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A PRIORI RESULTS IN LINEAR-QUADRATIC OPTIMAL CONTROL THEORY*

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In the present paper we shall see that philosophizing on the specific nature of Linear-Quadratic optimal Control Problems (LQCPs) yields several *a priori* statements that are valid for the entire *set* of these problems. For instance, the real symmetric matrix that represents the optimal cost for a particular LQCP necessarily is a *rank minimizing* solution of the dissipation inequality (DI). Since, in case of a positive definite input weighting matrix, the set of these solutions of the DI is equivalent to the set of real symmetric solutions of the algebraic Riccati equation (ARE), our result thus covers both the regular and the singular case. In addition, we will provide a *characterization* of the afore-mentioned set of solutions of the DI.

Next, a serious attempt is made at reducing general (indefinite) LQCPs to *nonnegative definite* LQCPs. Moreover, a *distributional* framework for *singular* LQCPs is proposed.

1. PRELIMINARIES

In this paper we will conider the linear time-invariant finite-dimensional system Σ:

$$\dot{x} = Ax + Bu\,, \quad x(0) = x_0\,, \tag{1.1a}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ for all $t \geqq 0$, together with the quadratic form in $(x, u) \in \in \mathbf{R}^{n+m}$

$$w(x, u) = x'Qx + 2u'Sx + u'Ru\,, \tag{1.1b}$$

with $Q = Q'$, $R = R'$. All matrices involved are real and constant.

The allowed inputs are assumed to be elements of $C^m_{sm} :=$

$$\{u: \mathbf{R}^+ \to \mathbf{R}^m \mid \underset{\varepsilon>0}{\exists} \; \underset{v\in C^\infty((-\varepsilon,\infty)\to\mathbf{R}^n)}{\exists} \; \underset{t\geqq 0}{\forall} : u(t) = v(t)\}\,, \tag{1.2}$$

the space of controls that are *smooth* on $[0, \infty)$. Now we introduce the *infinite horizon* cost criterion

$$J(x_0, u) := \textstyle\int_0^\infty w(x, u)\,\mathrm{d}t\,, \tag{1.3}$$

and here $\int_0^\infty w(x, u)\,\mathrm{d}t$ is understood to be $\lim_{T\to\infty} \int_0^T w(x, u)\,\mathrm{d}t$. The class of x_0-dependent

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elements of C^m_{sm} for which this limit exists in $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$, is denoted by $\mathbf{U}(x_0)$. With $x = x(x_0, u)$ we indicate the dependence of x on x_0 and u. Then, let $\mathbf{T} \subset \mathbf{R}^n$ be an arbitrary subspace. We define the *distance* from $x(x_0, u)$ to $\mathbf{T}$ at *infinity* by

$$d_\infty(x(x_0, u), \mathbf{T}) := \lim_{t \to \infty} d(x(x_0, u)(t), \mathbf{T}), \tag{1.4}$$

if this limit exists. Here $d(x, \mathbf{T})$, $x \in \mathbf{R}^n$, denotes the (Euclidean) distance from x to $\mathbf{T}$. Without loss of generality, we may assume that

$$[B' \; S \; R]' \quad \text{is of full column rank}. \tag{1.5}$$

The general infinite horizon Linear-Quadratic optimal Control Problem with *stability modulo* $\mathbf{T}$ $(\text{LQCP})_\mathbf{T}$ now is defined as follows:

For $x_0 \in \mathbf{R}^n$, determine

$$J_\mathbf{T}(x_0) := \inf \{J(x_0, u) \mid u \in \mathbf{U}(x_0) \text{ such that } d_\infty(x(x_0, u), \mathbf{T}) = 0\} \tag{1.6}$$

and, if for all x_0 $J_\mathbf{T}(x_0)$ is finite, then characterize, if one exists, all controls $u^* \in \mathbf{U}(x_0)$ (i.e., all inputs $u^* \in \mathbf{U}(x_0)$ for which $J(x_0, u^*) = J_\mathbf{T}(x_0)$).

Next, we introduce the *dissipation matrix*

$$F(K) := \begin{bmatrix} Q + A'K + KA & KB + S' \\ B'K + S & R \end{bmatrix}, \tag{1.7}$$

where K denotes any $n \times n$ real symmetric matrix. If $F(K) \geqq 0$, then K is said to satisfy the Dissipation Inequality (cf. [9]), abbreviated DI. We will define

$$\mathbf{\Gamma} := \{K \in \mathbf{R}^{n \times n} \mid K = K', F(K) \geqq 0\}, \tag{1.8}$$

the set of *solutions* of the DI.

If $(s_{1,2} \in \mathbf{C})$

$$H(s_1, s_2) := R + B'(Is_1 - A')^{-1} S' + S(Is_2 - A)^{-1} B + B'(Is_1 - A')^{-1} Q(Is_2 - A)^{-1} B, \tag{1.9}$$

then we may set

$$\varrho := \text{normal rank}\,(H(-s, s)). \tag{1.10}$$

Now Schumacher [8] established that

Lemma 1.1. If $K \in \mathbf{\Gamma}$, then $\operatorname{rank}(F(K)) \geqq \varrho$.

Hence we are invited to define

$$\mathbf{\Gamma}_{\min} := \{K \in \mathbf{\Gamma} \mid \operatorname{rank}(F(K)) = \varrho\}, \tag{1.11}$$

the set of *rank minimizing* solutions of the DI.

For every $K \in \mathbf{\Gamma}$ it is possible to find real constant matrices C_K and D_K such that $[C_K \; D_K]$ is of full row rank and such that $F(K) = [C_K \; D_K]' \, [C_K \; D_K]$. If, in addition, we define the linear system Σ_K by the system equation (1.1a) and the artificial *output* equation

$$y_K = C_K x + D_K u \tag{1.12}$$

($u \in C^m_{sm}$), then it is readily seen (cf. [9]) that for every x_0, every $T > 0$ and every smooth u,

$$\int_0^T w(x, u)\,dt + x'(T)\,K\,x(T) = \int_0^T y_K' y_K\,dt + x_0' K x_0\,, \tag{1.13}$$

with $x(T) = x(x_0, u)(T)$, of course. For further use, we set ($K \in \mathbf{\Gamma}$)

$$J_K(x_0, u) := \int_0^\infty y_K' y_K\,dt \tag{1.14}$$

(and we admit that this might cause some slight confusion). Moreover, we note that $[B' \; D_K']'$ is of full *column* rank (if $Bu = 0$ and $D_K u = 0$, then $Ru = 0$ and $0 = C_K' D_K u = (KB + S')\,u = S'u$, whence $u = 0$). Finally, we mention that, if

$$T_K(s) := D_K + C_K(sI - A)^{-1}\,B \tag{1.15}$$

($s \in \mathbf{C}$), then (1.10) $\varrho = \text{normal rank}\,(T_K(s))$ (cf. [9]). The relation (1.13) will be of paramount significance in the sequel, as it has been before in e.g. [1], [9].

Now we make the following

Standing Assumption. (A, B) is stabilizable and $\underset{K^0 \in \mathbf{\Gamma}}{\exists} : K^0 \leqq 0$.

Note that thus, in particular, $R \geqq 0$ and that K^0 is *not* necessarily required to be in $\mathbf{\Gamma}_{\min}$. Furthermore, we observe that

$$0 \in \mathbf{\Gamma} \Leftrightarrow \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} \geqq 0 \Leftrightarrow \underset{x}{\forall}\,\underset{u}{\forall}: w(x, u) \geqq 0 \tag{1.16}$$

and LQCPs with a nonnegative definite integrand will be called *nonnegative definite* LQCPs. The remaining ones will be called *indefinite*.

Proposition 1.2. For every subspace $\mathbf{T}$ and every x_0, $\mathbf{U}(x_0) \neq \varnothing$. Moreover, there exist real symmetric matrices M^+ and M^- such that, for all subspaces $\mathbf{T}$ and all x_0,

$$x_0' M^- x_0 \leqq J_{\mathbf{T}}(x_0) \leqq x_0' M^+ x_0\,.$$

Proof. Let $F \in \mathbf{R}^{m \times n}$ be such that $A_F := A + BF$ is asymptotically stable. By applying the feedback law $u = Fx$, we get that the solution of (1.1a) equals $\exp(A_F t)\,.\,x_0$ and thus $x(t) \to 0$ ($t \to \infty$). Hence, for all x_0, $J(x_0, u) = x_0' M^+ x_0$ with

$$M^+ = \int_0^\infty (\exp(A_F' t)\,[Q + F'S + SF + F'RF]\,\exp(A_F t))\,dt$$

and M^+ is clearly real and symmetric. We establish that $\mathbf{U}(x_0) \neq \varnothing$ and that for all $\mathbf{T}$, $J_{\mathbf{T}}(x_0) \leqq J_0(x_0) \leqq x_0' M^+ x_0$. On the other hand, it follows from (1.13) that for any $T > 0$ and any u, $\int_0^T w(x, u))\,dt \geqq x_0' K^0 x_0$, since $K^0 \leqq 0$. Hence for all x_0 and all $\mathbf{T}$, $J_{\mathbf{T}}(x_0) \geqq J_{\mathbf{R}^n}(x_0) \geqq x_0' M^- x_0$ with $M^- = K^0$.

Corollary 1.3 ([6], [7]). Consider $(\text{LQCP})_{\mathbf{T}}$. There exists a unique $K_{\mathbf{T}} \in \{K \in \mathbf{R}^{n \times n} \mid K = K'\}$ such that, for all x_0, $J_{\mathbf{T}}(x_0) = x_0' K_{\mathbf{T}} x_0$. Moreover, $K_{\mathbf{T}} \in \mathbf{\Gamma}$.

In Theorem 2.1 we will confirm an old conjecture concerning $K_{\mathbf{T}}$ raised in [9].

2. A GENERAL DETERMINATION OF $\mathbf{\Gamma}_{\min}$

Theorem 2.1. Consider $(\mathrm{LQCP})_{\mathbf{T}}$. There exists a unique $K_{\mathbf{T}} \in \mathbf{\Gamma}_{\min}$ such that, for all x_0, $J_{\mathbf{T}}(x_0) = x_0' K_{\mathbf{T}} x_0$.

Proof. See Theorem 2.1 in [4]. □

If $R > 0$ (the *regular* case), then we can define the quadratic matrix function

$$\phi(K) := Q + A'K + KA - (KB + S')\, R^{-1}\, (B'K + S) \tag{2.1}$$

(K an $n \times n$ real symmetric matrix), and it is immediately seen (cf. [9]) that then

$$\mathbf{\Gamma} = \{K \in \mathbf{R}^{n \times n} \mid K = K',\ \phi(K) \geqq 0\}\,, \tag{2.2}$$

$$\mathbf{\Gamma}_{\min} = \{K \in \mathbf{\Gamma} \mid \phi(K) = 0\}\,.$$

In other words, in the regular case the elements of $\mathbf{\Gamma}_{\min}$ are the real symmetric solutions of the *algebraic Riccati equation* (ARE) $\phi(K) = 0$.

In the *singular case* (R *not* positive definite) $\phi(K)$ is not defined. However, we will present a representation of $\mathbf{\Gamma}_{\min}$ that captures both the regular and the singular case.

For this we will need the following concepts. Let $K \in \mathbf{\Gamma}$ and Σ_K be the system described by (1.1a) and (1.12). Then the *weakly unobservable subspace* associated with Σ_K is defined by

$$\mathbf{V}_K = \mathbf{V}(\Sigma_K) := \{x_0 \in \mathbf{R}^n \mid \underset{u \in C_{sm}{}^m}{\exists} : y_K(x_0, u) \equiv 0\} \tag{2.3}$$

and it is the *largest* subspace $\mathbf{L}$ for which there exists an $F \in \mathbf{R}^{m \times n}$ such that $(A + BF)\,\mathbf{L} \subset \mathbf{L}$, $(C_K + D_K F)\,\mathbf{L} = 0$ (cf. [5]). Dually, $\mathbf{W}_K = \mathbf{W}(\Sigma_K)$ is the *smallest* subspace $\mathbf{S}$ for which there exists a $G \in \mathbf{R}^{n \times r_k}$ such that $(A + GC_K)\,\mathbf{S} \subset \mathbf{S}$, $\mathrm{im}\,(B + GD_K) \subset \mathbf{S}$. Here $r_k = \mathrm{rank}\,(F(K)) = \mathrm{rank}\,([C_K\ D_K])$. We state without proof that $\mathbf{W} = 0$ if and only if $\ker\,(D_K) = 0$. Finally, we introduce $R_K := \mathbf{V}_K \cap \mathbf{W}_K$. Set $\mathbf{W} := \mathbf{W}_{K^0}$, $\mathbf{R} := \mathbf{R}_{K^0}$. In Section 2.3 of [4] it is proven by direct computation that

Proposition 2.2. For every $K \in \mathbf{\Gamma}$, we have that $\mathbf{W}_K = \mathbf{W}$, $\mathbf{R}_K = \mathbf{R}$ and $(K - K^0)\,\mathbf{W} = 0$.

Next, if R^+ denotes the Moore-Penrose inverse of $R \geqq 0$, then for any real symmetric matrix K of dimension n we may define

$$\phi_0(K) := Q + A'K + KA - (KB + S')\, R^+ (B'K + S)\,. \tag{2.4}$$

If ($K \in \mathbf{\Gamma}$) $C_K^{-1}\,\mathrm{im}\,(D_K) := \{u \in \mathbf{R}^m \mid C_K u \in \mathrm{im}\,(D_K)\}$, then it is obvious that

$$C_K^{-1}\,\mathrm{im}\,(D_K) = \ker\,(\phi_0(K)) \tag{2.5a}$$

and hence, if

$$\mathbf{W}_{K_2} := \mathbf{W}_K \cap (C_K^{-1}\,\mathrm{im}\,(D_K))\,,\ \mathbf{W}_2 := \mathbf{W} \cap (C_{K^0}^{-1}\,\mathrm{im}\,(D_{K^0}))\,, \tag{2.5b}$$

then, by Proposition 2.2, for every $K \in \mathbf{\Gamma}$,

$$\mathbf{W}_{K_2} = \mathbf{W}_2 \,. \tag{2.5c}$$

We arrive at one of our main results.

Theorem 2.3. Let $\tilde{W}_1$ be any left invertible matrix such that $\operatorname{im}(\tilde{W}_1) \oplus \mathbf{W}_2 = \mathbf{W}$. Then

$$\mathbf{\Gamma} = \{K \in \mathbf{R}^{n \times n} \mid K = K', (K - K^0)\,\mathbf{W} = 0, \psi(K) \geqq 0\}$$

and

$$\mathbf{\Gamma}_{\min} = \{K \in \mathbf{R}^{n \times n} \mid K = K', (K - K^0)\,\mathbf{W} = 0, \psi(K) = 0\}$$

with, for every $n \times n$ real symmetric matrix K that satisfies $(K - K^0)\,\mathbf{W} = 0$,

$$\psi(K) := \phi_0(K) - (\phi_0(K))\,\tilde{W}_1(\tilde{W}_1'(\phi_0(K))\,\tilde{W}_1)^{-1}\,\tilde{W}_1'(\phi_0(K))\,.$$

and it holds that $\mathbf{W} \subset \ker(\psi(K))$.

Proof. Theorem 2.34 in [4]. □

For one thing, Theorem 2.3 expresses that $\psi(K)$ is *independent* of the choice for $\tilde{W}_1$. If $R > 0$. then $\mathbf{W} = 0$ and we reobtain the results in (2.2). If $\begin{bmatrix} Q & S' \\ S & R \end{bmatrix} \geqq 0$, i.e. if $0 \in \mathbf{\Gamma}$ (1.16), then Theorem 2.3 transforms into Theorem 3.3 of [3]. Theorem 2.3 can also be given in a form which is independent of K^0; in Section 2.3 of [4] the author describes in full detail a sequence of matrix computations, to be applied to the matrices A, B, Q, S and R. In fact, this technique is nothing else than the application of the *generalized dual structure algorithm* (cf. [2]) to a system Σ_K ($K \in \mathbf{\Gamma}$), *without actually knowing the matrices* C_K and D_K! This technique leads to matrices $\tilde{B}$, $\tilde{S}'$ and $\bar{B}$, $\bar{S}'$ and $\bar{R}$, where $\bar{R}$ is invertible, rank $(\bar{R}) = \varrho$ (1.10). Then, if for any real symmetric K of dimension n,

$$\tilde{\phi}(K) := Q + A'K + KA - (K\bar{B} + \bar{S}')\,\bar{R}^{-1}(\bar{B}'K + \bar{S})$$

and

$$\tilde{L}(K) := K\tilde{B} + \tilde{S}'\,,$$

it follows that $K \in \mathbf{\Gamma}$ if and only if $\tilde{L}(K) = 0$ and $\tilde{\phi}(K) \geqq 0$. Moreover, if $\tilde{L}(K) = 0$ then $(\tilde{\phi}(K))\,\tilde{B} = 0$. In addition, $K \in \mathbf{\Gamma}_{\min}$ if $\tilde{L}(K) = 0$, $\tilde{\phi}(K) = 0$ (see Proposition 2.31 (h) – (i) in [4]). Of course, if $R > 0$, then $\tilde{B}$, $\tilde{S}'$ are not appearing, $\bar{B} = B$, $\bar{S}' = S'$, $\bar{R} = R$. Hence, if for some real symmetric $K^0 \leqq 0$, $\tilde{L}(K^0) = 0$ and $\tilde{\phi}(K^0) \geqq 0$, then, apparently, there exists a negative semi-definite element of $\mathbf{\Gamma}$.

So much for the computational aspects of this paper. Now it is time for some analysis.

3. LINEAR-QUADRATIC CONTROL PROBLEMS IN A BROAD PERSPECTIVE

Let K be any real symmetric matrix of dimension n. Then, due to Theorem 2.1, there exists a unique $\hat{K} \in \mathbf{\Gamma}_{\min}$ such that, for all x_0, $J_{\ker(K)}(x_0) = x_0'\hat{K}x_0$. This defines a *function*

$$\eta: \{K \in \mathbf{R}^{n\times n} \mid K = K'\} \to \mathbf{\Gamma}_{\min} \tag{3.1}$$

with $\eta(K) := \hat{K}$.

Lemma 3.1. Let $K \in \mathbf{\Gamma}$. Then $\eta(K) \geqq K$.

Proof. Take any $x_0 \in \mathbf{R}^n$ and let $u = \mathrm{U}(x_0)$ be such that $d_\infty(x(x_0, u), \ker(K)) = 0$ (such a control exists!). Then (1.13)–(1.14) $J(x_0, u) = J_K(x_0, u) + x_0'Kx_0$ and thus $\eta(K) \geqq K$. □

If K is real and symmetric, but $K \notin \mathbf{\Gamma}$, then we cannot say that $\eta(K) \geqq K$! Recall (Theorem 2.1) that every subspace $\mathbf{T}$ generates an element $K_{\mathbf{T}}$ of $\mathbf{\Gamma}_{\min}$. Note that $\eta(0) = K_{\mathbf{R}^n}$, $\eta(I_n) = K_0$. More generally, let $\mathbf{T}$ be a given subspace, and let the matrix T (of full row rank) be such that $\ker(T) = \mathbf{T}$. Then $\ker(T) = \ker(K_T) = \mathbf{T}$ with $K_T := T'T$, and hence $\eta(K_T) = K_{\mathbf{T}}$. From this observation we derive directly that

Lemma 3.2.

$$\forall_{\bar{K} \in \{K \in \mathbf{R}^{n\times n} \mid K = K'\}} : \eta(\eta(\bar{K})) = \eta(\bar{K}) \Leftrightarrow \forall_{\mathbf{T} \subset \mathbf{R}^n} : \eta(K_{\mathbf{T}}) = K_{\mathbf{T}}\,.$$

We introduce

$$\mathbf{\Gamma}_{\min}^{\mathrm{eq}} := \{K \in \mathbf{\Gamma}_{\min} \mid \eta(K) = K\} \tag{3.2}$$

and note from the above that

$$\mathbf{\Gamma}_{\min}^{\mathrm{eq}} = \{K \in \mathbf{R}^{n\times n} \mid K = K', \eta(K) = K\}\,. \tag{3.3}$$

If, from now on,

$$K^- := K_{\mathbf{R}^n},\ K^+ := K_0\,, \tag{3.4}$$

then we find that $\mathbf{\Gamma}_{\min}^{\mathrm{eq}} \neq \emptyset$, since $K^+ \geqq \eta(K^+)$ $(0 \subset \ker(K^+))$ and $\eta(K^+) \geqq K^+$ (Lemma 3.1). It follows easily from Lemma 3.1 that K^+ is the *largest* element of $\mathbf{\Gamma}$ and thus K^+ is the largest element of $\mathbf{\Gamma}_{\min}^{\mathrm{eq}}$.

Now suppose that we are able to prove that for *every* $\mathbf{T} \subset \mathbf{R}^n$, $K_{\mathbf{T}} \in \mathbf{\Gamma}_{\min}^{\mathrm{eq}}$ (i.e., that $\eta^2 = \eta$, by Lemma 3.2). Then, clearly,

K^- is the smallest element of $\mathbf{\Gamma}_{\min}^{\mathrm{eq}}$.

If this turns out to be true, then it is the set $\mathbf{\Gamma}_{\min}^{\mathrm{eq}}$ rather than the set $\mathbf{\Gamma}_{\min}$ which appears to be the *pivot* in linear-quadratic optimal control theory:

Every $K_{\mathbf{T}} \in \mathbf{\Gamma}_{\min}^{\mathrm{eq}}$ and K^+ and K^- then are the *largest* and *smallest* element of this set, respectively.

But first, for something completely different. Recall (1.12)–(1.14) and read $K_{\mathbf{T}}$ instead of K there.

Theorem 3.3. Let $u \in \mathbf{U}(x_0)$ be such that $d_\infty(x(x_0, u), \mathbf{T}) = 0$. Then
(a) $J(x_0, u) \geqq J_{K_{\mathbf{T}}}(x_0, u) + x_0' K_{\mathbf{T}} x_0$.
Now assume that $J(x_0, u)$ is finite. Then the next statements are valid.
(b) The limit $(x'(\cdot) K_{\mathbf{T}} x(\cdot))_\infty := \lim_{T \to \infty} (x'(T) K_{\mathbf{T}} x(T))$ exists and it is smaller than or equal to zero.
(c) $J(x_0, u) = x_0' K_{\mathbf{T}} x_0 \Leftrightarrow \{x'(\cdot) K_{\mathbf{T}} x(\cdot))_\infty = 0$ and $y_{K_{\mathbf{T}}} \equiv 0\}$.
(d) $\operatorname{Inf}\{J_{K_{\mathbf{T}}}(x_0, u) \mid u \in C_{sm}^m$ such that $d_\infty(x(x_0, u), \mathbf{T}) = 0\} = 0$.
(e) If $\bar{K} \in \{K \in \mathbf{\Gamma} \mid K\mathbf{T} = 0\}$, then $\bar{K} \leqq K_{\mathbf{T}}$.
If $\mathbf{T} \subset \ker(K_{\mathbf{T}})$, then $K_{\mathbf{T}}$ is the largest element of the set $\{K \in \mathbf{\Gamma} \mid K\mathbf{T} = 0\}$.

Proof. Let $u \in \mathbf{U}(x_0)$ be such that $d_\infty(x(x_0, u), \mathbf{T}) = 0$. If $J(x_0, u) = +\infty$, then (a) is trivial. Since *always* $J(x_0, u) \geqq x_0' K^0 x_0$, we now assume that $J(x_0, u)$ is finite. Let $T > 0$, then (Corollary 1.3) $x'(T) K_{\mathbf{T}} x(T) \leqq \int_T^\infty w(x, u)\, dt$ $(x(T) = x(x_0, u)(T))$, and hence, by (1.13),

$$J(x_0, u) \geqq \int_0^T y_{K_{\mathbf{T}}}' y_{K_{\mathbf{T}}}\, dt + x_0' K_{\mathbf{T}} x_0 .$$

This yields (a). Next, from (a), $J_{K_{\mathbf{T}}}(x_0, u) < \infty$, and thus (1.13) $(x'(\cdot) K_{\mathbf{T}}(\cdot))_\infty$ exists. From the above it must be $\leqq 0$ and we have (b) and

$$J(x_0, u) + (x'(\cdot) K_{\mathbf{T}} x(\cdot))_\infty = J_{K_{\mathbf{T}}}(x_0, u) + x_0' K_{\mathbf{T}} x_0 .$$

Since $J_{K_{\mathbf{T}}}(x_0, u) \geqq 0$, we now establich (c), and (d) is immediate from (a). Finally, if $\bar{K}\mathbf{T} = 0$ and $u \in \mathbf{U}(x_0)$ is such that $d_\infty(x, \mathbf{T}) = 0$, then $x'(T) \bar{K} x(T) \to 0$ $(T \to \infty)$, and hence $J(x_0, u) = J_{\bar{K}}(x_0, u) + x_0' \bar{K} x_0$ (1.13). Thus, $K_{\mathbf{T}} \geqq \bar{K}$ and if, moreover, $\mathbf{T} \subset \ker(K_{\mathbf{T}})$ then $K_{\mathbf{T}} \in \{K \in \mathbf{\Gamma} \mid K\mathbf{T} = 0\}$. □

Consider Theorem 3.3 (e). It is clear that the first claim is a *generalization* of Lemma 3.1. Since $0 \in K$ for every $K \in \mathbf{\Gamma}$, we reobtain the well-known fact that $K^+ \geqq K$ for all $K \in \mathbf{\Gamma}$ from the second claim.

If $R > 0$, then there exists an invertible matrix D such that

$$F(K_{\mathbf{T}}) = [C_{K_{\mathbf{T}}} D]' [C_{K_{\mathbf{T}}} D]$$

with $C_{K_{\mathbf{T}}} = (D^{-1})' (B' K_{\mathbf{T}} + S)$, because (2.1)–(2.2) $\phi(K_{\mathbf{T}}) = 0$. It follows that

$$y_{K_{\mathbf{T}}}' y_{K_{\mathbf{T}}} = [u' + x'(K_{\mathbf{T}} B + S') R^{-1}]\, R[u + R^{-1}(B' K_{\mathbf{T}} + S)\, x]$$

and hence, by Theorem 3.3 (c) that

Corollary 3.4. If $R > 0$ and for a given x_0 there exists an optimal input for $(\mathrm{LQCP})_{\mathbf{T}}$ then this input is unique and it can be given by the state feedback law

$$u = -R^{-1}(B' K_{\mathbf{T}} + S)\, x .$$

The corresponding state trajectory $x(t) = \exp(A_{K_{\mathbf{T}}} t)\, x_0$ $(t \geqq 0)$, with

$$A_{K_{\mathbf{T}}} := A - BR^{-1}(B' K_{\mathbf{T}} + S) ,$$

is such that $x'(t) K_{\mathbf{T}} x(t) \to 0$ $(t \to \infty)$.

Hence, every optimal control for a regular LQCP can be implemented as a state feedback. This is in accordance with our expectations (e.g. [1], [9]).

If for some $\mathbf{T}$, $K_{\mathbf{T}} \geqq 0$, then Theorem 3.3 yields us

Corollary 3.5. Let $K_{\mathbf{T}} \geqq 0$. Then, for all x_0, $J_{(\ker(K_{\mathbf{T}}) \cap \mathbf{T})}(x_0) = J_{\ker(K_{\mathbf{T}})}(x_0) = J_{\mathbf{T}}(x_0)$. In particular, $K_{\mathbf{T}} \in \mathbf{\Gamma}^{eq}_{min}$.

Proof. Let x_0 be given and $u \in \mathrm{U}(x_0)$ be such that $d(x, \mathbf{T}) = 0$ and $J(x_0, u)$ is finite (and $\geqq 0$) Then (Theorem 3.3 (b)) $d_\infty(x, \ker(K_{\mathbf{T}})) = 0$ and hence $J_{(\ker(K_{\mathbf{T}}) \cap \mathbf{T})}(x_0) = J_{\mathbf{T}}(x_0)$. On the other hand, $J_{(\ker(K_{\mathbf{T}}) \cap \mathbf{T})}(x_0) \geqq J_{\ker(K_{\mathbf{T}})}(x_0) \geqq J_{\mathbf{T}}(x_0)$ by Lemma 3.1. □

Thus, if $0 \in \mathbf{\Gamma}$ (1.16), then for *all* $\mathbf{T}$, $K_{\mathbf{T}} \in \mathbf{\Gamma}^{eq}_{min}$. Now we are going to consider the general case. Analogously to the proof of Theorem 3.3, we can establish that if $u \in \mathrm{U}(x_0)$ is such that $J(x_0, u)$ is finite, then $J_{K^0}(x_0, u) < \infty$ and

$$(x'(\cdot)\, K^0\, x(\cdot))_\infty := \lim_{T \to \infty} x'(T)\, K^0\, x(T) \tag{3.5}$$

exists and it is $\leqq 0$ ($K^0 \leqq 0$!). In addition,

$$J(x_0, u) = J_{K^0}(x_0, u) - (x'(\cdot)\, K^0\, (x(\cdot))_\infty + x_0' K^0 x_0\,, \tag{3.6}$$

and thus we are motivated to investigate the *nonnegative definite* LQCP associated with Σ_{K^0}: For all x_0, determine

$$\hat{J}_{K^0}(x_0) := \inf\{\lim_{T\to\infty} (\textstyle\int_0^T y'_{K^0} y_{K^0}\, \mathrm{d}t - x'(T)\, K^0\, x(T)) \mid u \in C^m_{sm}\}\,. \tag{3.7}$$

Due to (A, B)-stabilizability, the optimal cost for this problem is finite for every x_0 and it can be proven (compare Lemmas 1, 3 in [7]) that there exists a real matrix $\hat{L}$ such that (for all x_0) $\hat{J}_{K^0}(x_0) = x_0' \hat{L} x_0$.

Moreover, if

$$F_{K^0}(L) := \begin{bmatrix} C'_{K^0} C_{K^0} + A'L + LA & LB + C'_{K^0} D_{K^0} \\ B'L + D'_{K^0} C_{K^0} & D'_{K^0} D_{K^0} \end{bmatrix}, \tag{3.8}$$

with L any $n \times n$ real symmetric matrix,

$$\mathbf{\Gamma}_{K^0} := \{L \in \mathbf{R}^{n\times n} \mid L = L', F_{K^0}(L) \geqq 0\}\,, \tag{3.9a}$$

and

$$\mathbf{\Gamma}_{K^0_{min}} := \{L \in \mathbf{\Gamma}_{K^0} \mid \operatorname{rank}(F_{K^0}(L)) = \text{normal rank}\,(T_{K^0}(s))\}\,, \tag{3.9b}$$

then it follows from [8] (or Theorem 2.1) that $\hat{L} \in \mathbf{\Gamma}_{K^0_{min}}$. But then, of course,

$$K^- = \hat{L} + K^0 \tag{3.10}$$

and $\hat{L} + K^0 \in \mathbf{\Gamma}_{min}$ (1.11), (1.15)! In fact, we have much more than that,

Proposition 3.6.

$$K \in \mathbf{\Gamma} \Leftrightarrow L = K - K^0 \in \mathbf{\Gamma}_{K^0}\,,$$

$$K \in \mathbf{\Gamma}_{min} \Leftrightarrow L = K - K^0 \in \mathbf{\Gamma}_{K^0_{min}}\,.$$

Now we make the following

Assumption 3.7. For every subspace $\mathbf{T}$ and every x_0,

$$\inf\{\lim_{T\to\infty}(\int_0^T y'_{K^0}y_{K^0}\,dt - x'(T)K^0x(T)) \mid u \in C^m_{sm} \text{ such that } d_\infty(x,\mathbf{T}) = 0\} =$$
$$\inf\{J_{K^0}(x_0,u) \mid u \in C^m_{sm} \text{ such that } d_\infty(x,(\ker(K^0)\cap\mathbf{T})) = 0\}\,.$$

The author believes that Assumption 3.7 is *generally* true, but he has not (yet) been able to prove this. Actually, he conjectures that even the next assumption is satisfied.

Assumption 3.8. Let the system Σ be described by $\dot{x} = Ax + Bu$, $x(0) = x_0$, and $y = Cx + Du$. The inputs are assumed to be smooth on $\mathbf{R}^+$, $J(x_0,u) = \int_0^\infty y'y\,dt$ and $M_0 \geqq 0$ is a given real symmetric matrix. Then, for all subspaces $\mathbf{T}$ and for x_0,

$$\inf\{\lim_{T\to\infty}(\int_0^T y'y\,dt + x'(T)M_0x(T)) \mid u \in C^m_{sm} \text{ such that } d_\infty(x(x_0,u),\mathbf{T}) = 0\}$$
$$= \inf\{J(x_0,u) \mid u \in C^m_{sm} \text{ such that } d_\infty(x(x_0,u),(\ker(M_0)\cap\mathbf{T})) = 0\}\,.$$

Anyway, let Assumption 3.7 be satisfied. Then, from (3.6)–(3.7), for every subspace $\mathbf{T}$ and every x_0,

$$J_{\mathbf{T}}(x_0) = \text{in}\{J_{K^0}(x_0,u) \mid u \in C^m_{sm} \text{ such that } d_\infty(x,(\ker(K^0)\cap\mathbf{T})) = 0\} + {} + x_0'K^0x_0 \tag{3.11}$$

(and thus, by definition (3.1), $\eta(K^0) = K^-$). Suppose that for all x_0,

$$\inf\{J_{K^0}(x_0,u) \mid u \in C^m_{sm} \text{ such that } d_\infty(x,(\ker(K^0)\cap\mathbf{T})) = 0\} = x_0'L^0_{\mathbf{T}}x_0 \tag{3.12}$$

with $L^0_{\mathbf{T}} \geqq 0$ and $L^0_{\mathbf{T}} \in \mathbf{\Gamma}_{K^0_{\min}}$ (3.9b). Then apparently,

$$K_{\mathbf{T}} = L^0_{\mathbf{T}} + K^0\,, \tag{3.13}$$

i.e., we have the optimal cost for the *general* $(\text{LQCP})_{\mathbf{T}}$ if the optimal cost for the *nonnegative definite* LQCP with stability modulo $(\ker(K^0)\cap\mathbf{T})$ is known. Next, we observe that $\ker(K^0)\cap\ker(K_{\mathbf{T}}) = \ker(K^0)\cap\ker(L^0_{\mathbf{T}})$. Now if u is such that $J_{K^0}(x_0,u) < \infty$ and $d_\infty(x(x_0,u),(\ker(K^0)\cap\mathbf{T})) = 0$, then (Theorem 3.3 (b)) *also* $d_\infty(x,\ker(L^0_{\mathbf{T}})) = 0$ and hence, by Corollary 3.5,

$$\inf\{J_{K^0}(x_0,u) \mid u \in C^m_{sm} \text{ such that } d_\infty(x,(\ker(K^0)\cap\ker(L^0_{\mathbf{T}}))) = 0\} =$$
$$= x_0'L^0_{\mathbf{T}}x_0\,,$$

for *all* x_0. But this implies that, for all x_0 (3.11), (3.13),

$$J_{\ker(K_{\mathbf{T}})}(x_0) = x_0'[L^0_{\mathbf{T}} + K^0]\,x_0 = J_{\mathbf{T}}(x_0)\,,$$

i.e.,

for every subspace $\mathbf{T}$, $K_{\mathbf{T}} \in \mathbf{\Gamma}^{eq}_{\min}$, and, in particular, K^- is the *smallest* element of $\mathbf{\Gamma}^{eq}_{\min}$.

Hence, if Assumption 3.7 is valid, then the set $\mathbf{\Gamma}_{\min}^{\text{eq}}$ contains all matrices that represent optimal costs for LQCPs and K^- is the smallest element of the set. Note that if $K^0 = 0$, then Assumption 3.7 is *automatically* satisfied (see also Corollary 3.5) and for $T = 0$ it is satisfied as well!

4. DISCUSSION

If Assumption 3.7 (or 3.8) is valid, then the above yields us a method for reducing *indefinite* LQCPs to *nonnegative definite* LQCPs. The idea runs as follows. Let the subspace $\mathbf{T}$ be given and assume for the moment that we can find the optimal cost for the nonnegative definite LQCP with stability modulo $(\ker(K^0) \cap \mathbf{T})$ associated with Σ_{K^0} (1.1a), (1.12). Let this optimal cost be denoted by $L_{\mathbf{T}}^0 \in \mathbf{\Gamma}_{K_{\min}^0}$ (3.9b), $L_{\mathbf{T}}^0 \geqq 0$. Then (3.13) $K_{\mathbf{T}} = L_{\mathbf{T}}^0 + K^0$.

Next, let $x_0 \in \boldsymbol{R}^n$ be given. If $u \in C_{sm}^m$ is such that $d_\infty(x(x_0, u), (\ker(K^0) \cap \mathbf{T})) = 0$, then (3.6) $J(x_0, u) = J_{K^0}(x_0, u) + x_0' K^0 x_0$. However, if R is not positive definite, then optimal controls within C_{sm}^m need not exist (see Example 2.11 in [5]). A reformulation in the style of [5] is needed incorporating *distributions* as allowed inputs. An appropriate distributional extension of C_{sm}^m is the input class C_{imp}^m, the space of *impulsive-smooth* distributions on $\boldsymbol{R}$ with support on $[0, \infty)$. Here an impulsive distribution is a linear combination of the Dirac δ distribution and its derivatives. If U_{Σ_K} ($K \in \mathbf{\Gamma}$, see (1.1a), (1.12)) denotes the space of controls $u \in C_{\text{imp}}^m$ for which y_K is *smooth* (i.e. has no impulsive component), then it turns out (Proposition 2.31 (e) in [4]) that for *every* $K \in \mathbf{\Gamma}$, $U_{\Sigma_K} = U_{\Sigma_{K^0}} =: U$ (compare with Proposition 2.2). Now if we *define*

$$J_{\mathbf{T}}(x_0) := \inf\{J_{K^0}(x_0, u) \mid u \in U \quad \text{such that} \quad d_\infty(x, (\ker(K^0) \cap \mathbf{T})) = 0\} + \\ + x_0' K^0 x_0$$

for every x_0, then this definition coincides with (3.11) if $R > 0$ and it is a *reasonable* extension of (3.11) if R is merely $\geqq 0$.

Note that if we would have chosen any *other* negative semi-definite element $\tilde{K}^0$ of $\mathbf{\Gamma}$, then the space of allowed distributional inputs remains the same.

Next, it is well known (see e.g. [4]), that the *existence* of optimal controls for nonnegative definite LQCPs associated with Σ_{K^0}, say is related to the question whether the intersection of the imaginary axis $\mathbf{C}^0$ and $\sigma^*(\Sigma_{K^0})$ is empty or not. Here the set $\sigma^*(\Sigma_{K^0})$ denotes the set of *invariant zeros* associated with Σ_{K^0} (cf. [10]). In Proposition 2.37 of [4] it is shown that if $K \in \mathbf{\Gamma}$ then $\sigma^*(\Sigma_K) \cap \mathbf{C}^0 = \emptyset$ if and only if $\sigma^*(\Sigma_{K^0}) \cap \mathbf{C}^0 = \emptyset$. Hence if for all x_0, optimal controls exist for the LQCP *with stability* ($\mathbf{T} = 0$) associated with Σ_{K^0}, then for all x_0 there exist optimal controls for the $(\text{LQCP})_0$ associated with $\Sigma_{\tilde{K}^0}$ as well *and vice versa*. Moreover (Proposition 2.2), $\mathbf{R}_{K^0} = 0 \Leftrightarrow \mathbf{R}_{\tilde{K}^0} = 0$ and hence (cf. [5]) optimal controls are *unique* for the former problem if and only if they are *unique* for the latter problem (if $\mathbf{R}_K = 0$, then Σ_K is called *left invertible*).

The reader will agree with the author, that the above-given strategy looks promising if (at least) we can solve nonnegative definite LQCPs with arbitrary stability requirements. These problems have been investigated in depth in [4]. Related material can be found in [2].

Briefly, our approach thus consists of the following steps. First, we must try to verify whether Assumption 3.7 (or 3.8) is valid or not. Then, we must find a negative semi-definite solution of the DI. Recall that at the end of Section 2 we mentioned that $K \in \mathbf{\Gamma} \Leftrightarrow \{\tilde{L}(K) = 0 \text{ and } \tilde{\phi}(K) \geqq 0\}$, with $\tilde{L}(K)$ and $\tilde{\phi}(K)$ a certain linear and a certain quadratic matrix function, respectively. Finally, with [4], the LQCP with stability modulo $\mathbf{T}$ is solvable.

Of course, many issues are not yet fully understood. To name but a few:

Suppose that, if $J_{K^0}(x_0, u) < \infty$, then *automatically* $x'(t) K^0 x(t) \to 0$. Hence, apparently, $L^0_{\mathbf{R}^n}$ is the *smallest positive semi-definite* element of $\mathbf{\Gamma}_{K^0_{\min}}$ (3.9b), by [2]. Thus K^- is the *smallest element of* $\mathbf{\Gamma}_{\min} \cap \{K \in \mathbf{\Gamma} \mid K \geqq K^0\}$, i.e. K^- is the *smallest element* K of $\mathbf{\Gamma}_{\min}$ *that satisfies* $K \geqq K^0$ (if $K^0 = 0$, then we reobtain Corollary 6.4 of [2].

If $J_{K^0}(x_0, u) < \infty$, but $x'(t) K^0 x(t)$ does *not* automatically converge to zero, then one might ask oneself whether the *choice* of K^0 matters or not. Assume that $K^0_1 \leqq K^0_2 \leqq 0$ and $K^0_{1,2} \in \mathbf{\Gamma}$, is it then sensible to choose K^0_2 instead of K^0_1 or is the choice irrelevant?

Yes, still a lot of work has to be done. Nevertheless the author has faith in the approach described above, not in the least because the easiest LQCP, the one with stability ($\mathbf{T} = 0$), has been solved along the lines of the above in Section 2.3 of [4].

5. CONCLUSIONS

Let us summarize the most relevant observations made in this paper. The real symmetric matrix that represents the optimal cost for any LQCP is necessarily a rank minimizing solution of the dissipation inequality. The set of these solutions can be characterized in an elegant way.

If Assumption 3.8 holds, then for every subspace $\mathbf{T}$, $K_{\mathbf{T}} \in \mathbf{\Gamma}^{\text{eq}}_{\min}$.

If $K_{\mathbf{T}} \geqq 0$, then $K_{\mathbf{T}} \in \mathbf{\Gamma}^{\text{eq}}_{\min}$.

Optimal controls for regular problems can always be implemented as state feedbacks

If $\mathbf{T} \subseteq \ker(K_{\mathbf{T}})$. then $K_{\mathbf{T}}$ is the largest element of the set $\{K \in \mathbf{\Gamma} \mid K\mathbf{T} = 0\}$.

Indefinite LQCPs can be reduced to nonnegative LQCPs.

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