KYBERNETIKA — VOLUME 27 (1991), NUMBER 4

FINITE SETTLING TIME STABILISATION OF A FAMILY OF DISCRETE TIME SYSTEMS

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The problem of finding a compensator C, that stabilises in the Finite Settling Time (FST) sense, a family of k distinct discrete time plants $\{P_i, i = 1, \ldots, k\}$, is referred to as Simultaneous FST Stabilisation Problem (S-FSTSP) and it is examined here. The general case of many input, many output plants is considered first and algebraic conditions for the existence of a S-FSTS controller are derived in terms of the properties of the plants family matrix. For the special case of single input many output (SIMO), and many input single output (MISO) plant families, testable necessary and sufficient conditions for solvability of S-SFTSP are derived and whenever a solution exists, the family of solutions is given. The nature of the results is algebraic, since they depend on the properties of a rational vector space associated with the family; however, the final conditions are expressed as standard linear algebra tests.

1. INTRODUCTION

The problem of stabilising a family of k distinct plants $\{P_i, i = 1, 2, \ldots, k\}$ with a common controller C is known as *Simultaneous Stabilisation Problem* (SSP) (cf. [2], [10], [11]). SSP is a type of robust stabilisation problem and arises naturally in the synthesis of control systems with different modes of operation, due for instance to some structural changes; SPP also naturally arises when $P_1, \ldots, P_k$ represent linearised models of a nonlinear plant, around a number of operating points, and a common controller C is required to stabilise the whole family. Necessary and sufficient conditions for solvability of SSP have been given in [2] and [11], but these conditions are not computationally verifiable. The aim of this paper is to examine SSP in the context of discrete time systems and for the special type of stabilisation, known as Finite Settling Time Stabilisation (FSTS); for this case, it is shown that testable solvability conditions may be derived.

The *Total (or State) Finite Settling Time Stabilisation Problem*, or simply FSTSP is unique in disctete time systems and it is a generalisation of the extensively studied dead-beat response (cf. [3], [6], [7], [8], [12]). In the FSTS case all internal and external signals of the system are required to settle to a new steady state after a finite time from the application of a step change to its inputs (cf. [5]). The simultaneous FSTS problem of $km \times l$ discrete time plants $\{P_i, i = 1, 2, \ldots, k\}$ is then referred to as S-FSTSP.

In this paper the general case of S-FSTSP is considered first for plants of $m \times l$ common dimension and the results are then specialised to the case of $m \times 1$, of $1 \times l$ families of plants for which testable necessary and sufficient conditions are derived. With a family of k plants of $m \times l$ dimension $\{P_i, i = 1, 2, \ldots, k\}$ we may associate a family matrix and its properties lead to a classification of the various types of families, as well as general conditions for solvability of S-FSTSS. In the special case of vector plant families ($l = 1$, or $m = 1$) testable necessary and sufficient conditions are given and when a solution exists, the family of S-FSTS controllers is derived. The necessary and sufficient conditions are expressed as properties of the plant family matrix and may be tested using tools of the minimal basis theory of rational vector spaces (cf. [1]), or equivalent standard linear algebra tests.

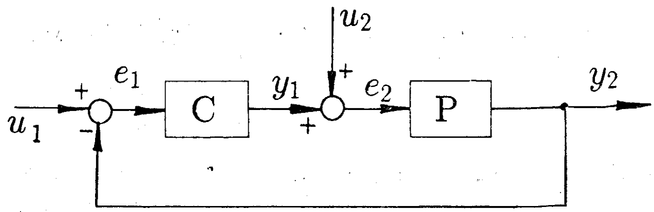

Fig. 2.1. Standard Feedback Configuration.

Throughout the paper, d denotes the delay operator ($d = z^{-1}$), $R(d)$, $R[d]$ are the sets of rational functions, polynomials in d respectively, $\mathscr{F}^{m \times n}[d]$ is the set of $m \times n$ matrices with elements from $\mathscr{F}(\mathscr{F} = R[d]$, or $R(d))$ and $\mathscr{U}_m[d]$ is the set of $m \times m$ $R[d]$ unimodular matrices. If $A \in \mathscr{F}^{m \times n}$, where $\mathscr{F}$ is a field, then $\varrho(A)$ denotes its rank and $\mathscr{N}_r(A)$, $\mathscr{N}_l(A)$ denotes the right, left null space over $\mathscr{F}$.

2. THE FINITE SETTLING TIME STABILISATION PROBLEM: DEFINITIONS AND BACKGROUND RESULTS

Consider the standard feedback configuration of Figure 2.1, where P and C are the pulse transfer function matrices of the discrete time plant and controller respectively, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are the externally applied inputs and $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ the outputs of the system. It is assumed that $P \in R^{m \times l}(d)$, $C \in R^{l \times m}(d)$ they are causal and that S_p, S_c are the state space descriptions of the plant and controller respectively. The finite settling time response of the standard feedback configuration is defined below (cf. [5]):

Definition 2.1. The discrete time feedback system of Figure 2.1 is said to exhibit: (i) an *External-Finite Settling Time* (E-FST) *Response*, or to be *Externally-FST* *stable* (E-FSTS), if for any step change in the components of the input vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$, all signals $\boldsymbol{y}_1, \boldsymbol{y}_2$ settle to a new steady state value in a finite number of steps.

(ii) an *Internal-Finite Settling Time* (I-FST) *Response, or to be Internally-FST stable*, if for every initial state vector and any step input, all states settle to a new steady state in finite time.

Note that in the above definition the values of the finite settling time and of the steady state are left free. The dead-beat response corresponds to the case where we have perfect tracking of step inputs and thus, it is a special case of the FST response.

Let $H(P, C)$ denote the transfer function matrix of the closed-loop feedback configuration from the input $\boldsymbol{u} = [\boldsymbol{u}_1^{\mathrm{T}}, \boldsymbol{u}_2^{\mathrm{T}}]^{\mathrm{T}}$ to the error vector $\boldsymbol{e} = [\boldsymbol{e}_1^{\mathrm{T}}, \boldsymbol{e}_2^{\mathrm{T}}]^{\mathrm{T}}$. If the feedback system is well formed ($|I + CP| = |I + PC| \neq 0$) it can be shown that

$$\boldsymbol{e}(d) = H(P, C)\, \boldsymbol{u}(d)\,, \quad H(P, C) = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix} \tag{1}$$

In the following we assume that the feedback configuration is also well-formed (i.e. composite state-space model is regular, not singular). The conditions for FST response of the feedback system are expressed by the following results (cf. [5], [9]):

Proposition 2.1. The feedback configuration of Figure 2.1 exhibits an external FST response, if and only if $H(P, C) \in R^{(m+l)\times(m+l)}[d]$, i.e. it is a polynomial matrix in d.

Remark 2.1. If S_P, S_C are stabilisable and detectable, then the condition that $H(P, C)$ is a polynomial matrix implies that the feedback system is internally stable (cf. [11]); however, the latter condition does not necessarily guarantee internal FST stability.

Proposition 2.2. If S_P, S_C are both controllable and observable, then the feedback configuration of Figure 2.1 exhibits a total (external, as well as internal) FST response, if and only if $H(P, C)$ is a polynomial matrix in d.

From the above result and using the standard results for the analysis of the feedback configuration (see e.g. [7], [11] we have the main result (cf. [9]):

Theorem 2.1. Let $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$, $C = N_C D_C^{-1} = \tilde{D}_C^{-1}\tilde{N}_C$ be $R[d]$-coprime MFD's for the plant and controller. The solution of the FSTSP exists, if and only if

$$\tilde{D}D_C + \tilde{N}N_C \in \mathcal{U}_m[d] \quad \text{or} \quad \tilde{D}_C D + \tilde{N}_C N \in \mathcal{U}_l[d] \tag{2}$$

Moreover, the family of all FSTS controllers is given by:

$$N_C = X + DR\,, \quad D_C = Y - NR\,, \quad R \in R^{l\times m}[d]\,, \quad |Y - NR| \neq 0 \tag{3}$$

$$\tilde{N}_C = \tilde{X} + \tilde{R}\tilde{D}\,, \quad \tilde{D}_C = \tilde{Y} - \tilde{R}\tilde{N}\,, \quad \tilde{R} \in R^{l\times m}[d]\,, \quad |\tilde{Y} - \tilde{R}\tilde{N}| \neq 0 \tag{4}$$

where R, $\tilde{R}$ are arbitrary and X, Y, $\tilde{X}$, $\tilde{Y}$ are appropriate $R[d]$ matrices satisfying the following Bezout identity

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & X \\ N & -Y \end{bmatrix} = I \tag{5}$$

The above family of FSTS controllers is identical to the Kučera-Bongiorno-Youla family. If the plant and controller are both stabilizable and detectable, then the above family defines the solution to External-FSTSP; when both plant and controller are controllable and observable, then the family defines the solution to Total-FSTSP.

3. THE SIMULTANEOUS FSTSP: STATEMENT OF THE PROBLEM AND BACKGROUND RESULTS

Let $\Sigma_k = \{P_i\colon P_i \in R^{m\times l}(d),\ i = 1, 2, \ldots, k\}$ be a k-family of discrete time controllable and observable plants, represented by their pulse transfer function matrices P_i, or by their $R[d]$-coprime MDF's $P_i = N_i D_i^{-1} = \tilde{D}_i^{-1}\tilde{N}_i$. The problem of finding the conditions under which there exists a controller C that stabilises in the FST sense all plants of the Σ_k family is referred to as *Simultaneous Finite Settling Time Stabilisation Problem* (S-FSTSP) and the controller that solves S-FSTSP will be called a Σ_k-S-FSTS controller.

If $C = N_C D_C^{-1} = \tilde{D}_C^{-1}\tilde{N}_C \in R^{l\times m}(d)$, then according to Theorem 2.1, C is a Σ_k-S-FSTS controller, if and only if

$$\tilde{D}_i D_C + \tilde{N}_i N_C = U_i \in \mathcal{U}_m[d]\,, \quad i = 1, 2, \ldots, k \tag{6}$$

or equivalently

$$\tilde{D}_C D_i + \tilde{N}_C N_i = \tilde{U}_i \in \mathcal{U}_l[d]\,, \quad i = 1, 2, \ldots, k\,. \tag{7}$$

The above conditions may be expressed as

$$T_k L = Q_u,\ T_k \equiv \begin{bmatrix} \tilde{D}_1 & \tilde{N}_1 \\ \vdots & \vdots \\ \tilde{D}_k & \tilde{N}_k \end{bmatrix}, \quad L \equiv \begin{bmatrix} D_C \\ N_C \end{bmatrix}, \quad Q_u \equiv \begin{bmatrix} U_1 \\ \vdots \\ U_k \end{bmatrix} \tag{8}$$

or

$$\tilde{L}\tilde{T}_k = \tilde{Q}_u\,, \quad \tilde{T}_k \equiv \begin{bmatrix} D_1, \ldots, D_k \\ N_1, \ldots, N_k \end{bmatrix}, \quad \tilde{L} \equiv [\tilde{D}_C, \tilde{N}_C]\,, \tag{9}$$

$$\tilde{Q}_u \equiv [\tilde{U}_1, \ldots, \tilde{U}_k]$$

where T_k, $\tilde{T}_k$ are referred to as *left-*, *right-plant Family matrix* (L-PFM, R-PFM) respectively; the matrices $Q_u \in R^{km\times m}[d]$, $Q_u \in R^{l\times kl}[d]$ are called *Partitioned Unimodular* and the corresponding sets will be denoted by $\mathcal{U}_{k,m}[d]$, $\tilde{\mathcal{U}}_{k,l}[d]$ respectively. The matrices $L, \tilde{L}$ are referred to as *Right, Left-Composite Representations* (R-CR, L-CR) and characterise the corresponding MFD's. In the study of S-FSTSP either (8), (9) may be used. We shall refer to (8) (9) as the *right*, *left* formulation of S-FSTSP respectively; in the following, we will work with the right formulation of S-FSTSP and all definitions and results can be translated to the left formulation in the obvious manner. We shall use the left formulation, whenewer (due to dimensions) there are certain advantages. Some preliminary definitions and results are considered first.

Remark 3.1. If $l \geqq m$, the Σ_k-S-FSTS controller represents a precompensator, in the standard configuration, whereas if $l \leqq m$ it represents an output feedback compensator.

Definition 3.1. A matrix $L = [D^T, N^T]^T \in R^{(m+l)\times m}[d]$, $D \in R^{m\times m}[d]$ with $|D| \not\equiv 0$ is called *column regular* (CR). If L is CR and $|D(0)| \not\equiv 0$, it will be called *column normal* (CN). Any $L \in R^{p\times k}[d]$ will be called *coprime*, if all its invariant polynomials are units of $R[d]$. A matrix $\tilde{L} = [\tilde{D}, \tilde{N}] \in R^{l\times(l+m)}[d]$ will be called *row regular* (RR), *row normal* (RN), if $\tilde{L}^T$ is CR, CN correspondingly.

Remark 3.2. If $L = [D^T, N^T]^T \in R^{(m+l)\times m}[d]$ is CR, then it defines a right MFD of a matrix $C = ND^{-1} \in R^{l\times m}[d]$; furthermore, C is causal is L is CN and the MFD is irreducible if L is coprime.

The solvability of S-FSTSP may be summarised as follows:

Remark 3.3. The S-FSTSP is solvable if and only if

$$T_k L = Q_u \tag{10}$$

has column regular solution L for some partitioned unimodular matrix Q_u. If S-FSTSP is solvable and L is also column normal, then the compensator C is causal and S-FSTSP will be called *Causal-S-FSTSP* (C-S-FSTSP).

The set of all families Σ_k of k, $m \times l$ systems will be denoted by $\mathscr{L}^k_{m,l}$. If $\Sigma_k \in \mathscr{L}^k_{m,l}$ and S-FSTSP is solvable then it will be called *S-FSTS family* and the set of all such families in $\mathscr{L}^k_{m,l}$ will be denoted by $\tilde{\mathscr{L}}^k_{m,l}$.

Remark 3.4. For any $\Sigma_k \in \mathscr{L}^k_{m,l}$ the L-PFM T_k is uniquely defined modulo permutations of the k row blocks and premultiplication by diag $\{U_1, \ldots, U_k\}$, where $U_i \in \mathscr{U}_m[d]$, $i = 1, 2, \ldots, k$.

A preliminary property of T_k matrices is given below.

Proposition 3.1. Let $\Sigma_k \in \mathscr{L}^k_{m,l}$ and T_k be a L-PFM. If $r = \operatorname{rank}_{R(d)}\{T_k\}$, then the following hold true:

(i) $$m \leqq r \leqq m + l \tag{11}$$

(ii) If $\{f_i(d), i \in \tilde{r}\}$ is the set of invariant polynomials of T_k, then the $f_1(d) = \ldots = f_m(d) = 1$.

Proof. Since $[\tilde{D}_i, \tilde{N}_i\}$ for any i is a rank m coprime matrix, part (ii) follows as well as that $r \geqq m$. The rest of the proof is obvious. □

Remark 3.5. For any $\Sigma_k \in \mathscr{L}^k_{m,l}$ the Smith form of any L-PFM, T_k, with $r = \operatorname{rank}_{R(d)}\{T_k\}$, is of the type:

$$S_k \equiv S(T_k) = \begin{bmatrix} I_m & & 0 \\ & S^* & \\ 0 & & 0 \end{bmatrix} = \begin{bmatrix} S_T^* & 0 \\ 0 & 0 \end{bmatrix}, \quad S^* = \operatorname{diag}\{f_{m+1}, \dots, f_r\} \tag{12}$$

According to the properties of the Smith form we may classify the families Σ_k as shown below:

Definition 3.2. Let $\Sigma_k \in \mathscr{L}^k_{m,l}$ and S_k be the Smith form of a L-PFM T_k, as in (12). The family Σ_k will be called:

(i) *Degenerate,* if $r < \min\{m + l, km\}$ and *strongly degenerate,* if $r = m$; otherwise, if $r = \min\{km, m + l\}$ it will be called *nondegenerate.*

(ii) *Coprime,* if $f_{m+1} = \dots = f_r = 1$; otherwise, it is called *noncoprime.*

(iii) *Complete,* if it is nondegenerate and coprime.

(iv) *Square* if $r = km = m + l$, *left regular* if $r = km \leqq m + l$ and *right regular,* if $r = m + l \leqq km$.

The analysis of S-FSTSP is based on the matrix equation (10) and some useful notions related to the analysis of such equations are defined below.

Definition 3.3. Let $A \in R^{p\times q}[d]$, and $r = \operatorname{rank}_{R(d)}\{A\} \leqq \min(p, q)$. Consider the Smith form decomposition of A defined by

$$\hat{U}_L A \hat{U}_R = \begin{bmatrix} S_A^* & 0 \\ 0 & 0 \end{bmatrix} = S_A \Leftrightarrow A = U_L S_A U_R \tag{13}$$

where S_A is the Smith form of A, S_A^* its $r \times r$ essential part and U_L, $\hat{U}_L = U_L^{-1} \in \mathscr{U}_p[d]$, U_R, $\hat{U}_R = U_R^{-1} \in \mathscr{U}_q[d]$. If $\hat{U}_L$, $\hat{U}_R$ are partitioned according to the partitioning of S_A, i.e.

$$\hat{U}_L = \begin{bmatrix} A_l^+ \\ A_l^\perp \end{bmatrix}, \quad \hat{U}_R = [A_r^+, A_r^\perp] \tag{14}$$

then $A_l^+ \in R^{r\times p}[d]$, $A_r^+ \in R^{q\times r}[d]$ are called *left, right-projectors* respectively and $A_l^\perp \in R^{(p-r)\times p}[d]$, $A_r^\perp \in R^{q\times(q-r)}[d]$ are called *left-, right-annihilators* correspondingly of A.

With the system $\Sigma_k \in \mathscr{L}^k_{m,l}$ and for the T_k L-PFM we may associate rational vector spaces and $R[d]$ modules as:

$$\mathscr{X}_r \equiv \text{row span}_{R(d)}\{T_k\}, \quad \mathscr{X}_c \equiv \text{col. span}_{R(d)}\{T_k\} \tag{15}$$

$$\mathscr{M}_r \equiv \text{row span}_{R[d]}\{T_k\}, \quad \mathscr{M}_c \equiv \text{col. span}_{R[d]}\{T_k\} \tag{16}$$

where $\mathscr{X}_r$, $\mathscr{X}_c$ are the *row, column-$R(d)$-spaces* and $\mathscr{M}_r$, $\mathscr{M}_c$ are the *row-, column-$R[d]$ modules* of T_k respectively. We shall denote by $\mathscr{M}_r^*$, $\mathscr{M}_c^*$ the maximal $R[d]$-modules in $\mathscr{X}_r$, $\mathscr{X}_c$ respectively. Using the same notation as in Definition 3.3, we may illustrate the significance of above defined concepts by the following result obtained in [4]:

Lemma 3.1. Let $A \in R^{p\times q}[d]$, $r = \operatorname{rank}_{R(d)}\{A\}$ and let $\{A_l^+, A_l^\perp\}$, $\{A_r^+ A_r^\perp\}$ be pairs of left projector, left annihilator, right projector, right annihilator of A respectively. Then,

$$A_l^+ A = \tilde{A}_l\,, \quad A_l^\perp A = 0 \tag{17}$$

$$AA_r^+ = \tilde{A}_r\,, \quad AA_r^\perp = 0 \tag{18}$$

where $\tilde{A}_l$, $\tilde{A}_r$ are basis matrices for the row, column $R[d]$-modules of A respectively and $A_l^\perp$, $A_r^\perp$ are least degree bases for the left, right $R(d)$-spaces $\mathcal{N}_l\{A\}$, $\mathcal{N}_r\{A\}$ correspondingly. Furthermore, we may write

$$\tilde{A}_l = Z_A^l A_l^*\,, \quad \tilde{A}_r = A_r^* Z_A^r \tag{19}$$

where A_l^*, A_r^* are minimal basis matrices for the row, column of A and Z_A^l, Z_A^r are $r \times r$ matrices $R[d]$-equivalent to S_A^* (essential part of the Smith form of A).

A matrix $Z \in R^{r\times r}[d]$, which is $R[d]$-equivalent to S_A^* (essential part of Smith form) is called a *Left-Right Divisor* (LRD) (cf. [4]; clearly, Z_A^r, Z_A^l above are LRD's.

Remark 3.6. For any pair $(A_l^+, A_l^\perp)$, or $(A_r^+, A_r^\perp)$ defined on A, we have that

$$Q_l = \begin{bmatrix} A_l^+ \\ A_l^\perp \end{bmatrix} \in \mathscr{U}_p[d]\,, \quad Q_r = [A_r^+, A_r^\perp] \in \mathscr{U}_q[d] \tag{20}$$

If A has full rank, then at least one of annihilators does not exist.

Using the above concepts we may express the solvability of matrix equations $AX = B$ in the following way [4]:

Lemma 3.2. Let $A \in R^{p\times q}[d]$, $B \in R^{p\times t}[d]$, $r = \operatorname{rank}_{R(d)}\{A\} \leqq \min\{p, q\}$ and consider the matrix equation over $R[d]$

$$AX = B\,, \quad X \in R^{q\times t}[d] \tag{21}$$

(i) For any $(A_l^+, A_l^\perp)$ pair, there exists a pair $(A_r^+, A_r^\perp)$ such that if

$$X = [A_r^+ \,\vdots\, A_r^\perp] \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = A_r^+ X_1 + A_r^\perp X_2 \tag{22}$$

the equation (21) is equivalent to (22) together with

$$A_l^\perp B = 0 \tag{23}$$

$$ZX_1 = A_l^+ B \tag{24}$$

where Z is a LRD of A.

(ii) Conditions (23), (24) are necessary and sufficient for solvability of (21). If these conditions are satisfied, then for any X_1 solving (24), there exists a family of X

matrices defined by (22), where X_2 is an appropriate dimensions arbitrary $R[d]$ matrix.

The above Lemma provides tools for the study of S-FSTSP. Some results on the general case are considered first.

Proposition 3.2. If Σ_k is strongly degenerate, then:

(i) For every pair of systems S_i, S_j described by $(\tilde{D}_i, \tilde{N}_i)$, $(\tilde{D}_j, \tilde{N}_j)$, there exists $Q_{ij} \in \mathcal{U}_m[d]$ such that

$$[\tilde{D}_j, \tilde{N}_j] = Q_{ij}[\tilde{D}_i, \tilde{N}_i] \quad \forall i, j, \in \tilde{k} = \{1, 2, \ldots, k\} \tag{25}$$

(ii) There exists a family of Σ_k-S-FSTS controllers, which is the family that stabilises any pair $(\tilde{D}_i, \tilde{N}_i) \in \Sigma_k$.

Proof. (i) If Σ_k is strongly degenerate, then any $[\tilde{D}_i, \tilde{N}_i]$ matrix, which by definition has full rank and it is coprime, defines a least degree basis matrix for the $\mathscr{X}_r$ space, or a basis for the maximal module $\mathscr{M}_r^*$. Clearly, any two bases of $\mathscr{M}_r^*$ are related by $R[d]$-unimodular matrices. Part (ii) readily follows from part (i).

Theorem 3.1. Let $\Sigma_k \in \mathscr{L}_{m,l}^k$ and assume that Σ_k is both left regular and coprime. The causal S-FSTSP is always solvable on the Σ_k family; furthermore, if T is a L-PFM, there exists a pair of right projector and annihilator $(T_r^+, T_r^\perp)$ such that the family of solutions of S-FSTSP is given by

$$L = T_r^+ Q_u + T_r^\perp X \tag{26}$$

where $Q_u \in \mathcal{U}_{k,m}[d]$ is an arbitrary partitioned unimodular and X an appropriate dimension but otherwise arbitrary $R[d]$-matrix.

Proof. If $T \in R^{km \times (m+l)}[d]$ is a L-PFM of a left regular and coprime matrix, there exists $U \in \mathcal{U}_{m+l}[d]$ such that

$$TU = [I_{km}\ 0] = S_T\,. \tag{27}$$

By partitioning U as $U = [T_r^+, T_r^\perp]$ according to the partitioning S_T and by writing

$$L = [T_r^+, T_r^\perp]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = T_r^+ L_1 + T_r^\perp L_2\,. \tag{28}$$

(10) is reduced to $L_1 = Q_u$ with L_2 arbitrary and this proves that L is written as in (26). Since Q_u is an arbitrary partitioned unimodular and X also arbitrarily selected, it is readily seen that there always exists a pair (Q_u, X) such that L is column normal and thus column regular. □

Remark 3.7. Let $U = [T_r^+, T_r^\perp]$ be the pair defined in Theorem 3.1 and partition $U(0)$

$$U(0) = \begin{bmatrix} N \\ \cdots \\ N' \end{bmatrix}, \quad N \in R^{m \times (m+l)} \neq 0 \tag{29}$$

For every pair of partitioned unimodular Q_u and arbitrary X in (26) such that

$$N\begin{bmatrix} Q_u(0) \\ X(0) \end{bmatrix} = D(0), \quad |D(0)| \neq 0 \tag{30}$$

The S-FSTSP defined on the left regular, coprime Σ_k, has a causal solution.

We consider now the case of right regular families, for which we may state the following result:

Theorem 3.2. Let $\Sigma_k \in \mathscr{L}^k_{m,l}$ and assume that Σ_k is right regular and coprime. If T is a L-PFM, $(T_l^+, T_l^\perp)$ a pair of left-projector, -annihilator of T and $\tilde{T}_l^+$ is first m-row block of T_l^+, then necessary and sufficient condition for the causal S-FSTSP to be solvable is that there exists $Q_u \in \mathscr{U}_{k,m}[d]$ such that

$$T_l^\perp Q_u = 0 \tag{31}$$

$$|\tilde{T}_l^+(0)\, Q_u(0)| \neq 0 \tag{32}$$

If the above conditions are satisfied, then the solution is given by

$$L = T_l^+ Q_u\,. \tag{33}$$

Proof. Since T is right regular and coprime, there exists $U \in \mathscr{U}_{km}[d]$ such that T is reduced to its Smith form as:

$$UT = \begin{bmatrix} I_{m+l} \\ 0 \end{bmatrix}, \quad U = \begin{bmatrix} T_l^+ \\ T_l^\perp \end{bmatrix} \tag{34}$$

By partitioning U as shown above it is clear that (10) is equivalent (after premultiplication by U) to

$$\begin{bmatrix} T_l^+ \\ T_l^\perp \end{bmatrix} TL = \begin{bmatrix} T_l^+ \\ T_l^\perp \end{bmatrix} Q_u \Leftrightarrow \begin{cases} L = T_l^+ Q_u \\ T_l^\perp Q_u = 0 \end{cases} \tag{35}$$

It is clear from (35) that (32) is the condition that guarantees that L is CN and thus also DR. □

Theorems 3.1, 3.2 cover the generic cases where either $km < m + l$, or where $km > m + l$, since nondegeneracy and coprimeness are generic properties. Using Lemma 3.2 we may state the conditions characterising the solvability conditions for the general case.

Theorem 3.3. Let $\Sigma_k \in \mathscr{L}^k_{m,l}$, T be a L-PFM, $\operatorname{rank}_{R(d)}(T) = r$. For any $(T_l^+, T_l^\perp)$ pair, there exists a pair $(T_r^+, T_r^\perp)$ such that if

$$L = [T_r^+, T_r^\perp]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = T_r^+ L_1 + T_r^\perp L_2 \tag{36}$$

then S-FSTSP is solvable if an only if there exist $Q_u \in \mathscr{U}_{km}[d]$ such that

$$T_l^\perp Q_u = 0 \tag{37}$$

$$Z L_1 = T_l^+ Q_u \tag{38}$$

where Z is a LRD of T. If (37), (38) are satisfied for some Q_u, then the family of solutions is given by (36), where L_2 is an arbitrary $R[d]$ matrix of appropriate dimensions.

The proof follows from Lemma 3.2, whereas the existence of column normal and column regular solutions may be argued as in Remark 3.7.

Remark 3.8. If $\mathcal{N}_l\{T\} \neq \{0\}$, condition (37) is present and expresses the fact that for solvability of S-FSTSP, it is necessary that the $R(d)$-column space $\mathcal{X}_c$ of T contains vectors which form a partitioned unimodular matrix. This alternative formulation of (37) is also valid, when also $\mathcal{N}_l\{T\} = \{0\}$. We shall refer to this condition as the *space structural condition* (SSC) of S-FSTSP.

Remark 3.9. Since Z, L_1 in (38) have dimensions $r \times r$, $r \times m$ respectively and $r \geqq m$, the solvability of (38) is not a trivial divisor condition, unless $r = m$. Note, that if $r = m$, then Σ_k is strongly degenerate and by Proposition 3.2 it follows that Z is $R[d]$-unimodular. Condition (38) is thus an essential condition when $r > m$ and shall refer to it as the *extended-divisor condition* (EDC) of S-FSTSP.

Remark 3.10. If the family Σ_k is degenerate, but coprime then any LRD Z of T is $R[d]$-unimodular and the *space* structural condition is the only condition which has to be tested.

The conditions (37) (38) may be combined to give the following alternative formulation of the problem.

Corollary 3.1. Let $\Sigma_k \in \mathcal{L}^k_{m,l}$, T be a L-PFM, $r = \operatorname{rank}_{R(d)} \{T\}$, $r \leqq \min\{km, m + l\}$. The S-FSTSP is solvable, if and only if for any pairs $(T_l^{\perp}, T_l^{+})$, $(T_r^{\perp}, T_r^{+})$ and associated LRD Z, the following conditions are satisfied:

(i) There exists a solution $X \in R^{(km+r)\times m}[d]$ of the equation

$$\begin{bmatrix} Z & -T_l^{+} \\ 0 & T_l^{\perp} \end{bmatrix} X = 0\,, \quad X = \begin{bmatrix} L_1 \\ \dots \\ Q_u \end{bmatrix}, \quad Q_u \in \mathcal{U}_{km}[d]\,. \tag{39}$$

(ii) For any L_1, Q_u solution of (39), there exists L_2 of appropriate dimensions such that

$$L = T_r^{+} L_1 + T_r^{\perp} L_2 \tag{40}$$

is column regular.

Equation (39) reduces the overall problem to an investigation of existence of a matrix, which is partially partitioned unimodular and has its columns from a given rational vector space. It is worth pointing out that if $km \neq m + l$, then generically, the families of $\mathcal{L}^k_{m,l}$ are nondegenerate and coprime. The space structural condition thus becomes the most significant. For special families of systems, this condition takes a rather simple form that allows the derivation of testable solvability conditions.

4. THE S-FSTSP ON FAMILIES OF VECTOR PLANTS

The analysis so far has shown that if $km < m + l$, then for a generic family Σ_k, the S-FSTSP is solvable, whereas if $km > m + l$, and the family Σ_k is once more generic, then solvability of S-FSTSP is reduced to a testing of the space-structural condition. The general problem associated with SSC, that is finding the conditions for the existence of partitioned unimodular matrices in a given rational vector space, is still open; however, this problem takes a rather simple form in certain special cases. From the formulation of S-FSTSP we note:

Remark 4.1. For families $\mathscr{L}^k_{1,l}$ and $\mathscr{L}^k_{m,1}$ the solvability of S-FSTSP is reduced to the study of the following equations:

(i) $\mathscr{L}^k_{1,l}$ families: From (8) we have:

$$T_k \boldsymbol{l} = \boldsymbol{q}_u\,, \quad T_k \in R^{k\times(l+1)}[d]\,, \quad \boldsymbol{l} \in R^{(l+1)}[d]\,, \quad \boldsymbol{q}_u \in R^k \tag{41}$$

where $\boldsymbol{q}_u^{\mathrm{T}} = [c_1, \ldots, c_k]$, $c_i \neq 0$, for all $i = 1, 2, \ldots, k$. In this case, $\boldsymbol{l}$ is the R-CR of the vector precompensator $\boldsymbol{c} = \boldsymbol{n}_c d_c^{-1}$.

(ii) $\mathscr{L}^k_{m,1}$ families: From (9) we have:

$$\tilde{\boldsymbol{l}}^{\mathrm{T}} \tilde{T}_k = \tilde{\boldsymbol{q}}_u^{\mathrm{T}}\,, \quad \tilde{T}_k \in R^{(m+1)\times k}[d]\,, \quad \tilde{\boldsymbol{l}} \in R^{1\times(m+1)}[d]\,, \quad \tilde{\boldsymbol{q}}_u \in R^k \tag{42}$$

where $\tilde{\boldsymbol{q}}_u^{\mathrm{T}} = [\tilde{c}_1, \ldots, \tilde{c}_k]$, $\tilde{c}_i \neq 0$ for all $i = 1, 2, \ldots, k$. In this case, $\tilde{\boldsymbol{l}}^{\mathrm{T}}$ is the L-CR of the vector feedback compensator $\boldsymbol{c}^{\mathrm{T}} = \tilde{d}_c^{-1} \tilde{\boldsymbol{n}}_c$.

The families $\mathscr{L}^k_{1,l}$, $\mathscr{L}^k_{m,1}$ contain systems with either one output, or one input and thus they have vector transfer functions; We shall refer to such families as families of *vector plants*. It is clear, that the study of S-FSTSP on such families is simpler, since the partitioned unimodular matrices become constant vectors with all components nonzero; we shall denote by R^k_0, all vectors of R^k with all coordinates nonzero. In the following, the case of many-input single output (MISO) families is considered, whereas the results for the single input many output (SIMO) case are similar. The case of left regular families has already been discussed. Since we want to explore space structure condition we shall assume throughout this section that the families are always coprime and nondegenerate.

Theorem 4.1. Let $\Sigma_k \in \mathscr{L}^k_{1,l}$ be a right regular coprime family and assume that $k > k + 1$. If T is a L-PFM and $(T_l^+, T_l^\perp)$ a pair of left projector, annihilator associated with T and $\boldsymbol{t}_1^{\mathrm{T}}$ is the first row of T_l^+, then the causal S-FSTSP is solvable on Σ_k if and only if there exists a $\boldsymbol{c} \in R^k_0$ such that

$$T_l^\perp \boldsymbol{c} = 0 \tag{43}$$

$$\boldsymbol{t}_1^{\mathrm{T}}(0)\, \boldsymbol{c} \neq 0 \tag{44}$$

If the above conditions are satisfied, then the solution is given by

$$\boldsymbol{l} = \begin{bmatrix} d_c \\ \boldsymbol{n}_c \end{bmatrix} = T_l^+ \boldsymbol{c} \tag{45}$$

This result follows from Theorem 3.2. The significance of condition (43) is emphasised by the following result.

Corollary 4.1. Let $\Sigma_k \in \mathscr{L}^k_{1,l}$, T be a L-PFM and assume that $\mathcal{N}_l\{T\} \neq \{0\}$. Necessary conditions for S-FSTSP to be solvable on Σ_k is that either of the following equivalent conditions hold true:

(i) If $\mathscr{M}_c$ is the column module of T, then $\mathscr{M}_c$ has at least a zero dynamical index; furthermore, id $\mathscr{M}_c^0$ is the submodule characterised by the zero dynamical indices, then $\mathscr{M}_c^0 \cap R_0^k \neq 0$.

(ii) If $T_l^\perp[d] = T_0 + dT_1 + \ldots + d^n T_n$ and $\hat{T} = [T_0^T, T_1^T, \ldots, T_n^T]^T$, then $\mathcal{N}_r\{\hat{T}\} \cap R_0^k \neq 0$.

(iii) If $T_l^1(d) = [\boldsymbol{t}_1(d), \ldots, \boldsymbol{t}_k(d)]$, then the set $\{\boldsymbol{t}_i(d), i = 1, 2, \ldots, k\}$ is completely dependent over R (i.e. no coefficient in the relationship is zero).

Proof. Note that $T_l^\perp(d)\,\boldsymbol{c} = 0$ is a necessary condition for solvability of S-FSTSP, when $\mathcal{N}_l\{T\} \neq \{0\}$. Clearly conditions (ii) and (iii) express the $T_l^\perp(d)\,\boldsymbol{c} = \mathbf{0}$ condition. Condition (i) is also obvious and expresses the formulation of the problem as in (4.1).

Note that part (i), or (ii) also provide tools for the computation of the vectors $\boldsymbol{c} \in R_0^k$ which satisfy the space structure condition.

Remark 4.2. If N is a basis matrix for $\mathcal{N}_r\{\hat{T}\}$, then the space structure condition is satisfied, if and only if the matrix N has no zero rows.

5. CONCLUSIONS

The S-FSTSP has been addressed and necessary and sufficient conditions for its solvability have been given. It has been shown that for the left regular and coprime families a solution always exists, whereas for the left singular case of plant families the space structure condition is the key one. For the case of families of vector plants the latter condition is readily testable using standard linear algebra tools. The derivation of testable criteria for the space structure condition in the general case is under investigation.

(Received December 13, 1990.)

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