KYBERNETIKA — VOLUME *28* (1992), NUMBER 2, PAGES 81–89

FUNDAMENTAL THEOREM OF PROPORTIONAL STATE FEEDBACK FOR DESCRIPTOR SYSTEMS

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The limits of proportional state feedback are studied in altering the dynamics of linear descriptor systems. A necessary and sufficient condition is given for a list of polynomials and integers to represent the finite and infinite pole structure of a system obtained by proportional state feedback from the given system.

1. INTRODUCTION

We shall consider linear time-invariant systems described by

$$E\dot{x} = Ax + Bu \tag{1}$$

where E, A are $n \times n$ matrices and B is an $n \times m$ matrix over $\mathbb{R}$, the field of reals, with E possibly singular.

It is our intent to study the dynamics of the systems obtainable from (1) by applying state feedback

$$u = Fx + v \tag{2}$$

where F is an $m \times n$ matrix over $\mathbb{R}$. In particular, we are interested in determining the limits of state feedback (2) in assigning a specified eigenstructure to the closed-loop system

$$E\dot{x} = (A + BF)x + Bv. \tag{3}$$

This problem has had a long history. Rosenbrock [5] was the first who gave a result for (1) with E nonsingular, when the dynamics is fully described by finite poles. This result was extended by Kučera and Zagalak [1] to a general system (1). The result is as follows.

Suppose (1) is regular and controllable with controllability indices $n_1 \geq n_2 \geq \dots \geq n_m$. Let $c_1(s), c_2(s), \dots, c_m(s)$ be monic polynomials such that $c_{i+1}(s)$ divides $c_i(s)$, $i = 1, 2, \dots, m-1$, and

$$\sum_{i=1}^{m} \deg c_i(s) = \sum_{i=1}^{m} n_i.$$

Then there exists a state feedback (2) such that the system (3) is regular, proper (no infinite poles) and its invariant polynomials are $c_1(s), c_2(s), ..., c_m(s)$ if and only if

$$\sum_{i=1}^{j} \deg c_i(s) \geq \sum_{i=1}^{j} n_i, \quad j = 1, 2, ..., m.$$

Thus poles can be shifted to arbitrary locations but the sizes of the associated cyclic chains are limited. This is one extreme. The other extreme is where a state feedback (2) is applied to (1) so as to shift all its poles to infinity. The result, given by Zagalak and Kučera [6], is as follows.

Let (1) be regular and controllable, $n_1 \geq n_2 \geq ... \geq n_m$ the list of its controllability indices, and $z_1 \geq z_2 \geq ... \geq z_q$ the structure of its infinite zero. Let $p_1 \geq p_2 \geq ... \geq p_k$ be positive integers such that

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{m} n_i.$$

Then there exists a state feedback (2) such that the system (3) is regular, polynomial (no finite poles) and its infinite pole structure is given by $p_1, p_2, ..., p_k$ if and only if

$$m \geq k + q$$

and

$$\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} n_i, \quad j = 1, 2, ..., m$$

where $p_i = 0$ for $i > k$.

This time, not only the sizes of the cyclic chains are limited but so is the number of the cyclic chains.

In this paper we will extend the above results so as to obtain a result which will include, as a special case, the two above extremes. Such a result will be called the fundamental theorem of proportional state feedback for descriptor systems.

2. BASIC CONCEPTS

Let us first recall some notions that will be frequently used throughout the paper.

The *finite poles* of (1) are the zeros of the invariant polynomials of $sE - A$, taken all together. The structure of the *infinite pole* of (1) is given by an ordered list of integers $p_1 \geq p_2 \geq ... \geq p_k$ that appear as the positive powers of s in the Smith-McMillan form of $sE - A$ over the ring of proper rational functions in s.

The *finite zeros* of (1) are the zeros of the invariant polynomials of $\begin{bmatrix} sE - A & B \\ I_n & 0 \end{bmatrix}$, taken all together. The structure of the *infinite zero* of (1) is given by an ordered list of integers $z_1 \geq z_2 \geq ... \geq z_q$ that appear as the negative powers of s in the Smith-McMillan form of $\begin{bmatrix} sE - A & B \\ I_n & 0 \end{bmatrix}$ over the ring of proper rational functions in s.

The system (1) is said to be *regular* if $sE - A$ is nosingular. Regular systems give rise to a well defined transfer function,

$$T(s) = (sE - A)^{-1}B.$$

Closely related to the problem of pole placement is the concept of controllability. Following Lewis [3], we shall say that a regular system (1) is *controllable* if the matrix $\begin{bmatrix} sE - A, & B \end{bmatrix}$ has no finite and infinite zeros.

Let $N(s), D(s)$ be polynomial matrices over $\mathbb{R}[s]$, the ring of polynomials in the indeterminate s over $\mathbb{R}$, of respective sizes $n \times m$ and $m \times m$ such that

$$\begin{bmatrix} sE - A, & -B \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0.$$

Then the matrices $N(s), D(s)$ are said to form a (right) *normal external description* of (1) if

(a) $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is a minimal polynomial basis for $\mathrm{Ker}\begin{bmatrix} sE - A, & -B \end{bmatrix}$;

(b) $N(s)$ is a minimal polynomial basis for $P(sE - A)$ where P is a maximal anihilator of B;

(c) $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is decreasingly column-degree ordered.

See [4] for details.

The *controllability indices* of (1) are defined to be the column degrees of any normal external description of (1).

3. PROBLEM STATEMENT

Consider a regular system (1) where, to avoid trivia, we shall assume that $E \neq 0$ and $\mathrm{rank}\, B = m$. Let $c_1(s), c_2(s), ..., c_m(s)$ be monic polynomials having coefficients in $\mathbb{R}$ such that

$$c_{i+1}(s) \text{ divides } c_i(s), \quad i = 1, 2, ..., m-1. \tag{4}$$

Let further $p_1 \geq p_2 \geq ... \geq p_k$ be positive integers satisfying

$$\sum_{i=1}^{m} \deg c_i(s) + \sum_{i=1}^{k} p_i = \sum_{i=1}^{m} n_i \tag{5}$$

where $n_1, n_2, ..., n_m$ are the controllability indices of (1).

We shall consider the following problem. Does there exist a state feedback (2) such that the system (3) is regular with infinite pole structure given by $p_1, p_2, ..., p_k$ and finite pole structure given by $c_1(s), c_2(s), ..., c_m(s)$? If so, give conditions for existence and a procedure to calculate F.

4. PRELIMINARY RESULTS

This section contains lemmas that are needed to prove the fundamental theorem and, at the same time, seem to be of independent interest.

Lemma 1. Let $N(s), D(s)$ be a normal external description of a regular and controllable system (1). Then, for any $m \times n$ matrix F over $\mathbb{R}$ such that either of the matrices $D(s) - FN(s)$ or $sE - (A + BF)$ is nonsingular, the other matrix is also nonsingular and both have the same structure of finite and infinite zeros.

Proof. See [1,6]. □

Lemma 2. Let $C(s)$ be a column reduced, polynomial $m \times m$ matrix with $a_1 \geq a_2 \geq \ldots \geq a_m$ as column degrees. Let $b_1 \geq b_2 \geq \ldots \geq b_m$ be nonnegative integers satisfying

$$\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i, \quad j = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Then there exist unimodular matrices $U_1(s)$ and $U_2(s)$ such that the matrix

$$\bar{C}(s) = U_1(s) C(s) U_2(s)$$

is column reduced with column degrees $b_1, b_2, \ldots, b_m$.

Proof. If $a_i = b_i$, $i = 1, 2, \ldots, m$, put $U_1(s) = U_2(s) = I_m$. If there exists $a_i > b_i$ for some i, then there must exist $a_j < b_j$ for some $j > i$ since the sums of the two list are equal.

Then we apply Rosenbrock's lemma, see [5, Chap. 5, Lemma 4.1] or [1, Lemma 2], several times if necessary, to bring $C(s)$ to $\bar{C}(s)$. The matrices $U_1(s)$ and $U_2(s)$ are implied by this procedure. □

Lemma 3. Let $N(s), D(s)$ and $C(s)$ be $n \times m$, $m \times m$ and $m \times m$ polynomial matrices over $\mathbb{R}[s]$.

Then the equation

$$XD(s) + YN(s) = C(s)$$

has a constant solution X, Y over $\mathbb{R}$ such that X is invertible if and only if the rows of the matrices

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}, \quad \begin{bmatrix} N(s) \\ C(s) \end{bmatrix}$$

span the same $\mathbb{R}$-linear space of polynomial m-tuples.

Proof. See [2, Thm. 2]. □

5. FUNDAMENTAL THEOREM

The main result of the paper is given below and extends the fundamental theorems established in [1, Thm. 2] and [6, Thm. 1].

Theorem 1. Let (1) be a regular and controllable system, $n_1 \geq n_2 \geq \ldots \geq n_m$ the list of its controllability indices and $z_1 \geq z_2 \geq \ldots \geq z_q$ the structure of its infinite zero. Let $p_1 \geq p_2 \geq \ldots \geq p_k$ be a list of positive integers and $c_1(s), c_2(s), \ldots, c_m(s)$ a list of monic polynomials satisfying (4) and (5).

Then there exists a state feedback (2) such that (3) is regular, has the infinite pole structure given by $p_1, p_2, \ldots, p_k$ and the finite pole structure given by $c_1(s), c_2(s), \ldots, c_m(s)$ if and only if

$$m \geq k + q \tag{6}$$

and

$$\sum_{i=1}^{j} \deg c_i(s) + p_i \geq \sum_{i=1}^{j} n_i, \quad j = 1, 2, \ldots, m \tag{7}$$

where, by convention, $p_i = 0$ for $i > k$.

Proof. We shall prove the necessity of (6) and (7) first. Suppose there exists a state feedback (2) such that (3) is regular and its eigenstructuare is given by $c_1(s), c_2(s), \ldots, c_m(s)$ and $p_1, p_2, \ldots, p_k$.

To this end, consider the relation

$$\begin{bmatrix} sE - A - BF, & -B \\ I_n, & 0 \end{bmatrix} = \begin{bmatrix} sE - A, & -B \\ I_n, & 0 \end{bmatrix} \begin{bmatrix} I_n, & 0 \\ F & I_m \end{bmatrix}$$

which shows that the zero structure and the controllability of (1) are unaffected by state feedback (2).

Let $N(s), D(s)$ be a normal external description of (1) and consider the matrix $D(s) - FN(s)$. By Lemma 1, $D(s) - FN(s)$ is nonsingular and has the zero structure (finite and infinite) given by $c_1(s), c_2(s), \ldots, c_m(s)$ and $p_1, p_2, \ldots, p_k$. Moreover, the column degrees of $D(s) - FN(s)$ are the same as those of $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$, i.e., $n_1, n_2, \ldots, n_m$.

Let $\begin{bmatrix} P(w) \\ Q(w) \end{bmatrix}$ be defined by

$$\begin{bmatrix} P(w) \\ Q(w) \end{bmatrix} := \begin{bmatrix} N\left(\frac{bw}{w-a}\right) \\ D\left(\frac{bw}{w-a}\right) \quad -FN\left(\frac{bw}{w-a}\right) \end{bmatrix} \operatorname{diag}\left[\ (w-a)^{n_1}, \ldots, (w-a)^{n_m}\ \right]$$

where

$$w = \frac{as}{s-b}, \quad a \neq 0, \ b \neq 0 \tag{8}$$

defines a conformal mapping that sends the point $s = \infty$ to the point $w = a$ and the point $s = b$ to the point $w = \infty$. If $s = b$ is not a pole or a zero of (3), then

$P(w), Q(w)$ shall reflect the entire pole-zero structure of (3). Hence its transfer function $T(w) = P(w)Q^{-1}(w)$, when brought to the Smith-McMillan form, can be written as

$$\begin{aligned}&\operatorname{diag}\left[\ \frac{e_1(w)}{f_1(w)},\ \frac{e_2(w)}{f_2(w)}\ ,\dots,\ \frac{e_m(w)}{f_m(w)}\ \right]\times\\ &\operatorname{diag}\left[\ (w-a)^{-p_1},\dots,(w-a)^{-p_k},1,\dots,1,(w-a)^{z_q},\dots,(w-a)^{z_1}\ \right]\end{aligned}\qquad(9)$$

where $f_{i+1}(w)$ divides $f_i(w)$ and $e_i(w)$ divides $e_{i+1}(w)$, $i = 1,2,\dots,m-1$. Neither $e_i(w)$ nor $f_i(w)$ has a root at $w = a$. Hence it follows from (9) that $k + q \le m$, which proves (6).

By (9), the invariant polynomials of $Q(w)$ are $\bar{c}_i(w) = f_i(w)(w-a)^{p_i}$, $i = 1,2,\dots,m$ where $p_i = 0$, $i > k$. Since the product $\bar{c}_{k+1}(w)\dots\bar{c}_m(w)$ is the greatest common divisor of all minors of order $m-k$ in $Q(w)$, it easily follows that

$$\sum_{i=j+1}^{m} \deg \bar{c}_i(w) \le \sum_{i=j+1}^{m} n_i, \quad j = 0,1,\dots,m-1$$

or

$$\sum_{i=j+1}^{m} \deg f_i(w) + p_i \le \sum_{i=j+1}^{m} n_i, \quad j = 0,1,\dots,m-1. \qquad (10)$$

Now, in view of (5), the inequalities (10) can be reordererd to yield (7) on noting that $\deg f_i(w) = \deg c_i(s)$, $i = 1,2,\dots,m$.

To prove sufficiency, let $c_1(s), c_2(s),\dots,c_m(s)$ be a list of monic polynomials and $p_1,\dots,p_k$ a list of positive integers satisfying (4) and (5). We shall construct a state feedback gain F such that the system (3) will be regular with the pole structure given by $c_1(s), c_2(s),\dots,c_m(s)$ and $p_1, p_2,\dots,p_k$.

Consider again $N(s)$ and $D(s)$, a normal external description of (1), and let $s = b$ be not a root of $c_1(s), c_2(s),\dots,c_m(s)$. Applying the conformal mapping (8) to the list $c_1(s), c_2(s),\dots,c_m(s)$ and to the matrix $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$, we obtain a new list

$$f_i(w) = t_i(w-a)^{\deg c_i(s)} c_i\left(\frac{bw}{w-a}\right), \quad i = 1,2,\dots,m$$

where $t_i \in \mathbb{R}$ is introduced to make the polynomial $f_i(w)$ monic, and new matrices $N'(w), D'(w)$ defined by

$$\begin{bmatrix} N'(w) \\ D'(w) \end{bmatrix} = \begin{bmatrix} N\left(\frac{bw}{w-a}\right) \\ D\left(\frac{bw}{w-a}\right) \end{bmatrix} \quad \operatorname{diag}\left[\ (w-a)^{n_1},(w-a)^{n_2},\dots,(w-a)^{n_m}\ \right].$$

It is to be noted that

$$\left[\ \frac{bw}{w-a}E - A,\ \ -B\ \right]\begin{bmatrix} N'(w) \\ D'(w) \end{bmatrix} = 0.$$

Clearly, the matrix $\begin{bmatrix} N'(w) \\ D'(w) \end{bmatrix}$ is irreducible and column reduced with column degrees $n_1, n_2, ..., n_m$.

Now we form the $m \times m$ matrix

$$\bar{C}(w) = \operatorname{diag}\left[\ \bar{c}_1(w), \bar{c}_2(w), ..., \bar{c}_m(w)\ \right]$$

where $\bar{c}_i(w) = f_i(w)(w-a)^{p_i}$, $\quad i = 1, 2, ..., m$, and $p_i = 0$ for $i > k$. If $\deg \bar{c}_i(w) = n_i$, $\quad i = 1, 2, ..., m$, we put $\hat{C}(w) = \bar{C}(w)$. If not, then Lemma 2 implies the existence of unimodular matrices $U_1(w), U_2(w)$ such that the matrix $\hat{C}(w) := U_1(w)\bar{C}(w)U_2(w)$ is column reduced with column degrees $n_1, n_2, ..., n_m$.

Now if $\hat{C}(w)$ is such that $\begin{bmatrix} N'(w) \\ \hat{C}(w) \end{bmatrix}$ is irreducible, we put $C'(w) := \hat{C}(w)$. If not, there is a zero at $w = a$ common to $N'(w)$ and $\hat{C}(w)$. It follows that, possibly after constant column operations, the matrix $\begin{bmatrix} N'(a) \\ \hat{C}(a) \end{bmatrix}$ has a zero column. Since the unimodular operations implied by Lemma 2 keep the last $m-k$ columns of $\hat{C}(a)$ nonzero and $\mathbb{R}$-linearly independent, we conclude that only the first k columns of $\begin{bmatrix} N'(a) \\ \hat{C}(a) \end{bmatrix}$ can be zero. Hence there exists a matrix, say S, over $\mathbb{R}$ such that the matrix $\begin{bmatrix} N'(a) \\ \hat{C}(a)S \end{bmatrix}$ is of full rank. We then put $C'(w) = \hat{C}(w)S$.

The matrices $\begin{bmatrix} N'(w) \\ C'(w) \end{bmatrix}$ and $C'(w)$ are both column reduced with column degrees $n_1, n_2, ..., n_m$. Moreover, the matrix $\begin{bmatrix} N'(w) \\ C'(w) \end{bmatrix}$ is irreducible. This implies that the rows of $\begin{bmatrix} N'(w) \\ C'(w) \end{bmatrix}$ and $\begin{bmatrix} N'(w) \\ D'(w) \end{bmatrix}$ span the same $\mathbb{R}$-linear space. Then, by Lemma 3, the equation

$$XD'(w) + YN'(w) = C'(w)$$

possesses a constant solution pair X, Y such that X is nonsingular. Using (8), it can be readily seen that X, Y is also a constant solution to $XD(s) + YN(s) = C(s)$ where

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} N'\left(\dfrac{sa}{s-b}\right) \\ D'\left(\dfrac{sa}{s-b}\right) \end{bmatrix} \operatorname{diag}\left[\ (s-b)^{n_1}, (s-b)^{n_2}, ..., (s-b)^{n_m}\ \right]$$

and

$$C(s) = C'\left(\frac{sa}{s-b}\right) \operatorname{diag}\left[\ (s-b)^{n_1}, (s-b)^{n_2}, ..., (s-b)^{n_m}\ \right].$$

Then

$$F = -X^{-1}Y$$

is a state feedback gain that makes (3) regular with the desired pole structure. This immediately follows from Lemma 1. □

6. CONSTRUCTION

The major steps of the sufficiency part of the proof of Theorem 1 are summarized below.

Given $E, A, B,\ c_1(s), c_2(s), \ldots, c_m(s)$ and $p_1, p_2, \ldots, p_k$, find F.

Step 1: Calculate $N'(w)$ and $D'(w)$ such that

$$\left[\frac{bw}{w-a}E - A, \quad -B \right] \left[\begin{array}{c} N'(w) \\ D'(w) \end{array} \right] = 0$$

where $\left[\begin{array}{c} N'(w) \\ D'(w) \end{array} \right]$ is polynomial, irreducible, column reduced and decreasingly column-degree ordered.

Step 2: Read out $n_1, n_2, \ldots, n_m$, the column degrees of $\left[\begin{array}{c} N'(w) \\ D'(w) \end{array} \right]$ and q, the defect of $N'(a)$.

Step 3: Check the conditions (6) and (7).

Step 4: Construct $C'(w)$ that has the zero structure given by $c_1(s), c_2(s), \ldots, c_m(s)$ and $p_1, p_2, \ldots, p_k$ and that makes the matrix $\left[\begin{array}{c} N'(w) \\ C'(w) \end{array} \right]$ irreducible and column reduced.

Step 5: Find a constant solution pair X, Y with X nonsingular of the equation

$$XD'(w) + YN'(w) = C'(w).$$

Step 6: Put $F = -X^{-1}Y$.

Remark. The matrix F yielded by the above procedure is not the only one that solves the problem.

7. CONCLUSIONS

The dynamics of all regular systems obtainable from a given regular and controllable system by proportional state feedback have been established in Theorem 1. This result generalizes those obtained by Kučera and Zagalak [1] and Zagalak and Kučera [6] where the cases $p_1 = p_2 = \ldots = p_k = 0$ and $c_1(s) = c_2(s) = \ldots = c_m(s) = 1$, respectively, have been investigated.

We finally note that the assumption of regularity for (1) can be replaced by a weaker assumption of regularizability under the proportional state feedback (2).

(Received November 6, 1990.)

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