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A STRUCTURAL ANALYSIS OF THE POLE SHIFTING PROBLEM

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Pole assignability and stabilizability problems are investigated qualitatively. Both problems are first formulated algebraically and then translated to a structural setting via graph-theoretic formulations. Graphical sufficient conditions concerning the closed-loop system digraph which determine pole assignable and stabilizable structures are developed.

1. INTRODUCTION

The concept of structure was first introduced by Lin [1] in his characterization of structural controllability for single-input systems. Since then many researchers have demonstrated (see, for example, [2] and the references therein) the use of a qualitative approach based on system structure in the analysis of such properties as controllability, observability, existence of fixed modes, etc., which describe latent qualities possessed by a system. Such an approach is consistent with physical reality since system parameter are never known precisely. On the other hand, the insight into the system structure gained by a qualitative analysis is often helpful in establishing a way-out through certain problems that arise due to features as high-dimensionality, uncertainty in system parameters and constraints on information structure. A qualitative analysis also enables investigation of general system properties from the genericity point of view.

In this paper, we present a qualitative analysis of the pole assignment and stabilization problems. We assume that the nonzero system parameters are algebraically independent, and realize a qualitative analysis of the problems of pole assignability and stabilizability in terms of the structure of the closed-loop system. Using graph theoretic formulations due to Reinschke [3], we develop graphical results which can be used to characterize certain pole assignable and stabilizable structures.

We first present an algebraic formulation of the two problems combined as the general problem of pole-shifting. The preliminaries for the establishment of the structural framework needed for our graph-theoretical approach is followed by the main results concerning the two problems. Proofs, which are avoided in this paper, can be found in [4].

2. ALGEBRAIC CHARACTERIZATION OF THE POLE-SHIFTING PROBLEM

Consider a linear, time-invariant system described as

$$\mathscr{S}: \quad \dot{x} = Ax + Bu \qquad y = Cx \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^r$ are the state, input, and output of $\mathscr{S}$, and A, B, and C are real, constant matrices of appropriate dimensions. Applying a constant output feedback

$$\mathscr{F}: \quad u = Fy \tag{2}$$

to $\mathscr{S}$ in (1) results in a closed-loop system

$$\mathscr{S}(\mathscr{F}): \quad \dot{x} = (A + BFC)\,x \qquad y = Cx \tag{3}$$

having a characteristic polynomial

$$p(s) = \det(sI - A - BFC) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n\,. \tag{4}$$

Let the $m \times r$ feedback matrix F have $\mu \leqq mr$ nonzero elements which can be chosen arbitrarily. If $f = (f_1, f_2, \ldots, f_\mu) \in \mathbb{R}^\mu$ and $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ represent the nonzero elements of F and the coefficients of $p(s)$ in (4) respectively, then the relation between p and f can be represented by a smooth mapping $g\colon \mathbb{R}^\mu \to \mathbb{R}^n$ as

$$p = g(f)\,. \tag{5}$$

The problem of pole-shifting is concerned with the existence of a solution $f \in \mathbb{R}^\mu$ of (5) for every given $p \in \mathbb{R}^n$ (arbitrary pole assignment problem) or for some $p \in \mathbb{R}^n$ corresponding to a stable polynomial (stabilization problem).

We assume that $\mu \geqq n$ (note that this is a necessary condition for arbitrary pole assignability), and partition the feedback variables $f_1, f_2, \ldots, f_\mu$ into two disjoint subsets f_v and f_c, containing n and $\mu - n$ elements respectively. Fixing the variables in f_c at particular real values, (5) reduced to

$$p = \bar{g}(f_v)\,, \tag{6}$$

where $\bar{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a restriction of g to $\mathbb{R}^n$. The following results give sufficient conditions for pole assignability and stabilizability.

Lemma 1 [3]. Suppose that after appropriately fixing the elements of f_c, the derivative $\bar{g}_{f_v}$ of $\bar{g}$ in (6) is unimodular. Then the system $\mathscr{S}$ is arbitrarily pole assignable by the feedback $\mathscr{F}$.

Note that unimodularity of $\bar{g}_{f_v}$ implies that $\bar{g}$ is a homeomorphism, and hence for every $p \in \mathbb{R}^n$, there exists a unique $f_v \in \mathbb{R}^n$ satisfying $\bar{g}(f_v) = g(f_v, f_c) = p$.

Lemma 2 [4]. Let the feedback variables in f_v be renumbered as $f_1, f_2, \ldots, f_n$. Suppose that the mapping $\bar{g}$ between p and f can be written in the 'staircase' form

$$
\begin{aligned}
p_1 &= \bar{g}_1(f_v) = \alpha_1 + (a_1 f_1 + c_1)\, b_{11} + \ldots + (a_k f_k + c_k)\, b_{k1} + \ldots + (a_n f_n + c_n)\, b_{n1} \\
&\;\;\vdots \\
p_k &= \bar{g}_k(f_v) = \alpha_k + (a_k f_k + c_k)\, b_{kk} + \ldots + (a_n f_n + c_n)\, b_{nk} \\
&\;\;\vdots \\
p_n &= \bar{g}_n(f_v) = \alpha_n + (a_n f_n + c_n)\, b_{nn}
\end{aligned} \tag{7}
$$

where a_k, b_{kj}, c_k are polynomials in $f_1, \ldots, f_{k-1}$, $1 \leqq k \leqq n$, $1 \leqq j \leqq k - 1$, with $a_k \neq 0 \neq b_{kk}$ and α_k are constants. Then $\mathcal{S}$ is stabilizable with $\mathcal{F}$.

The result of this lemma depends on the fact that the structure in (7) allows for stabilization of $p(s)$ in (4) by a recursive root-locus technique, utilizing arbitrarily high feedback gains of appropriate sign at each step [4].

3. PRELIMINARIES

A *structured matrix* $\mathbf{M}$ [5] is a matrix whose entries are either fixed zeros or algebraically independent parameters in $\mathbb{R}$. If $\mathbf{M}$ has μ nonzero entries, then we associate with these a parameter space $\mathbb{R}^\mu$ such that every data point $d \in \mathbb{R}^\mu$ defines a matrix $M = \mathbf{M}(d)$. Two matrices M_1 and M_2 are said to be structurally equivalent if there corresponds the same structured matrix $\mathbf{M}$ to both, and $\mathbf{M}$ represents the equivalence class of structurally equivalent matrices.

Let Π be a property asserted about the structured matrix $\mathbf{M}$. Then it is a mapping Π: $\mathbb{R}^\mu \to \{0, 1\}$ defined as

$$\Pi(d) = \begin{cases} 1\,, & \text{if } \Pi \text{ holds for } \mathbf{M}(d) \\ 0\,, & \text{otherwise} \end{cases}$$

If the set $\{d \in \mathbb{R}^\mu \mid \Pi(d) = 0\}$ is included in a variety in $\mathbb{R}^\mu$ characterized as the set of zeros of a nonzero polynomial in d, then Π is said to be generic. A generic property holds almost everywhere in $\mathbb{R}^\mu$.

A *digraph* [6] is an ordered pair $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of *vertices* and $\mathcal{E}$ a set of oriented *edges*. And edge oriented from $v_j \in \mathcal{V}$ to $v_i \in \mathcal{V}$ is denoted by the ordered pair (v_j, v_i). If $(v_j, v_i) \in \mathcal{E}$, then v_j is said to be *adjacent* to v_i, and v_i adjacent from v_j. Adjacency relation can be described by a square binary matrix, $\mathbf{R} = (\mathbf{r}_{ij})$ such that $\mathbf{r}_{ij} = 1$ if and only if $(v_j, v_i) \in \mathcal{E}$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$ where all vertices are distinct is called a *path* from v_1 to v_k. If v_k coincides with v_1, then the path is called a *cycle*. Any two cycles are said to be disjoint if they have no common vertices. A collection of disjoint cycles is called a *cycle family*.

Associated with the system $\mathcal{S}$ of (1), we define a square structured matrix as

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{8}$$

called the system structure matrix. Viewing the matrix $\mathbf{S}$ as a binary matrix with zero and nonzero elements, we define the digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ which assumes $\mathbf{S}$ as its adjacency matrix to be the digraph of the system $\mathcal{S}$. For convenience, the vertex set of $\mathcal{D}$ can be partitioned as $\mathcal{V} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are the sets of input, state and output variables, respectively. The system digraph $\mathcal{D}$ completely characterizes the structure of $\mathcal{S}$. We say that two dynamic systems are structurally equivalent if their digraphs are the same up to an enumeration of $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$.

When a feedback of the from (2) is applied to $\mathcal{S}$ of (1), the resulting closed loop-system of (3) has the system structure matrix

$$\mathbf{S}(\mathbf{F}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{9}$$

Accordingly, the system digraph becomes $\mathcal{D}(\mathcal{F}) = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_F)$, where $\mathcal{E}_F = \{(y_j, u_i) \mid \mathbf{f}_{ij} \neq 0\}$ is the set of feedback edges. For convenience, the edges in $\mathcal{E}$ are called the d-edges and those in $\mathcal{E}_{\mathcal{F}}$ the f-edges. Accordingly, a cycle is called an f-cycle if it contains at least one f-edge and a d-cycle otherwise. Similarly, a cycle family is called an f-cycle family if it contains at least one f-edge, a simple f-cycle family if it contains one and only one f-edge, and d-cycle family otherwise. Where necessary, we will represent a cycle family as a collection of cycles which in turn are represented by the set of the weights of the edges contained. Note that if a feedback variable f_{ij} is given a fixed nonzero value, then the corresponding f-edge (y_j, u_i) becomes a d-edge as f_{ij} is no more different from a nonzero parameter of A, B or C.

4. STRUCTURAL FORMULATION OF THE POLE-SHIFTING PROBLEM

We first state our definition of a structurally pole assignable (respectively stabilizable) system.

Definition 1. A system $\mathcal{S}$ of (1) is said to be structurally pole assignable (respectively stabilizable) by a feedback $\mathcal{F}$ of (2) if there exists a system structurally equivalent to $\mathcal{S}$ which is pole assignable (respectively stabilizable) by $\mathcal{F}$.

Let us assume, that the nonzero parameters of the system structure matrix $\mathbf{S}$ in (8) are algebraically independent, and correspond to a data point $d \in \mathbb{R}^{\mu}$. Then, the relation in (5) can be expressed as

$$p = g(d, f) \tag{10}$$

to indicate the dependence of g on the system parameters. Clearly, a system is structurally pole assignable if for some $d^* \in \mathbb{R}^u$ the equation

$$p = g(d^*, f) = g^*(f) \tag{11}$$

has a solution for every given $p \in \mathbb{R}^n$. Similarly, structural stabilizability is concerned with the existence of a particular $d^* \in \mathbb{R}^n$ for which one can find an f that satisfies (11) for some $p \in \mathbb{R}^n$ corresponding to a stable polynomial. We note that, due to the nonlinearity of (10), neither structural pole assignability not stabilizability are generic properties, in general. However, in this paper, we aim at obtaining graphical conditions which guarantee genericity of the two structural properties. The formulation due to Reinschke [7], which is summarized below, serves this purpose.

Consider the closed loop system digraph $\mathscr{D}(\mathscr{F}) = (\mathscr{V}, \mathscr{E} \cup \mathscr{E}_F)$ associated with the system structure matrix $\mathbf{S}(\mathbf{F})$ of (9). By assigning a weight to every edge, $\mathscr{D}(\mathscr{F})$ becomes a weighted digraph. The weight of a d-edge is the corresponding nonzero parameter value of A, B or C, and the weight of an f-edge is the corresponding variable feedback gain. Accordingly, the weight of a path, a cycle or a cycle family is the product of weights of all edges involved. Denoting the number of cycles in a cycle family $\mathscr{CF}$ by $\sigma(\mathscr{CF})$, the weight of $\mathscr{CF}$ by $\omega(\mathscr{CF})$, and defining the width $\gamma(\mathscr{CF})$ of $\mathscr{CF}$ to be the total number of state vertices covered by $\mathscr{CF}$, Reinschke proved the following:

Lemma 3. The coefficients $p_k = g_k(f)$, $k = 1, 2, \ldots, n$, of the closed loop characteristic polynomial are given as

$$g_k(f) = \sum_{\gamma(\mathscr{CF}) = k} (-1)^{\sigma(\mathscr{CF})} \omega(\mathscr{CF}) \tag{12}$$

where the summation is carried over all cycle families of width k.

5. GENERIC POLE ASSIGNABILITY

The following result, which is a special of Lemma 1, gives algebraic sufficient conditions for generic pole assignability:

Corollary 1. Let f_v and f_c be as defined in Lemma 1, with the feedback variables in f_v renumbered as $f_1, f_2, \ldots, f_n$. For a partitioning $\mathscr{N} = \mathscr{I} \cup (\mathscr{N} - \mathscr{I})$, with $\mathscr{I} \neq \emptyset$, of the index set $\mathscr{N} = \{1, 2, \ldots, n\}$, let the auxiliary variables $\tilde{f}_k$ be defined as

$$\tilde{f}_k = \begin{cases} f_k, & k \in \mathscr{I} \\ \theta_k f_k + \psi_k, & k \in \mathscr{N} - \mathscr{I} \end{cases} \tag{13}$$

where $\theta_k = \theta_k(d)$ are nonzero polynomials in d, and $\psi_k = \psi_k(d, f_{\mathscr{I}})$ are polynomials in f_l, $l \in \mathscr{I}$, with coefficients being polynomials in d. Suppose that the restriction $\bar{g}$

of g in (6) to $\mathbb{R}^n$ is given by

$$\bar{g}_k(d, f_v) = \tilde{g}_k(d; \tilde{f}) = \alpha_k + \sum_{l=1}^{n} e_{kl}\tilde{f}_l, \quad k = 1, 2, \dots, n \tag{14}$$

where $\alpha_k = \alpha_k(d)$ and $e_{kl} = e_{kl}(d)$. Then, $\mathcal{S}$ is generically pole assignable by $\mathcal{F}$ if the coefficient matrix $E = E(d) = (e_{kl})$ has full generic rank.

Under the conditions of Corollary 1, the mapping $\bar{g}$ can be decomposed as $\bar{g} = \tilde{g} \circ h$, where $\tilde{g}$: $\mathbb{R}^n \to \mathbb{R}^n$ is the affine mapping defined in (14), and h: $\mathbb{R}^n \to \mathbb{R}^n$ is defined in (13), both mappings being homeomorphisms. With Lemma 3 on hand, the assumptions and the full generic rank condition on the matrix E of the corollary can be characterized in terms of the closed-loop system digraph. This leads us to the following two main results which give sufficient conditions for generic pole assignability.

Theorem 1. Suppose that in $\mathcal{D}(\mathcal{F})$ there exists a choice of n distinct f-edges, renumbered conveniently as $f_1, f_2, \dots, f_n$, which after converting the remaining f-edges into d-edges by fixing their weights at arbitrary values, satisfy the following conditions.

(i) No two f-edges occur in the same cycle;
(ii) All f-cycles have a vertex in common;
(iii) For $k = 1, 2, \dots, n$, there exist particular simple f-cycle families of width k, denoted by $\mathcal{CF}_k^*$, such that
  (a) $f_k \in \mathcal{CF}_k^*$, and
  (b) any other simple f-cycle family of width k which contains an f-edge f_l, $l \leq k$, also contains a d-edge which appears in no $\mathcal{CF}_j^*$, $j \leq k$.

Then $\mathcal{S}$ is generically pole assignable with $\mathcal{F}$.

Note that conditions (i) and (ii) in the statement of Theorem 1 guarantee that every f-cycle family is a simple f-cycle family. Thus, each g_k in (12) is an affine function of $f_1, f_2, \dots, f_n$ as in (14), so that $\bar{g}$ has the structure in Corollary 1 with $\mathcal{I} = \mathcal{N}$. Conditions (iii) establish the generic nonsingularity of the coefficient matrix E.

The second result characterizes a larger class of pole assignable structures:

Theorem 2. The result of Theorem 1 remains valid if condition (ii) is replaced by

(ii)′ To any two f-edges f_p and f_q that appear in disjoint cycles there corresponds a unique pair of edges f_r and d_r such that
  (a) d_r appears in every cycle of f_r but in no cycle of f_p or f_q, and
  (b) to any two disjoint cycles $\mathcal{C}_p$ and $\mathcal{C}_q$ of f_p and f_q there corresponds a cycle $\mathcal{C}_r$ of f_r which covers exactly the same state vertices as $\mathcal{C}_p$ and $\mathcal{C}_q$ cover, and vice versa.

Under condition (ii)′ of Theorem 2, $\mathcal{D}(\mathcal{F})$ may contain at most two pairwise disjoint f-cycles, and hence each product term in $\omega(\mathcal{CF})$ of (12) contains at most two variable weights. Also the correspondence between the pair (f_r, d_r) and the pair (f_p, f_q) in the statement of condition (ii)′ is one-to-one, and if f_r appears in a product term in some $g_k(f)$ of (12), then so does the product $f_p f_q$, and vice versa. Moreover, all the product terms that contain f_r in any $g_k(f)$ are of the form $e_{kr}(e_r f_r + e_{pq} f_p f_q)$, where e_{kr}, e_r, and e_{pq} are polynomials in d with e_r and e_{pq} being the same in all such expressions. These allow for defining the index set $\mathscr{I}$ and the auxiliary variables $\tilde{f}_k$ in Corollary 1 uniquely such that the indices of all such e_r's are included in $\mathscr{N} - \mathscr{I}$ and $\theta_r = e_r$, $\psi_r = e_{pq} f_p f_q$ in (13).

The usefulness of these two theorems is due to the fact that they can easily be translated to an algorithm which identifies a possible choice of n feedback gains to be included in f_v, as well as the fixed values to be assigned to those in the corresponding f_c.

5.1. Examples of Generically Pole Assignable Systems

The following discussion on examples of classes of pole assignable structures demonstrates the nontriviality of our results.

It is a fact that pole assignability under state feedback is a generic property of a structurally controllable system. Consider such a system described by

$$\mathscr{S}: \quad \dot{x} = Ax + Bu\,, \tag{15}$$

and a full state feedback law

$$\mathscr{F}: \quad u = Fx\,, \tag{16}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The resulting closed-loop system $\mathscr{S}(\mathscr{F})$ can be represented by the reduced system structure matrix

$$\mathbf{S}(\mathbf{F}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{F} & \mathbf{0} \end{bmatrix} \tag{17}$$

Let $\mathscr{D}_{xu}(\mathscr{F}) = (\mathscr{X} \cup \mathscr{U}, \mathscr{E}_{ux} \cup \mathscr{E}_F)$ denote the closed-loop system digraph. Then, we have the following result, which relates the pole assignability of $\mathscr{S}(\mathscr{F})$ to Theorem 1.

Theorem 3. The following are equivalent.

(a) $\mathscr{S}$ is structurally controllable.

(b) $\mathscr{S}(\mathscr{F})$ is generically pole-assignable.

(c) There exists a choice of n feedback edges such that when the remaining feedback edges are assigned suitable fixed weights, $\mathscr{D}_{xu}(\mathscr{F})$ satisfies the conditions of Theorem 1.

The proof of this theorem hinges on the fact that the special structure of the system digraph implied by structural controllability allows for a systematic selection of the feedback edges $f_1, f_2, \ldots, f_n$ that satisfy the conditions of Theorem 1.

Our second example is a characterization of a class of structurally controllable and observable systems with dynamic output feedback which satisfy conditions of Theorem 2. For this, consider a structurally controllable and observable single-input/single-output plant

$$\mathscr{S}\colon\quad \begin{aligned} \dot{x} &= Ax + bu \\ y &= c^{\mathrm{T}}x \end{aligned} \tag{18}$$

to be controlled by a dynamic output feedback of the form

$$\hat{\mathscr{S}}\colon\quad \begin{aligned} \dot{\hat{x}} &= \hat{A}x + \hat{b}y \\ u &= \hat{c}^{\mathrm{T}}\hat{x} + \hat{f}y \end{aligned} \tag{19}$$

where $\hat{x} \in \mathbb{R}^{\hat{n}}$ is the state of the controller $\hat{\mathscr{S}}$. We know [8] that the problem of assigning the poles of $\mathscr{S}$ using $\hat{\mathscr{S}}$ is equivalent to the problem of assigning the poles of the augmented system

$$\mathscr{S}_a\colon\quad \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ u_c \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} y \\ y_c \end{bmatrix} = \begin{bmatrix} c^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

by using constant output feedback of the form

$$\mathscr{F}_a\colon\quad \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} \hat{f} & \hat{c}^{\mathrm{T}} \\ \hat{b} & \hat{A} \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} \tag{21}$$

Thus, our graph-theoretic approach can be used to attack the problem. Let $\mathscr{D}$ denote the digraph corresponding to the plant $\mathscr{S}$ and $\mathscr{D}(f)$ the digraph of the closed-loop system consisting of $\mathscr{S}$ and the (scalar) constant output feedback

$$\mathscr{F}\colon\quad u = fy\,.$$

Since $\mathscr{S}$ is structurally controllable and observable, it has no structurally fixed modes [9]. Equivalently, $\mathscr{D}(f)$ is strongly connected and contains a cycle family of width n [10].

We choose the order of the controller $\hat{\mathscr{S}}$ to be $\hat{n} = n - 1$, and fix the structure of $(\hat{A}, \hat{c}^{\mathrm{T}})$ to be in the observability canonical form. Thus, of the n^2 elements of $\mathscr{F}_a$ in (21), $n^2 - (2n - 1)$ are fixed at 0 or 1 with the remaining $2n - 1$ left as variable parameters.

We have the following result about the pole assignability of $\mathscr{S}_a(\mathscr{F}_a)$.

Theorem 4. Suppose that $\mathscr{D}(f)$ contains a cycle family of width n, each cycle of which has a vertex in common with some input-output path in $\mathscr{D}$. Then $\mathscr{D}_a(\mathscr{F}_a)$ satisfies the conditions of Theorem 2 with n replaced by $n_a = 2n - 1$.

Under the conditions of Theorem 4, each of the n_a feedback edges can be associated in a systematic way with one of the particular cycle families $\mathscr{C}\mathscr{F}_k^*$, $1 \leqq k \leqq n_a$ in the statement of Theorem 1. Note that the assumption of the theorem which puts a constraint on the structure of $\mathscr{D}(f)$ is not essential for structural pole assignability of $\mathscr{S}$ using dynamic output feedback controller $\hat{\mathscr{S}}$. However, it is needed for proving generic pole assignability using Theorem 2. On the other hand, the restriction of $\mathscr{S}$ to be single-input/single-output can be relaxed since preliminary constant output feedback can be employed to reduce the system to a single-input/single-output one without destroying structural controllability and observability.

6. GENERIC STABILIZABILITY

We first state a result which is the structural counterpart of Lemma 2.

Lemma 4. Let $f_v = (f_1, f_2, \ldots, f_n)$ and f_c be as in Lemma 2 and suppose that the relation $p = \bar{g}(d, f_v)$ can be written as in (7), where $a_k = a_k(d, \bar{f}_k)$, $b_{kj}(d, \bar{f}_k)$ and $c_k = c_k(d, \bar{f}_k)$ are polynomials in d with $\bar{f}_k = (f_1, \ldots, f_{k-1})$, $1 \leqq k \leqq n$, $1 \leqq j \leqq k$; and $\alpha_k = \alpha_k(d)$ are polynomials in d with a_k and b_{kk} being nonzero. Then, $\mathscr{S}$ is structurally stabilizable by $\mathscr{F}$.

Now, let $\mathscr{D}(\mathscr{F}) = (\mathscr{V}, \mathscr{E} \cup \mathscr{E}_F)$ be the digraph associated with the closed-loop system $\mathscr{S}(\mathscr{F})$, in the usual way. The definition below is a generalization of condition (ii)′ of Theorem 2 to any pair of edges of the digraph.

Definition 2. In $\mathscr{D}(\mathscr{F})$, consider a pair of edges, denoted by $\{e_p, e_q\}$ which never appear in the same cycle. Suppose that there corresponds to the pair $\{e_p, e_q\}$ a unique ordered pair of edges (e_r, e_s) such that,

(a) e_s appears in every cycle of e_r but in no cycle of e_p or e_q, and

(b) to any two disjoint cycles $\mathscr{C}_p$ and $\mathscr{C}_q$ of e_p and e_q, there corresponds a cycle $\mathscr{C}_r$ of e_r which covers exactly the same state vertices as $\mathscr{C}_p$ and $\mathscr{C}_q$ cover, with no input and/or output vertices that occur in $\mathscr{C}_p \cup \mathscr{C}_q$ taking part in a cycle disjoint from $\mathscr{C}_r$, and vice versa.

Then. we say that $\{e_p, e_q\}$ is a pair biased to (e_r, e_s) and that any cycle family of $\{e_p, e_q\}$ in an accompanying cycle family of e_r.

We now state our first result on stabilizability:

Theorem 5. Suppose that in $\mathscr{D}(\mathscr{F})$ there exists a choice of n distinct f-edges, renumbered conveniently as $f_1, f_2, \ldots, f_n$, which after converting the remaining f-edges by fixing their weights at arbitrary values, satisfy the conditions listed below. Then, $\mathscr{S}$ is structurally (generically) stabilizable with $\mathscr{F}$.

There exists an integer $\bar{n}$, $1 \leqq \bar{n} \leqq n$, such that,

(i) for $k = n, n-1, \ldots, \bar{n}$, there exist particular cycle families of width k, denoted by $\mathscr{CF}_k^*$ such that $f_k \in \mathscr{CF}_k^*$, $f_j \notin \mathscr{CF}_k^*$, $j > k$ and either of the following holds.

  (a) Any cycle family of width $l < k$ which contains f_k either contains or is an accompanying cycle family of some $f_j, j > k$, or

  (b) Any other cycle family of width k which neither contains nor is an accompanying cycle family of any f_j, $j > k$, contains either f_k or a pair of edges $\{e_p, e_q\}$ biased to (f_k, e) for some e such that if $e_p = f_l$ (respectively $e_q = f_l$), then $\mathscr{CF}_l^*$ does not contain e_q (respectively e_p), $l < k$.

(ii) With f_k and all $\{e_p, e_q\}$ which are biased to (f_k, e) for some e removed for $k \geqq \bar{n}$, the remaining digraph satisfies Theorem 1, with n replaced by $\bar{n} - 1$.

Under the conditions of Theorem 5, (6) takes the special form

$$\begin{aligned}
\bar{g}_1(f_v) &= h_1(f_1, \ldots, f_{\bar{n}-1}) + (a_{\bar{n}} f_{\bar{n}} + c_{\bar{n}})\, b_{\bar{n}1} + \cdots + (a_n f_n + c_n)\, b_{n1} \\
&\;\vdots \\
\bar{g}_{\bar{n}-1}(f_v) &= h_{\bar{n}-1}(f_1, \ldots, f_{\bar{n}-1}) + (a_{\bar{n}} f_{\bar{n}} + c_{\bar{n}})\, b_{\bar{n},\bar{n}-1} + \cdots + (a_n f_n + c_n)\, b_{n,\bar{n}-1} \\
\bar{g}_{\bar{n}}(f_v) &= \alpha_n + (a_{\bar{n}} f_{\bar{n}} + c_{\bar{n}})\, b_{\bar{n},\bar{n}} + \cdots + (a_n f_n + c_n)\, b_{n,\bar{n}} \\
&\;\vdots \\
\bar{g}_n(f_v) &= \alpha_n + (a_n f_n + c_n)\, b_{nn}
\end{aligned} \tag{22}$$

where c_k contain the edges biased to f_k, $k \geqq \bar{n}$, and $h_k(f_1, \ldots, f_{\bar{n}-1})$ are linear in $f_1, \ldots, f_{\bar{n}-1}$, $1 \leqq k \leqq \bar{n} - 1$ as in Theorem 1. The result then follows from a combination of Lemma 2 and Theorem 1.

The next two results, which are based on an asymptotic approach justified by the use of high gains in stabilization by Lemma 2, characterize systems whose characteristic polynomial coefficients are not in the form of (6) but can be effectively brought into that form by neglecting certain system parameters.

Theorem 6. Suppose that, for $k = 1, 2, \ldots, n$, there exist particular cycle families of width k, denoted by $\mathscr{CF}_k^*$ in $\mathscr{D}(\mathscr{F})$ such that

(i) $f_k \in \mathscr{CF}_k^*$ and $f_j \notin \mathscr{CF}_k^*, j > k$;

(ii) for any other f-cycle family $\mathscr{CF}_k$ of width k, $\#f(\mathscr{CF}_k) \leqq \#f(\mathscr{CF}_k^*)$, with strict inequality if $\mathscr{CF}_k$ contains no $f_j, j > k$, where $\#f(\cdot)$ denotes the number of variable f-edges in $(\cdot)$.

Then $\mathscr{S}$ is generically stabilizable by $\mathscr{F}$.

The proof of this theorem relies on the fact that with $f_k = \varrho \bar{f}_k$, those f-edges that are included in $\mathscr{CF}_k^*$ become dominant in $\bar{g}_k(f_v)$ as $\varrho \to \infty$, which, in turn, satisfy conditions of Theorem 1.

Our last result on generic stabilizability of $\mathscr{S}(\mathscr{F})$ depends on the following recursive reduction process applied to the closed-loop digraph $\mathscr{D}(\mathscr{F})$.

(i) Delete from $\mathscr{D}(\mathscr{F})$ all edges that do not appear in any cycle.

(ii) Let a be a d-edge such that to every cycle $\mathscr{C}_l^a$, $l = 1, 2, \ldots,$ that contains a, there corresponds a cycle family $\mathscr{CF}_l^a$ with the following properties.

(a) $\mathscr{CF}_l^a$ covers the same state vertices as $\mathscr{C}_l^a$ does,

(b) $\mathscr{CF}_l^a$ covers no input or output vertices which are covered by some f-cycle disjoint from $\mathscr{C}_l^a$,

(c) $\mathscr{CF}_l^a$ includes all the f-edges that appear in $\mathscr{C}_l^a$ and at least one additional f-edge.

Let $\mathscr{E}_l^a(f)$ denote the set of the additional f-edges in $\mathscr{C}\mathscr{F}_l^a$, but not in $\mathscr{C}_l^a$. Delete a_l and record $\mathscr{E}_l^a(f)$.

Let the digraph obtained from $\mathscr{D}(\mathscr{F})$ by successive application of (i) and (ii) above be denoted by $\overline{\mathscr{D}}(\mathscr{F})$. We state the following.

Theorem 7. Suppose $\overline{\mathscr{D}}(\mathscr{F})$ satisfies either Theorem 5 or Theorem 6 with at least one f-edge from each $\mathscr{E}_l^a(f)$ included in f_v, $l = 1, 2, \ldots$. Then $\mathscr{S}(\mathscr{F})$ is generically stabilizable.

6.1. A Class of Generically Stabilizable Systems

Consider a system $\mathscr{S}$ composed on $\mathscr{N}$ interconnected structurally controllable sub-systems described by

$$\mathscr{S}_i\colon\ \dot{x}_i = A_i x_i + \sum_{i=1}^{N} A_{ij} x_j + b_i u_i\,, \quad i = 1, 2, \ldots, N\,. \tag{23}$$

Suppose that local state feedback law

$$\mathscr{F}_i\colon\ u_i = f_i^{\mathrm{T}} x_i\,, \tag{24}$$

is applied to the decoupled subsystems

$$\mathscr{S}_i^{\mathrm{D}}\colon\ \dot{x}_i = A_i x_i + b_i u_i\,, \tag{25}$$

where $x_i \in \mathbb{R}^n$, and $u \in \mathbb{R}$, with $\sum_{i=1}^{N} n_i = n$, and $f_i^{\mathrm{T}} = (f_{i1}, f_{i2}, \ldots, f_{in_i})$. We know from our results on pole assignability that each decoupled subsystem $\mathscr{S}_i^{\mathrm{D}}(\mathscr{F}_i)$ is generically pole assignable. We assume that each (A_i, b_i) is controllable canonical form, and that the interaction between the states of the subsystem satisfies the following condition:

$$\operatorname{Im} A_{ij} \subset \operatorname{Im} b_i\,, \quad i \neq j\,, \quad i, j = 1, 2, \ldots, N \tag{26}$$

With $\mathscr{D}(\mathscr{F})$ denoting the closed-loop system digraph, the following result concerns the generic stabilizability under this well known restriction on the interconnection structure.

Theorem 8. All the d-edges of $\mathscr{D}(\mathscr{F})$ corresponding to the interconnection matrices A_{ij} of (25) can be deleted by the reduction process. The resulting digraph $\overline{\mathscr{D}}(\mathscr{F})$ consists of decoupled components $\overline{\mathscr{D}}_i(\mathscr{F}_i)$ associated with the decoupled systems $\mathscr{S}_i^{D}(\mathscr{F}_i)$. Since $\mathscr{S}_i^{D}(\mathscr{F}_i)$ are generically stabilizable by Theorem 3, then so is $\mathscr{S}(\mathscr{F})$ by Theorem 7.

7. CONCLUSIONS

A qualitative investigation of the two aspects of the basic problem of pole-shifting, namely, arbitrary pole assignability and stabilizability is realized. The analysis depends on the closed-loop system structure. The results obtained are essentially algebraic ones, which are stated in graph-theoretical terms. The graphical nature of the sufficient conditions guarantees genericity of the results and allows for translation to an algorithm.

An observation about the results on generic pole assignability is remarkable: In some systems for which the renowned results of Brasch an Pearson [11] and of Kimura [12] require dynamic compensation in order to place all the poles at desired locations, our results show that constant output feedback or at least a dynamic compensator of smaller order is sufficient for the job.

We note that the closed-loop system structures required either for generic pole assignability or generic stabilizability are characterized as special cases of more general results which have possible hints for characterizing broader classes of pole assignable or stabilizable structures.

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