ON THE NATURE OF DESCRIPTOR SYSTEMS

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In this paper we clarify the nature of description systems. In order to do that we introduce the notion of transfer-like sum. Another ingredient in our clarification is the pair causality-reversed causality.

1. INTRODUCTION

We will consider discrete-time lumped dynamical systems in the framework put forward in [1]. In this view, a dynamical system is a triple $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathfrak{B})$ with $\mathbf{Z}$ the time axis. $\mathbf{R}^q$ the signal space, and $\mathfrak{B} \subseteq (\mathbf{R}^q)^{\mathbf{Z}}$ the behavior. We will assume that the system is linear ($\mathfrak{B}$ is a linear subspace), time-invariant ($\sigma\mathfrak{B} = \mathfrak{B}$) with the shift: $(\sigma f)(t) = f(t + 1)$), and complete (see [1]). Equivalently, that $\mathfrak{B} \in \mathfrak{L}^q$ with $\mathfrak{L}^q$ the set of all linear shift-invariant closed subspaces of $(\mathbf{R}^q)^{\mathbf{Z}}$. equipped with the topology of pointwise convergence. It is well-known (see [1]) that $\mathfrak{L}^q$ coincides with the kernels of the polynomials in the shift, i.e., $\mathfrak{B} \in \mathfrak{L}^q$ if and only if there exists for some g a polynomial matrix $R(s, s^{-1}) \in \mathbf{R}^{g\times q}[s, s^{-1}]$ such that $\mathfrak{B} = \ker R(\sigma, \sigma^{-1})$ with $R(\sigma, \sigma^{-1})$ viewed as a map (from $(\mathbf{R}^q)^{\mathbf{Z}}$ to $(\mathbf{R}^g)^{\mathbf{Z}}$. In the language of [1] this means that Σ is described by the behavioral equations

$$R(\sigma, \sigma^{-1})\, w = 0 \qquad \text{(AR)}$$

Without loss of generality one can take g such that $R(s, s^{-1})$ has full row rank, hence such that $g \leqq q$.

2. LATENT VARIABLES

Consider the following set of behavioral equations

$$R(\sigma, \sigma^{-1})\, w = M(\sigma, \sigma^{-1})\, a \qquad \text{(L)}$$

with $M(s, s^{-1}) \in \mathbf{R}^{g \times k}[s, s^{-1}]$. The variables in $\boldsymbol{a}$ are called *auxiliary* or *latent* and help to describe the behavior of the variables in $\boldsymbol{w}$. Let $\mathfrak{B}$ be defined as follows:

$$\mathfrak{B} = \{\boldsymbol{w} \in (\mathbf{R}^q)^{\mathbf{Z}} \mid \exists \boldsymbol{a} \in (\mathbf{R}^k)^{\mathbf{Z}} \text{ such that (L) holds}\} .$$

One easily sees that $\mathfrak{B} \in \mathfrak{L}^q$. This follows from the following observations:

(1) Let $U(s, s^{-1}) \in \mathbf{R}^{g \times g}[s, s^{-1}]$ and $V(s, s^{-1}) \in \mathbf{R}^{k \times k}[s, s^{-1}]$ be unimodular, then $\mathfrak{B}$ is also described by the following behavioral equations

$$U(\sigma, \sigma^{-1})\, R(\sigma, \sigma^{-1})\, \boldsymbol{w} = U(\sigma, \sigma^{-1})\, M(\sigma, \sigma^{-1})\, V(\sigma, \sigma^{-1})\, \boldsymbol{a} .$$

(2) In (1) one can take $U(s, s^{-1})$ and $V(s, s^{-1})$ such that $U(s, s^{-1})\, M(s, s^{-1})\, V(s, s^{-1})$ is diagonal (Smith-form).

(3) Let $0 \neq p(s, s^{-1}) \in \mathbf{R}[s, s^{-1}]$, then $p(\sigma, \sigma^{-1}]: (\mathbf{R})^{\mathbf{Z}} \to (\mathbf{R})^{\mathbf{Z}}$ is surjective.

Based on the above observations one can eliminate the auxiliary variables and write $\mathfrak{B}$ as the kernel of a polynomial matrix in the shift, and hence $\mathfrak{B} \in \mathfrak{L}^q$.

We call, in (L), *$\boldsymbol{a}$ observable from $\boldsymbol{w}$*, if $R(\sigma, \sigma^{-1})\, \boldsymbol{w}_1 = R(\sigma, \sigma^{-1})\, \boldsymbol{w}_2 = M(\sigma, \sigma^{-1})\, \boldsymbol{a}$ implies that $\boldsymbol{w}_1 = \boldsymbol{w}_2$. One easily sees that $\boldsymbol{a}$ is observable from $\boldsymbol{w}$ if rank $M(\lambda, \lambda^{-1}) = h, \forall 0 \neq \lambda \in \mathbb{C}$.

A special type of latent variables is considered in the next section.

3. STATE EQUATIONS

Consider the following set of behavioral equations

$$E\sigma x + Fx + Gw = 0 \tag{S}$$

with $E, F \in \mathbf{R}^{f \times n}$ and $G \in \mathbf{R}^{f \times q}$. The distinguishing feature of this system is that, as far as the shift is concerned, it is *first order* in $\boldsymbol{x}$ and *zero-th order* in $\boldsymbol{w}$. In [1] it is shown that this corresponds exactly to linear time-invariant complete systems in which $\boldsymbol{x}$ plays the role of state variable (see [1] for a formal definition).

The *external behavior* of (S) is defined by

$$\mathfrak{B} = \{w: \mathbf{Z} \to \mathbf{R}^q \mid \exists x: \mathbf{Z} \to \mathbf{R}^n \text{ such that (S) is satisfied}\}$$

From part 2 it follows that $\mathfrak{B} \in \mathfrak{L}^q$ and conversely in [1] it is shown that if $\mathfrak{B} \in \mathfrak{L}^q$ there will exist E, F, G such that $\mathfrak{B}$ is the external behavior of (S).

4. DESCRIPTOR SYSTEMS

We now introduce a special type of state representations. Let $\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix}$ a partition of $\boldsymbol{w}$ into (its first) q_1 and (its last) q_2 components. Correspondingly $\mathbf{R}^q \cong \mathbf{R}^{q_1} \times \mathbf{R}^{q_2}$. Now consider the set of behavioral equations

$$\begin{aligned} E\sigma x &= Ax + Bw_1 \\ w_2 &= Cx + Dw_1 \end{aligned} \tag{DS}$$

with $E, A \in \boldsymbol{R}^{f \times n}$, $B \in \boldsymbol{R}^{f \times q_1}$, $C \in \boldsymbol{R}^{q_2 \times n}$ and $D \in \boldsymbol{R}^{q_2 \times q_1}$. In a recent paper Kuijper and Schumacher [2] prove that (with this pre-imposed partition) each (!) $\mathfrak{B} \in \mathfrak{L}^q$ admits such a representation. In the next section, we will give a short proof of this result.

We will call a system of the type (DS) a *descriptor system*. As already mentioned each $\mathfrak{B} \in \mathfrak{L}^q$ may be represented this way. Such systems acquire more structure if we assume more properties of w_1 and/or w_2. In particular we will investigate what representations correspond to the case that w_2 processes w_2, that w_2 is maximally free, and that (DS) is a non-anticipating input/output representation with w_1 input and w_2 output. See [1] for formal definitions of these concepts.

5. DESCRIPTOR REPRESENTATIONS OF LINEAR SYSTEMS.

The following theorem gives a broad classification of descriptor systems.

Theorem. Let w be partitioned as $w = \begin{bmatrix} w_1 \\ \hdashline w_2 \end{bmatrix}$ and $\mathfrak{B} \in \mathfrak{L}^q$. Then:

1. $\mathfrak{B}$ admits a representation (DS);
2. $\mathfrak{B}$ admits a representation (DS) with $Es - A$ of full column rank if and only if w_2 processes w_1;
3. $\mathfrak{B}$ admits a representation (DS) with E, A square and $\det(Es - A) \neq 0$ if and only if w_1 is maximally free;
4. $\mathfrak{B}$ admits a representation (DS) with E square and $\det E \neq 0$ if and only if w_1 is a non-anticipating input for the output w_2.

Proof. 1. Start from a representation (S). Introduce $\tilde{x} = \begin{bmatrix} x \\ \hdashline w \end{bmatrix}$ and write (S) as as $\sigma\tilde{E}\tilde{x} = \tilde{A}\tilde{x}$; $w = \tilde{C}\tilde{x}$. Now observe that this is of the form (DS).

2. (*only if*): take $w_1 = \mathbf{0}$. Then the corresponding x-behavior satisfies $E\sigma x = Ax$ and is thus finite-dimensional [1]. Hence also the possible w_2's: $E\sigma x = Ax$; $w_2 = Cx$ forms also a finite-dimensional space, equivalently, w_2 processes w_1. (*if*): $\mathfrak{B}$ admits a representation $R_1(\sigma, \sigma^{-1}) w_1 = \mathbf{0}$; $P(\sigma, \sigma^{-1}) w_2 = Q(\sigma, \sigma^{-1}) w_1$ with P square and $\det P \neq 0$. Now wr shall see while proving 3 that this second relation may be represented as

$$E_2\sigma x_2 = A_2 x_2 + B_2 w_1$$
$$w_2 = C_2 x_2 + D_2 w_1$$

with $\det(E_2 s - A_2) \neq 0$. The first relation may be represented as

$$E_1 \sigma x_1 = A_1 x_1 + B_1 w_1$$

with $E_1 s - A_1$ of full column rank [1]. Defining $x = \begin{bmatrix} x_1 \\ \hdashline x_2 \end{bmatrix}$ yields the result.

3. (*only if*): if $\det(Es - A) \neq 0$, then $E\sigma - A$ is surjective (this follows from the observations in part 2.), whence w_1 is free. By (2) w_2 also processes w_1. Hence w_1 is maximally free. (*if*): such representations will be studied in Section 7.

4. This is the classical case studied in linear systems theory. □

6. TRANSFER-LIKE SUMS

Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^q$, and w be partitioned as $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Then the *transfer-like sum* of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, denited as $\mathfrak{B}_1 \dotplus \mathfrak{B}_2$, is defined as

$$\mathfrak{B} = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \middle| \exists \begin{bmatrix} w_1 \\ w_2' \end{bmatrix} \in \mathfrak{B}_1 \quad \text{and} \quad \begin{bmatrix} w_1 \\ w_2'' \end{bmatrix} \in \mathfrak{B}_2 \quad \text{with} \quad w_2 = w_2' + w_2'' \right\}$$

If $\mathfrak{B}_i$ is described by $P_i(\sigma, \sigma^{-1}) w_2 = Q_i(\sigma, \sigma^{-1}) w_1$ with P_i square and $\det P_i \neq 0$, with transfer function $G_i(s) = P_i^{-1}(s, s^{-1}) Q_i(s, s^{-1})$ then it is easy to see that in $\mathfrak{B}$ w_1 will also be maximally free and that the corresponding transfer function $G(s)$ satisfies $G(s) = G_1(s) + G_2(s)$. However our notion of transfer-like sum also concerns the non-controllable part of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. In fact, if Q_1 and Q_2 are zero, then $\mathfrak{B}_i \dotplus \mathfrak{B}_2 = \mathfrak{B}_1 + \mathfrak{B}_2$.

Representations of transfer-like sums will be studied in detail elsewhere.

7. SPLITTING THE BEHAVIOR IN A CAUSAL AND REVERSED CAUSAL PART

Let $\mathfrak{B} \in \mathfrak{L}^q$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, and assume that w_1 is maximally free. Let $P(\sigma, \sigma^{-1}) \cdot w_2 = Q(\sigma, \sigma^{-1}) w_1$ with $\det P \neq 0$ be an AR-representation of $\mathfrak{B}$. Then [1] w_2 does not anticipate w_1 if the matrix of rational functions $P^{-1}(s, s^{-1}) Q(s, s^{-1})$ is proper. We will call such systems *causal*. If $P^{-1}(s, s^{-1}) Q(s, s^{-1})$ is strictly proper, then we will call the system *strictly causal*. Let rev: $(\mathbf{R}^q)^{\mathbf{Z}} \to (\mathbf{R}^q)^{\mathbf{Z}}$ be the time-reversal operator: $(\text{rev} f)(t) = f(-t)$. If rev $\mathfrak{B}$ is causal, then $\mathfrak{B}$ will be called *reversed causal*. This requires that $P^{-1}(s^{-1}, s) Q(s^{-1}, s)$ is proper. If $P^{-1}(s^{-1}, s) Q(s^{-1}, s)$ is strictly proper, then we will call $\mathfrak{B}$ *reversed strictly causal*.

We will now show how one can split a given behavior $\mathfrak{B} \in \mathfrak{L}^q$ with $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and w_1 free into the transfer-like sum of a causal and a reversed strictly causal part. Let $\mathfrak{B}$ be represented by

$$P(\sigma, \sigma^{-1}) w_2 = Q(\sigma, \sigma^{-1}) w_1 ,$$

with $\det P \neq 0$. Write $P = U\Sigma_1\Sigma_2 V = U\Sigma_2\Sigma_1 V$ with U, V unimodular, and Σ_1, Σ_2 coprime diagonal polynomial matrices with nonnegative powers in s only and $\Sigma_1(0)$

non-singular. Now define $P_1 = \Sigma_2 V$, $P_2 = \Sigma_1 V$, and let D_1, D_2 be diagonal polynomial matrices such that $D_1\Sigma_1 + D_2\Sigma_2 = I$. Observe that

$$\begin{bmatrix} U\Sigma_1 & U\Sigma_2 \\ -V^{-1}D_2 & V^{-1}D_1 \end{bmatrix}\begin{bmatrix} D_1U^{-1} & -P_1 \\ D_2U^{-1} & P_2 \end{bmatrix} = I$$

Now define, for an arbitrary constant matrix C of appropriate dimensions, the polynomials Q_1 and Q_2 as:

$$\begin{bmatrix} U\Sigma_1 & U\Sigma_2 \\ -V^{-1}D_2 & V^{-1}D_1 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\begin{bmatrix} Q \\ C \end{bmatrix}$$

Now define $\mathfrak{B}_i$ as the system with AR-representation

$$P_i(\sigma, \sigma^{-1})\, w_2 = Q_i(\sigma, \sigma^{-1})\, w_1$$

Define $\tilde{\mathfrak{B}} := \mathfrak{B}_1 + \mathfrak{B}_2$, hence $\tilde{\mathfrak{B}} = \Bigg\{ (w_1, w_2) \mid \exists (w_{21}, w_{22})$ such that

$$\left.\begin{pmatrix} Q_1 & 0 \\ Q_2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \\ I & I \end{pmatrix}\begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix}\right\}.$$

Notice that $\tilde{\mathfrak{B}}$ is expressed in terms of the latent variables (w_{21}, w_{22}). In the following steps we will eliminate these variables, see also part 2.

$$\tilde{\mathfrak{B}} = \Bigg\{ (w_1, w_2) \mid \text{such that}$$

$$\left.\begin{pmatrix} Q_1 & -P_1 \\ Q_2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & -P_1 \\ 0 & 0 \\ I & I \end{pmatrix}\begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix} \text{ for some } (w_{21}, w_{22})\right\}.$$

$$= \Bigg\{ (w_1, w_2) \mid \text{such that}$$

$$\begin{pmatrix} U\Sigma_1 & U\Sigma_2 & 0 \\ -V^{-1}D_2 & -V^{-1}D_1 & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} Q_1 & -P_1 \\ Q_2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =$$

$$\left.\begin{pmatrix} U\Sigma_1 & U\Sigma_2 & 0 \\ -V^{-1}D_2 & -V^{-1}D_1 & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 0 & -P_1 \\ 0 & P_2 \\ I & I \end{pmatrix}\begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix} \text{ for some } (w_{21}, w_{22})\right\}$$

$$= \Bigg\{ (w_1, w_2) \mid \text{such that} \begin{pmatrix} Q & -U\Sigma_1 P_1 \\ C & V_1 D_2 P_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ 0 & I \\ I & I \end{pmatrix} \begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix} \text{ for some } (w_{21}, w_{22}) \Bigg\}$$

$$= \{(w_1, w_2) \mid Qw_1 = U\Sigma_1 P_1 w_2 = U\Sigma_1\Sigma_2 V w_2 = Pw_2\} = \mathfrak{B}\,.$$

So we proved that $\mathfrak{B} = \mathfrak{B}_1 \dotplus \mathfrak{B}_2$.

It is easy to see that $\mathfrak{B}_1 \dotplus \mathfrak{B}_2 = \mathfrak{B}$ and that $P_1^{-1}Q_1 = P_1^{-1}D_1U^{-1}Q - C$, $P_2^{-1}Q_2 = P_2^{-1}D_2U^{-1}Q + C$. By a proper choice of C we can make $P_2^{-1}(s^{-1}, s)\,.$ $.\, Q_2(s^{-1}, s)$ strictly proper and $\mathfrak{B}_1$ causal. Now observe that $\mathfrak{B}_2$ is reversed strictly causal. In addition, if we take $\Sigma_1 = I$, $\mathfrak{B}_2$ will be FIR (finite impulse response), equivalently $P_2^{-1}(s, s^{-1})\, Q_2(s, s^{-1})$ is a polynomial, while if we take $\Sigma_2 = I$. $\mathfrak{B}_2$ will be FIR, equivalently $P_1^{-1}(s, s^{-1})\, Q_1(s, s^{-1})$ a polynomial in s^{-1}. It is also easy to calculate that $P_1^{-1}Q_1 + P_2^{-1}Q_2 = P^{-1}Q$. Suppose now that $\mathfrak{B}$ is causal. Then we are allowed to take $\Sigma_1 = I$, $D_1 = I$, $D_2 = 0$ and $C = 0$. It is easy to see that in this case $\mathfrak{B}_1 = \mathfrak{B}$ and $\mathfrak{B}_2 = \{(w_2, w_1) \mid w_2 = 0\}$.

8. DESCRIPTOR REPRESENTATIONS WHEN w_1 IS MAXIMALLY FREE

If w_1 is maximally free, then $\mathfrak{B}$ admits a representation of the type

$$P(\sigma, \sigma^{-1})\, w_2 = Q(\sigma, \sigma^{-1})\, w_1$$

with P square and $\det P(s, s^{-1}) \neq 0$. Note that the transfer function $G(s) =$ $= P^{-1}(s, s^{-1})\, Q(s, s^{-1})$ need not be proper. The representation question is close to what has been studied by Conte and Perdon [3] with the proviso that we will also consider the non-controllable case and not only the transfer function.

In order to obtain a (DS)-representation, write $\mathfrak{B}$ (as explained in Section 7) as a transfer-like sum $\mathfrak{B} = \mathfrak{B}_1 \dotplus \mathfrak{B}_2$ with $\mathfrak{B}_1$ causal and $\mathfrak{B}_2$ reverse (strictly) causal. Write a non-anticipating input/state/output-representation for $\mathfrak{B}_1$:

$$\sigma x_1 = A_1 x_1 + B_1 w_1\,; \quad w_2 = C_1 x_1 + F_1 w_1$$

Next consider rev $\mathfrak{B}_2$ and write a non-anticipating input/state/output-representation for

$$\mathfrak{B}_2' := \begin{bmatrix} I & 0 \\ 0 & \sigma^{-1} \end{bmatrix} \operatorname{rev} \mathfrak{B}_2\,.$$

Note that $\mathfrak{B}_2'$ is always strictly causal.

$$\sigma x_2 = A_2 x_2 + B_2 w_1\,; \quad \sigma^{-1} w_2 = C_2 x_2$$

This yields, by defining $\tilde{x}_2 = \sigma^{-1} \operatorname{rev} x_2$, the following state representation for $\mathfrak{B}_2$:

$$A_2 \sigma \tilde{x}_2 = \tilde{x}_2 + B_2 w_1\,; \quad w_2 = C_2 \tilde{x}_2$$

Now define $x = \begin{bmatrix} x_1 \\ \tilde{x}_2 \end{bmatrix}$ and observe that

$$\begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \sigma x = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} w_1$$

$$w_2 = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x + D w_1$$

yields the desired descriptor representation. Notice that

$$s \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$$

is a regular matrix pencil. When $\mathfrak{B}$ is given by a descriptor representation it is in principle quite easy to write $\mathfrak{B}$ as the transfer-sum of a causal- and a reversed causal part. In order to do that one brings the matrix pencil on Kronecker canonical form, [4], and then one easily reads off a causal- and a reversed causal part.

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