic and Q invertible such that

$$N(s1 - A)K = P \cdot N'(s1 - A')K' \cdot Q^{-1}$$

if and only if there exist a feedback F and G monic such that $(C,\, A + BF,\, BG)$ and $(C',\, A',\, B')$ have the same Morse's lists.

If then we assume that (C, A, B) is right invertible (that is $N(s1 - A)K$ has only column minimal indices $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$, finite invariant factors $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and infinite elementary divisors with degrees $n_1 \geq n_2 \geq \cdots \geq n_p$) and that (C', A', B') is invertible ($N'(s1 - A')K'$ has only elementary divisors, the finite ones form invariant factors $\alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_n$, the infinite ones have degrees $n'_1, n'_2, \dots, n'_p$), we can apply Theorem 5.

Corollary 3. There exist G monic and F such that the Morse's lists of $(C,\, A + BF,\, BG)$ are those of (C', A', B') if and only if the following conditions hold

$$\alpha'_i \geq \alpha_i \geq \alpha'_{i+m} \qquad \text{for } i = 1, 2, \dots \tag{18}$$

$$n'_i \geq n_i \geq n'_{i+m} \qquad \text{for } i = 1, 2, \dots \tag{19}$$

$$\sum_{j=1}^{m} (\sigma_j + 1) = \sum_{j=1}^{n} \zeta_j \tag{20}$$

and

$$\sum_{j=1}^{i} (\sigma_j + 1) \leq \sum_{j=1}^{i} \zeta_j, \qquad \text{for } i = 1, 2, \dots \tag{21}$$

where

$$\begin{aligned}
\zeta_j &= d(\eta^j/\eta^{j+1}) + \psi^j - \psi^{j+1}, \\
\eta^j &= \eta_1^j \cdot \eta_2^j \cdot \dots \cdot \eta_n^j, \\
\psi^j &= \psi_1^j + \psi_2^j + \dots + \psi_n^j, \\
\eta_i^j &= \ell\text{cm}\,(\alpha_i,\, \alpha'_{i+j-1}), \\
\psi_i^j &= \max\,(n_i,\, n'_{i+j-1}) \qquad \text{for } i, j = 1, 2, \dots .
\end{aligned}$$

The proof of Corollary 3 is reported in the Appendix.

Corollary 3 provides necessary and sufficient conditions for the freedom in assigning by feedback the Morse's lists of a right-invertible system (C, A, B), when the closed-loop system has to be invertible. This certainty provides a good way to solve Morgan's problem with stability. These conditions also have been used in the field of model following problem [8].

5. CONCLUSIONS

The equation

$$(sE - H) = P \cdot (sE' - H') \cdot Q^{-1} \tag{22}$$

where P is epic and Q is invertible, is the link between the different results discussed in this paper.

Rosenbrock's Theorem concern the case where $(sE - H)$ has only column minimal indices and $(sE' - H')$ has only finite elementary divisors. Zaballa's result generalizes it when $(sE - H)$ has also finite elementary divisors. Theorem 4 and Corollary 3 show how this can be extended to the case where $(sE - H)$ and $(sE' - H')$ have also infinite divisors. Finally, Heymann's Theorem corresponds to the case where the two pencil have only column minimal indices.

This unified point of view permits us to generalize Rosenbrock's result for singular systems (Theorem 4) and to generalize Theorem 7 to the simultaneous placement of finite and infinite structures of a (C, A, B) triple (Corollary 3).

It also permits us to point out some open problems. A first challenge is to give for problem (22) necessary and sufficient conditions of solvability in terms of the Kronecker's invariants of the pencil $(sE - H)$ and $(sE' - H')$, in the general case.

A second axis of research is offered when some rank conditions – the invertibility of Q-fall down, in the spirit of Theorem 3, Theorem 4 and Theorem 7.

One can hope for generalizations of these concepts to more general systems. Theorem 4 gives an example of such extension for singular systems. The formulation of the solution to control problems in terms of structure of systems offer real perspectives for non-linear systems.

Let us finally point out the importance of problem (22) for control theory. In [6], Kučera shows how the construction of linear regulators, optimal estimators or dead-beat controllers can be reduced to the problem of pole placement. Descusse, Lafay and

Malabre [1] used the zero placement [9] as a key-point to study Morgan's problem. Corollary 3 is the starting point of [8], in which is studied the model following problem. We hope for more developments in this area.

APPENDIX

Lemma 1 (cf. [10]). Two systems (A, B) and (A', B') are equivalent under proportional and derivative feedback if and only if their restricted pencils $N(s1-A)$ and $N'(s1-A')$ are equivalent in the sense of Kronecker.

Lemma 2. Let $(sH - J)$ be a matrix pencil having

- finite elementary divisors $(s-\gamma_1)^{k_{11}}, (s-\gamma_1)^{k_{12}}, \dots (s-\gamma_1)^{k_{1n}}, (s-\gamma_2)^{k_{21}}, \dots,$ $(s-\gamma_1)^{k_{tn}}$ which correspond to the invariant factors $\alpha_1, \alpha_2, \dots, \alpha_n$ like follows
$$\alpha_i = (s-\gamma_1)^{k_{1i}} \cdot (s-\gamma_2)^{k_{2i}}, \dots, (s-\gamma_t)^{k_{ti}} \qquad \text{for } i = 1, 2, \dots,$$
- infinite elementary divisors with degrees $p_1, p_2, \dots, p_n$,
- column minimal indices $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$,
- row minimal indices $\eta_1, \eta_2, \dots, \eta_p$.

Consider now the pencil $(p(kH - J) + H)$ obtained by the change of variable
$$s = (1 + kp)/p. \tag{23}$$

The Kronecker's invariants of $(p(kH - J) + H)$ are

- finite elementary divisors $(p-\phi_1)^{k_{11}}, (p-\phi_1)^{k_{12}}, \dots, (p-\phi_1)^{k_{1n}}, (p-\phi_2)^{k_{21}}, \dots,$ $(p-\phi_t)^{k_{tn}}, p^{p_1}, p^{p_2}, \dots, p^{p_n}$, where $\phi_i = \frac{1}{\alpha_i - k}$ for each i so that $\alpha_i \neq k$,
- infinite elementary divisors with degrees $k_{i1}, k_{i2}, \dots, k_{in}$, if for some i, $\alpha_i = k$,
- column minimal indices $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$,
- row minimal indices $\eta_1, \eta_2, \dots, \eta_p$.

Proof of Theorem 4. We are now able to prove Theorem 4. Using Lemma 1 and the change of variable (12), our problem comes to find P epic and Q isomorphic such that
$$N(p(kE - A) + E) = P(p(kE' - A') + E')\,Q \ .$$
where $sE' - A'$ is any matrix pencil having invariant factors $\alpha_1, \alpha_2, \dots, \alpha_n$ infinite elementary divisors with degrees $p_1, p_2, \dots, p_n$.

Without any limitation, we can assume that k is neither in the spectrum of $sE - A$ nor in the spectrum of $sE' - A'$. $kE - A$ and $kE' - A'$ are then invertible, so we are searching P epic and an isomorphism Q_1 so that
$$N(p1 + E(kE - A)^{-1}) = P(p1 - E'\left(kE' - A'\right)^{-1})\; Q_1$$

where

$$Q_1 = (kE' - A')\,Q(kE - A)^{-1}.$$

Lemma 2 and Corollary 1 then give the conclusion. □

Proof of Corollary 3. Using Proposition 2 and the change of variable (23), find T, E and G as in Corollary 3 comes to find P epic and Q invertible so that

$$P(kNK - NAK) + NK = P(p(kN'K' - N'A'K') + N'K')\,Q.$$

The conclusion then follows from Lemma 2, Proposition 1 and Theorem 5. □

(Received November 30, 1990.)

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