KYBERNETIKA — VOLUME 28 (1992), NUMBER 2, PAGES 140 – 154

SUPPORT SEPARATION THEOREMS AND THEIR APPLICATIONS TO VECTOR SURROGATE REVERSE DUALITY

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Given a closed convex subset A of a Hausdorff locally convex space X and a point $x \notin A$, does there exist a nonzero continuous linear functional $\varphi \in X^*$ such that $\varphi(x) = \sup \varphi(A)$? In this work the just defined problem is dealt with and obtained results are then applied to establish some strong duality principles concerning the surrogate reverse duality.

1. INTRODUCTION

It is well known that separation theorems play a crucial part in optimization theory (see [1 – 14] and the references therein). One of the most known separation theorems is the following:

Given a nonempty closed convex subset A of Hausdorff locally convex space X and a point $x \in X \setminus A$, there always exists a nonzero continuous linear functions $\varphi \in X^*$ such that $\varphi(x) \geq \sup \varphi(A)$.

In the last time the so-called surrogate quality has been widely used to establish dual problems to convex infimization problems, e. g. convex infimization problems with reverse convex constraints or concave infimization problems (see [15 – 22] and references therein). It is just the "unusual" surrogate duality [15, 20, 21, 22] that gave rise to the following question:

Given a nonempty closed convex subset A of Hausdorff locally convex space X and a point $x \in X \setminus A$, does there exist a nonzero continuous linear functional $\varphi \in X^*$ such that $\varphi(x) = \sup \varphi(A)$?

This problem was first studied by I. Singer (see [20 – 22]). In [21, Corollary 2] he proved the existence of such a functional under the assumptions that the space X is a normed linear space and A is a solid subset of X. Further, in [20] it was shown that the assertion still holds if A is a bounded set and X is a normable linear space.

In the present paper we shall give a satisfying answer to this question for a general case when X is a Hausdorff locally convex space.

Theorem 2.4 is an analogue to the Singer's results [21 – 22] in Hausdorff locally convex spaces. The main contribution of this work is, however, Theorem 2.16 which deals with unbounded sets A.

It should be stressed that the case of unbounded sets A has not been considered in any published work and the spaces we are concerned with here are general Hausdorff spaces and we need no normalization condition (e. g. existence of a bounded solid subset as it was implicitly supposed in [20]).

The obtained results are then used to establish some strong duality principles concerning the surrogate reverse duality in vector optimization in the last section.

2. SUPPORT SEPARATION THEOREMS

Let X be a Hausdorff locally convex space. Given a nonempty convex closed set $A \subset X$ and a point $x_0 \notin A$ we know that there exists a nonzero continuous linear functional $\psi \in X^*$ such that (see [1, § 12, F, Corollary])

$$\psi(x_0) \geq \sup \psi(A). \tag{2.1}$$

In this section we will find out conditions under which the inequality (2.1) becomes equality, i. e.

$$\psi(x_0) = \sup \psi(A). \tag{2.2}$$

It is trivially seen that if $\dim(X) = 1$ then (2.2) does not generally hold. The case $\dim(X) = 0$ is also trivial. So we shall suppose that

$$\dim(X) \geq 2. \tag{2.3}$$

2.1. Theorem. If A is a nonempty bounded closed and convex set and $x_0 \notin A$, then there exists nonzero $\psi \in X^*$ satisfying (2.2).

Proof. For the sake of simplicity we can suppose that $x_0 = \theta$ (otherwise consider the set $A - x_0$ and θ instead). Since $\theta \notin A$ there exists nonzero $\varphi_1 \in X^*$ such that

$$0 \geq \sup \varphi_1(A).$$

If $0 = \sup \varphi_1(A)$ then clearly $\psi \equiv \varphi_1$ satisfies (2.2) and we are done. Otherwise we have

$$0 > \sup \varphi_1(A). \tag{2.4}$$

Choose a point $a \in A \neq \emptyset$. Since the line

$$L(\theta, a) = \{t \cdot a : \ t \in \mathbb{R}\}$$

is a proper closed subspace of X ($\dim L(0, A) = 1 < \dim X$) we can separate it from an outside point by a nonzero $\varphi_2 \in X^*$. It is then easily checked that

$$\varphi_2(x) = 0 \qquad \forall x \in L(\theta, a). \tag{2.5}$$

If $0 = \sup \varphi_2(A)$, then clearly $\psi \equiv \varphi_2$ satisfied (2.2) and we are done. Otherwise from (2.5) it follows

$$\sup \varphi_2(A) > 0. \tag{2.6}$$

Consider the following function

$$g(t) = \sup \{t \cdot \varphi_1(x) + (1-t) \cdot \varphi_2(X) \mid x \in A\}.$$

The function g is the composition of the support functional of the nonempty closed convex set A (hence, a finite-valued convex function) with an affine function on $\mathbb{R}$ and hence is continuous. In virtue of (2.4) and (2.6) we have

$$g(0) > 0 > g(1).$$

Consequently, since $g(t)$ is continuous on $[0,1]$ there exists $t \in [0,1]$ such that $g(t) = 0$. Then obviously the following functional

$$\psi \equiv t\,\varphi_1 + (1-t)\,\varphi_2$$

in a nonzero continuous linear functional ($\psi(a) \neq 0$) and satisfies (2.2). The proof is complete. □

Now let us consider the case when A is not supposed to be bounded.

First, let us recall that a *recession cone* of a subset A of X is the set

$$C_A = \{x \in X \mid \forall a \in A\ \forall t \geq 0:\ a + tx \in A\}.$$

2.2. Lemma. If A is a nonempty closed convex set, then

$$C_A = \{x \in X \mid \exists a \in A\ \forall t \geq 0:\ a + tx \in A\}. \tag{2.7}$$

Proof. Denote the right part of (2.7) by B. Obviously $C_A \subset B$. Conversely, for $q \in B$ there exists $a \in A$ such that

$$a + t \cdot q \in A \qquad \forall t \geq 0. \tag{2.8}$$

Given an arbitrary point $b \in A$ we have to prove that

$$b + t \cdot q \in A \qquad \forall t \geq 0. \tag{2.9}$$

So, fix $c = b + t_0 \cdot q$ for some $t_0 > 0$. Let $d = (1-\alpha)\,a + \alpha \cdot c$ $(0 < \alpha < 1)$ be a point of the segment (a, c). Setting

$$t_1 = t_0 \cdot \frac{\alpha}{(1-\alpha)} \qquad \text{and} \qquad e = a + t_1 \cdot q,$$

we have, by (2.8), $e \in A$ and

$$\begin{aligned} A \ni \alpha \cdot b + (1-\alpha)\,e &= \alpha \cdot b + (1-\alpha)\left(a + t_0 \cdot \frac{\alpha}{1-\alpha} q\right) = (1-\alpha)\,a + \alpha\,(b + t_0 q) \\ &= (1-\alpha)\,a + \alpha \cdot c = d. \end{aligned}$$

Hence $d \in A$ for all $d \in [a, c)$ which entails $c \in \operatorname{lin}[A] \subset A$. Since t_0 is arbitrarily chosen, we obtain (2.9) and the proof is complete. □

2.3. Lemma. Suppose that A is a nonempty convex subset of X, $x_0 \in X \setminus \overline{A}$, $\varphi \in X^* \setminus \{\theta\}$ satisfies

$$\varphi(x_0) < \inf \varphi(\overline{A})$$

and

$$E = B \cap \{x \in X \mid \varphi(x) = \beta\},$$

where

$$B = \big\{(1-\alpha)\, x_0 + \alpha \cdot a \mid a \in \overline{A},\ 0 \le \alpha \le 1\big\}$$

and

$$\varphi(x_0) < \beta < \inf \varphi(\overline{A}).$$

Then, if E contains a half-line

$$\mathbf{h}(b; u) = \{b + t \cdot u \mid t \ge 0\}$$

the recession cone $C_{\overline{A}}$ also contains the vector u.

Proof. Since $b = b + 0 \cdot u \in E \subset B$, there exists $a \in \overline{A}$, $0 < \alpha < 1$ (for $b \ne x_0$ and $b \ne a$) such that $b = (1-\alpha)\, x_0 + \alpha \cdot a$. By Lemma 2.2 it suffices to prove that $\mathbf{h}(a; u) \subset \overline{A}$. Suppose, on the contrary, that there exists a real $t > 0$ such that $c = a + t \cdot u \notin \overline{A}$. Then by a separation Theorem (see [1, § 12, F]) there exists $\psi \in X^* \setminus \{\theta\}$ such that $\psi(c) < \inf \psi(\overline{A})$. There may be two cases.

(i) $\psi(x_0) < \inf \psi(\overline{A})$ (see Fig. 1)

Choose a real γ such that

$$\max\{\psi(c),\, \psi(x_0)\} < \gamma < \inf \psi(\overline{A}).$$

Then the hyperplane

$$H(\psi, \gamma) = \{x \in X \mid \psi(x) = \gamma\}$$

intersects $\mathbf{h}(a, u)$ and $\mathbf{h}(b, u)$ at d and e, respectively. It is easily seen that all

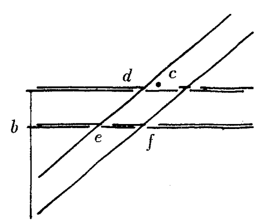

Fig. 1.

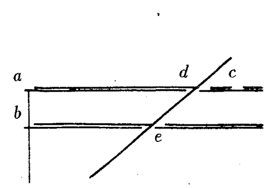

Fig. 2.

points x_0, a, b, c, d and e lie in a two-dimensional affine subspace. Hence the half line $\mathbf{h}(x_0,\, d-e)$ necessarily intersects $\mathbf{h}(b,u)$ at the point f. On the other hand, since the value of ψ on $\mathbf{h}(x_0,\, d-e)$ is $\psi(x_0) < \gamma < \inf \psi(\overline{A})$, $\mathbf{h}(x_0,\, d-e)$ does not cut through $\overline{A}$. So, the point f cannot belong to B that contradicts the hypothesis $E \supset \mathbf{h}(b,u)$.

(ii) $\psi(x_0) \geq \inf \psi(\overline{A})$ (see Fig. 2)

Choose a real γ such that

$$\psi(c) < \gamma < \inf \psi(\overline{A}).$$

The hyperplane $H(\psi,\gamma)$ intersects $\mathbf{h}(a,u)$ and $\mathbf{h}(b,u)$ at d and e, respectively. Since $\psi(a) > \gamma$, $\psi(x_0) > \gamma$ and $\psi(c) < \gamma$, we have $\psi(b) > \gamma$ and

$$\psi(u) = \psi\left(\frac{c-a}{t}\right) = \frac{1}{t}\left(\psi(c) - \psi(a)\right) < 0.$$

Hence, clearly $e = b + \tau \cdot u$ for some $\tau > 0$. Then, the half-line $\mathbf{h}(x_0,\, e-x_0)$ does not cut through $\overline{A}$ that contradicts $e \in E \subset B$. The proof is complete. □

2.4. Lemma. Suppose that all assumptions and notations of Lemma 2.3 remain valid. Then for each $b \in (\operatorname{lin} E) \setminus E$ we have

$$b - x_0 \in C_{\overline{A}}.$$

Proof. Let $b \in (\operatorname{lin} E) \setminus E$. Then there exists $c \in E$ such that the segment $[c,b) \subset E$. Since $c \in E$ there exists $a \in \overline{A}$ such that $c \in (x_0, a)$. By Lemma 2.2, to prove $b - x_0 \in C_{\overline{A}}$ it suffices to show that the half-line $\mathbf{h}(a,\, b-x_0) \subset \overline{A}$. Suppose, on the contrary, that there exists $d = a + t\cdot u \notin \overline{A}$ for $u = b - x_0$ and some $t > 0$. Then, since $d \notin \overline{A}$, there exists $\psi \in X^* \setminus \{\theta\}$ such that $\psi(d) < \inf \psi(\overline{A})$ (see [1, § 12, F]). Choose a real γ satisfying

$$\psi(d) < \gamma < \inf \psi(\overline{A}).$$

The hyperplane $H(\psi,\gamma)$ intersects $\mathbf{h}(a,u)$ and $\mathbf{h}(x_0,u)$ at e and f, respectively. There may occur the following cases.

(i) $b \in [f, x_0)$ (see Fig. 3).

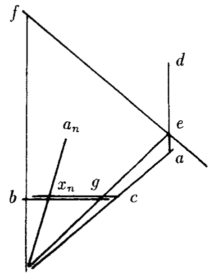

Fig. 3.

We note that the points x_0, a, b, c, d, e and f lie in a two-dimensional affine subspace. So, the segment $[x_0, e)$ necessarily intersects the segment $[c, b)$ at a point $g \in [c, b)$. Choose a sequence $\{x_n\} \subset (g, b)$ converging to b on the segment (g, b). For each x_n there exists $a_n \in \overline{A}$ such that $x_n \in (x_0, a_n)$.
Since a_n belong to the two-dimensional affine subspace containing x_0, a, b, c, d, e, f, g and $\psi(a_n) > \gamma$, a_n necessarily lie in the "quadrangle" $b\,g\,e\,f$ for all n. Consequently for each a_n, there exists nonnegative reals μ_b^n, μ_g^n, μ_e^n, μ_f^n such that

$$\mu_b^n + \mu_g^n + \mu_e^n + \mu_f^n = 1$$

and

$$\mu_b^n \cdot b + \mu_g^n \cdot g + \mu_e^n \cdot e + \mu_f^n \cdot f = a_n.$$

Since μ_b^n, μ_g^n, μ_e^n, μ_f^n are bounded for all n, we can suppose, without loss of generality, that $\mu_b^n \longrightarrow \mu_b$, $\mu_g^n \longrightarrow \mu_g$, $\mu_e^n \longrightarrow \mu_e$ and $\mu_f^n \longrightarrow \mu_f$. Setting $h = \mu_b \cdot b + \mu_g \cdot g + \mu_e \cdot e + \mu_f \cdot f$, it is clear that $a_n \longrightarrow h$ (note that X is a locally convex space), where $h \in \overline{A}$. Then, since $x_n \in (x_0, a_n)$, $x_n \longrightarrow b$ and $a_n \longrightarrow h$, we have $b \in (x_0, h)$ which contradicts $b \notin E$.

(ii) $f \in [x_0, b)$ (see Fig. 4)

Let $[e, f]$ intersects $[c, b)$ at g. Then clearly there exists no $h \in \overline{A}$ such that $g \in (x_0, h)$, a contradiction to $g \in [c, b) \subset E \subset B$.

(iii) $x \in (f, b)$ (see Fig. 5)

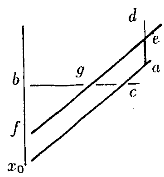

Fig. 4.

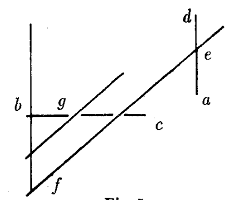

Fig. 5.

Set $v = e - f$. The half-line $\mathbf{h}(x_0,\, v)$ necessarily intersects $[c, b)$ at a point $g \in [c, b)$. It is easily seen that there exists no $h \in \overline{A}$ with $g \in (x_0,\, h)$ (for the value of ψ on $\mathbf{h}(x_0,\, v)$ equal to $\psi(x_0) < \gamma < \inf \psi(\overline{A})$), which contradicts $g \in [c, b) \subset E \subset B$.

We have exhausted all possible cases and the achieved contradictions prove that the assumption

$$a + t\,u \notin \overline{A} \qquad \text{for some } t > 0,\ u = b - x_0$$

fails. Hence $\mathbf{h}(a, u) \subset \overline{A}$. The proof is complete. □

2.5. Lemma. Suppose that all assumptions and notations of Lemma 2.3 remain valid. Furthermore, let B be solid and E contain a boundary point of B. There exists $\psi \in X^* \setminus \{\theta\}$ such that

$$\psi(x_0) = \sup \psi(\overline{A}) = \sup \psi(B).$$

Proof. Let $b \in E$ be a boundary point of B. Then, since B is solid and convex (B is nothing else than the convex hull of $\{x_0\} \cup \overline{A}$), there exists $\psi \in X^* \setminus \{\theta\}$ such that $\psi(b) = \sup \psi(B)$. Let $a \in \overline{A}$ such that $b \in (x_0, a)$. Then from $\psi(a) \le \psi(b)$, $\psi(x_0) \le \psi(b)$ and $\psi(b) = \alpha \cdot \psi(x_0) + (1 - \alpha) \cdot \psi(a)$ for some $0 < \alpha < 1$ it follows

$$\sup \psi(\overline{A}) \ge \psi(a) = \psi(b) = \sup \psi(B) \ge \sup(\overline{A}).$$

Hence $\psi(x_0) = \psi(a) = \sup \psi(\overline{A}) = \sup \psi(B)$. □

2.6. Theorem. Suppose that A is a nonempty convex subset of X, $x_0 \notin \overline{A}$, the convex hull of x_0 and $\overline{A}$, denoted by $\operatorname{co}(x_0, \overline{A})$, is solid, $C_{\overline{A}}$ does not contain a hyperplane and $\overline{A} \notin \operatorname{int}(x_0 + C_{\overline{A}})$. Then there exists $\psi \in X^* \setminus \{\theta\}$ such that $\psi(x_0) = \sup \psi(\overline{A})$.

Proof. Since all hypotheses of Lemma 2.3 are satisfied we can choose a $\varphi \in X^* \setminus \{\theta\}$ and construct the sets B and E as in Lemma 2.3. Note that $B = \operatorname{co}(x_0, \overline{A})$. Now, since B is solid and $C_{\overline{A}}$ does not contain a hyperplane, the convex set E has a nonempty relative interior, i.e. rel-int $(E) \neq \emptyset$ and aff $(E) \setminus \bar{E} \neq \emptyset$. So, by [1, § 11, A],

$\text{lin}(E) = \bar{E} \neq \text{icr}(E) = \text{rel-int}(E)$ and the topological boundary of E coincides with its algebraic boundary. If E does not contain any of its boundary points, by Lemma 2.3 and 2.4 we have $\overline{A} \subset \text{int}(x_0, C_{\overline{A}})$, a contradiction to the assumption $A \not\subset \text{int}(x_0, C_{\overline{A}})$. Consequently, E must contain a boundary point of itself. It is clear that this boundary point is also a boundary point of the set B and by Lemma 2.5 we obtain the assertion.□

2.7. Remark 1. Condition $A \not\subset \text{int}(x_0 + C_{\overline{A}})$ is not necessary as it is shown by the following example. Let $X = \mathbb{R}^2$, $x_0 = (0,0)$ and

$$\overline{A} = A = \{(x,y) \mid x > 0,\ y > 0,\ x \cdot y \geq 1\}$$

(see Fig. 6).

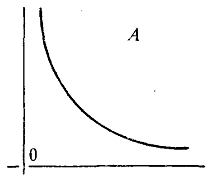

Fig. 6.

Obviously $C_{\overline{A}} = \mathbb{R}^2$, and $\overline{A} \subset \text{int}(x_0 + C_{\overline{A}}) = \text{int}\,\mathbb{R}^2_+ = \{(x,y) \mid x > 0,\ y > 0\}$, i.e. the mentioned condition fails. But the continuous linear nonzero functional $\psi(x,y) = -y$ satisfies $\psi(x_0) = 0 = \sup \psi(\overline{A})$.

2.8. Remark 2. The statement of Lemma 2.3 may be reduced to say, since $E \subseteq \text{co}\left(x_0 \cup \overline{A}\right)$,

$$C_E \subseteq C_{\overline{\text{co}}(x_0 \cup \overline{A})} = C_{\overline{A}}.$$

In the latter equality, the inclusion $\supseteq$ is obvious, while $\subseteq$ follows from the following reasoning.

Let $u \in C_{\overline{\text{co}}(x_0 \cup \overline{A})}$, $a \in \overline{A}$ and let V be a balanced neighborhood of 0. Take a convex neighborhood V' of 0 such that $V' + V' \subseteq V$ and let $\lambda > 1$ be such that

$$\lambda^{-1}(x_0 - a) \in V'.$$

Since $a + \lambda \cdot u \in \overline{\text{co}}\left(x_0 \cup \overline{A}\right)$, there exist $\alpha \in [0,1]$, $a' \in \overline{A}$ and $v' \in V'$ such that

$$a + \lambda \cdot u = (1 - \alpha) \cdot x_0 + \alpha \cdot a' + \lambda \cdot v'.$$

Then

$$a + u = \left(1 - \alpha \cdot \lambda^{-1}\right) a + \alpha \cdot \lambda^{-1} a' + (1 - \alpha)\,\lambda^{-1}(x_0 - a) + v',$$

which shows that $a+u+v$ intersects $\overline{\text{co}}\,(x_0 \cup \overline{A})$. Thus,

$$a+u \in \overline{\text{co}}\,(x_0 \cup \overline{A}).$$

This proves that $u \in C_{\overline{\text{co}}(x_0 \cup \overline{A})}$.

3. AN APPLICATION TO VECTOR SURROGATE REVERSE DUALITY

First, let us introduce some necessary notions and notations. In the sequel $\overline{\mathbb{R}}^m$ denotes the extension of $\mathbb{R}^m$, i.e.

$$\overline{\mathbb{R}}^m = \{(u_1,\dots,u_m) \mid u_i \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}\ \forall i=1,\dots,m\}.$$

Further let us denote

$$\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\} \qquad \text{and} \qquad \mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}.$$

For $u=(u_1,\dots,u_m)$ and $v=(v_1,\dots,v_m)$ from $\overline{\mathbb{R}}^m$ we define

$$\begin{aligned} u \geq v \quad &\text{iff} \quad u_i \geq v_i \quad \forall i=1,\dots,m;\\ u \gg v \quad &\text{iff} \quad u_i > v_i \quad \forall i=1,\dots,m;\\ u > v \quad &\text{iff} \quad u \geq v \quad \text{and } u \neq v. \end{aligned}$$

Let $A \subset \overline{\mathbb{R}}^m$ be a nonempty subset. We denote

$$\begin{aligned} {}^{\leq}\!A &= \{u \in \overline{\mathbb{R}}^m \mid \exists v \in A:\ u \leq v\}\\ \text{acun}\,A &= {}^{\leq}\!A \cup \{u \in \overline{\mathbb{R}}^m \mid \exists v \ll u \ \&\, \forall v \ll u\, \exists w \in A:\ v \ll w\} \end{aligned}$$

and

$$\text{Sup}\,A = \{u \in \text{acun}\,A \mid \forall v \in \text{acun}\,A:\ \neg(v > u)\}.$$

Interpretation: If A is a set of vector utility disposals, then acun A can be interpreted as the set of "free and approximately admissible disposals" and Sup A consists of optimal disposals in the sense of the following properties.

3.1. Lemma. Sup A is inside-stable, i.e.

$$\forall u, v \in \text{Sup}\,A:\ \neg(u > v).$$

Proof. This property is clear from definition. □

3.2. Lemma. If $A \subset \mathbb{R}^m_{+\infty}$, then $\operatorname{Sup} A$ is sup-stable to acun A, i. e.

$$\forall a \in \operatorname{acun} A \; \exists u \in \operatorname{Sup} A : \; u \geq a.$$

Proof. Let $a \in \operatorname{acun} A$. Consider the set $B = \operatorname{acun} A \cap \, \overset{\geq}{}\{a\}$.
Let $C \subset B$ be a nonempty chain, i. e.

$$\forall u, v \in C : \; u \geq v \quad \text{or} \quad v \geq u.$$

For $i = 1, \ldots, m$ we define

$$u_i^* = \sup \{u_i \,|\, \exists u \in C : \; u_i \text{ is the } i\text{th component of } u\}$$

and set

$$u^* = (u_1^*, \ldots, u_m^*).$$

It is clear that

$$\forall u \in C : \; u \leq u^*. \tag{3.1}$$

We show that $u^* \in B$. Indeed, since $A \subset \mathbb{R}^m_{+\infty}$, u^* also belongs to $\mathbb{R}^m_{+\infty}$. Consequently there exists $u \in \mathbb{R}^m$, $u \ll u^*$. Taking into consideration that C is a chain, from definition of u^* we deduce the existence of a point $u^1 \in C$, with $u \ll u^1$. Since $u^1 \in C \subset B \subset$ acun A there exists a point $u^2 \in A$ with $u \ll u^2$, which means u^*, in its turn, also belongs to acun A. The fact that $u^* \geq a$ is obvious. Consequently

$$u^* \in \operatorname{acun} A \cap \, \overset{\geq}{}\{a\} = B.$$

So, by (3.1), u^* is an upper bound of C in B. From the Zorn Lemma it follows that B possesses a maximal element that clearly belongs to $\operatorname{Sup} A$ and is obviously greater than or equal to a. The proof is complete. □

Now given a set X, a subset $C \subset X$ and a map

$$F = (f_1, \ldots, f_m) : X \longrightarrow \overline{\mathbb{R}}^m,$$

we shall be concerned with the global vector supremal Problem 3.3

$$\mathcal{A} = \operatorname{Sup} F(C).$$

Let

$$\{\Delta_d \,|\, d \in \mathcal{D}\}$$

be a family of subsets of X. Then the vector supremal Problem 3.4

$$\mathcal{B} = \operatorname{Sup}_{d \in \mathcal{D}} \operatorname{Sup}_{x \in \Delta_d} F(x)$$

is called a *first type surrogage reverse dual problem* to (3.3). Problems 3.3 and 3.4 are *placed in strong reverse duality* if $\mathcal{A} = \mathcal{B}$.

With help of results of Section 2 we shall establish the strong reverse duality for a particular case when Δ_d, $d \in \mathcal{D}$, are closed hyperplanes and C is a complement of a convex set.

But first let us formulate a general criterion.

3.5. Lemma. Let $\{A_\lambda \,|\, \lambda \in \Lambda\}$ be a (finite or infinite) family of subsets of $\overline{\mathbb{R}}^m$. Then

$$\text{acun}\left[\bigcup_\Lambda A_\lambda\right] = \text{acun}\left[\bigcup_\Lambda \text{Sup}\, A_\lambda\right] \tag{3.2}$$

and, hence,

$$\text{Sup}\left[\bigcup_\Lambda A_\lambda\right] = \text{Sup}\left[\bigcup_\Lambda \text{Sup}\, A_\lambda\right]. \tag{3.3}$$

Proof. (i) Let $y_0 \in$ acun $[\bigcup_\Lambda A_\lambda]$. Then either $y_0 \in {}^{\leq}A_\lambda$, for some $\lambda' \in \Lambda$, or

$$\exists\, y \ll y_0 \;\&\; \forall\, y \ll y_0 \;\exists\, \lambda' \in \Lambda \;\exists\, y' \in A_{\lambda'} :\; y \ll y'. \tag{3.4}$$

In the first case, since Sup A_λ, is sup-stable to A_λ, by Lemma 3.2, we have

$$y_0 \in {}^{\leq}\text{Sup}\, A_{\lambda'} \subset \text{acun}\left[\bigcup_\Lambda \text{Sup}\, A_{\lambda'}\right].$$

In the second case, from (3.4) and the sup-stability of Sup A_λ to A_λ for all $\lambda \in \Lambda$, it follows

$$\exists\, y \ll y_0 \;\&\; \forall\, y \ll y_0 \;\exists\, \lambda' \in \Lambda \;\exists\, y'' \in \text{Sup}\, A_{\lambda'} :\; y_0 \ll y'', \tag{3.5}$$

which means

$$y_0 \in \text{acun}\left[\bigcup_\Lambda \text{Sup}\, A_\lambda\right].$$

(ii) Conversely, let $y_0 \in \text{acun}\left[\bigcup_\Lambda \text{Sup}\, A_\lambda\right]$. Then either $y_0 \in {}^{\leq}\text{Sup}\, A_\lambda$, for some $\lambda' \in \Lambda$, or

$$\exists\, y \ll y_0 \;\&\; \forall\, y \ll y_0 \;\exists\, \lambda' \in \Lambda \;\exists\, y'' \in \text{Sup}\, A_{\lambda'} :\; y \ll y''.$$

In the first case

$$y_0 \in \text{acun}\, A_{\lambda'} \subset \text{acun}\left[\bigcup_\Lambda A_\lambda\right].$$

In the second case, since $y'' \in \text{Sup}\, A_{\lambda'} \subset \text{acun}\, A_\lambda$, there exists $y' \in A_{\lambda'}$ such that $y \ll y'$. Consequently, we obtain again (3.4) which means

$$y_0 \in \text{acun}\left[\bigcup_\Lambda A_\lambda\right].$$

□

From Lemma 3.5 we immediately obtain the following statement.

3.6. Lemma. For any family $\{\Delta_d \,|\, d \in \mathcal{D}\}$ we have

$$\mathcal{B} = \text{Sup}\left[F\left(\bigcup_{\mathcal{D}} \Delta_d\right)\right]. \tag{3.6}$$

3.7. Theorem. Let X be a topological space. If either

(i) $C = \bigcup_{\mathcal{D}} \Delta_d$
or

(ii) $C \subset \bigcup_{\mathcal{D}} \Delta_d \subset \overline{C}$, f_i are lower semicontinuous for all $i = 1, \ldots, m$, and $F(\overline{C}) \subset \mathbb{R}^m_{+\infty}$.

Then Problems 3.3 and 3.4 are placed in strong reverse duality.

Proof. If (i) holds then by Lemma 3.6 it is clear that $\mathcal{A} = \mathcal{B}$.
If (ii) holds then from definition it follows

$$\text{acun}\, F(C) \subset \text{acun}\, F\left(\bigcup_{\mathcal{D}} \Delta_d\right) \subset \text{acun}\, F(\overline{C}) \subset \text{acun}\, F(C),$$

hence

$$\text{acun}\, F(C) = \text{acun}\, F\left(\bigcup_{\mathcal{D}} \Delta_d\right) = \text{acun}\, F(\overline{C}),$$

which entails $\mathcal{A} = \mathcal{B}$. □

Now suppose that X is a linear topological space and $\mathcal{D}$ is a subset of $X^* \setminus \{0\}$. Let us choose the following families of subsets of X

$$\Delta^s_d = \{x \in X \mid d(x) = \sup d(X \setminus C)\} \qquad d \in \mathcal{D} \tag{3.7}$$

and

$$\Delta^i_d = \{x \in X \mid d(x) = \inf d(X \setminus C)\} \qquad d \in \mathcal{D}. \tag{3.8}$$

The associate first type surrogate reverse dual problems are

$$\operatorname*{Sup}_{d \in \mathcal{D}} \quad \operatorname*{Sup}_{\substack{x \in X \\ d(x) = \sup d(X \setminus C)}} \quad F(x) \tag{3.9}$$

and

$$\operatorname*{Sup}_{d \in \mathcal{D}} \quad \operatorname*{Sup}_{\substack{x \in X \\ d(x) = \inf d(X \setminus C)}} \quad F(x). \tag{3.10}$$

respectively.

3.8. Definition. The set $\mathcal{D} \subset X^* \setminus \{0\}$ is said to be quasiabsorbing if

$$\forall \varphi \in X^* \setminus \{0\}\ \exists t > 0 : \ t \cdot \varphi \in \mathcal{D}.$$

3.9. Theorem. Suppose that X is a Hausdorff locally convex space, $\mathcal{D}$ is quasiabsorbing, $X \setminus C$ is nonempty convex bounded and closed and for each $i = 1, \dots, m$, f_i is lower semicontinuous and $f_i(x) > -\infty\ \forall x \in \overline{C}$.

Then we have

$$\begin{aligned} \operatorname*{Sup}_{x\in C} F(x) &= \operatorname*{Sup}_{d\in\mathcal{D}} \operatorname*{Sup}_{\substack{x\in X \\ d(x)=\sup d(X\setminus C)}} F(x) \\ &= \operatorname*{Sup}_{d\in\mathcal{D}} \operatorname*{Sup}_{\substack{x\in X \\ d(x)=\inf d(X\setminus C)}} F(x). \end{aligned} \tag{3.11}$$

Proof. For $x \in C$ we have $x \notin X\setminus C$, whence by Theorem 2.1 there exists $\varphi \in X^*\setminus\{0\}$ such that $\varphi(x) = \sup \varphi(X \setminus C)$. Since $\mathcal{D}$ is quasiabsorbing there exists $t > 0$ such that $d = t \cdot \varphi \in \mathcal{D}$. Clearly, $x \in \Delta^s_d$. We have proved $C \subset \bigcup_{\mathcal{D}} \Delta^s_d$.

On the other hand for $x \in \Delta^s_d$, $d \in \mathcal{D}$ it is clear that $x \notin \operatorname{int}(X \setminus C)$ which implies $x \in \overline{C}$. Finally we have $C \subset \bigcup_{\mathcal{D}} \Delta^s_d \subset \overline{C}$.

Now it suffices to apply Theorem 3.7 and we obtain the first equality. The second equality can be proved analogously. □

3.10. Theorem. Suppose that X is a Hausdorff locally convex space, $\mathcal{D}$ is quasiabsorbing, $X \setminus C$ is convex, solid, $C_{\overline{X\setminus C}}$ does not contain a hyperplane and there holds

$$\overline{X \setminus C} \not\subset x + \operatorname{int} C_{\overline{X\setminus C}} \qquad \forall x \in C. \tag{3.12}$$

Further, let either

$$C = \overline{C} \tag{3.13}$$

or

$$f_i \text{ is lower semicontinuous on } \overline{C} \text{ and } f_i(x) > -\infty \quad \forall x \in C \quad \forall i = 1, \dots, m. \tag{3.14}$$

Then the equalities (3.17) hold.

Proof. For $x \in C$, we have $x \notin X \setminus C$. If $x \in \overline{X \setminus C}$, since $\overline{X \setminus C}$ is convex and solid, x is a support point of $\overline{X \setminus C}$ (see [1, § 12, E]), whence there exists $\Psi \in X^* \setminus \{0\}$ satisfying $\Psi(x) = \sup \Psi\left(\overline{X \setminus C}\right) = \sup \Psi(X \setminus C)$. Then since $\mathcal{D}$ is quasiabsorbing, there exists $t > 0$ such that $d = t \cdot \psi \in \mathcal{D}$ and we have $x \in \Delta^s_d$.
If $x \notin \overline{X \setminus C}$, then all hypotheses of Theorem 2.6 are satisfied. So there exists again $\Psi \in X^* \setminus \{0\}$ satisfying $\Psi(x) = \sup \Psi\left(\overline{X \setminus C}\right) = \sup \Psi(X \setminus C)$, and by the same consideration as above we have $x \in \Delta^s_d$ for some $d \in \mathcal{D}$.

So

$$C \subset \bigcup_{\mathcal{D}} \Delta^s_d.$$

On the other hand for $x \in \Delta_d^s$, $d \in \mathcal{D}$ it is clear that $x \notin \text{int}(X \setminus C)$ which entails $x \in \overline{C}$. Hence

$$C \subset \bigcup_{\mathcal{D}} \Delta_d^s \subset \overline{C}.$$

Now it suffices to apply Theorem 3.10 and we obtain the first equality. The second equality can be proved analogously. □

3.11. Remark. In comparison with similar results in scalar case of I. Singer [15], our results are considerably more general, e.g. X need not be a normed space. In addition, our hypotheses are much weaker than those of I. Singer [15], e.g. the set $X \setminus C$ need not be bounded and the map F need not always be lower semicontinuous.

(Received November 9, 1990.)

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