KYBERNETIKA — VOLUME 27 (1991), NUMBER 3

ON THE REPRESENTATION OF 2-D SYSTEMS

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2-D systems are here considered from a behavioral point of view. The main purpose is to express structural properties of behaviors in terms of their mathematical descriptions. Special attention is paid to two kinds of controllability which will be defined as properties of the system signals.

1. INTRODUCTION

The behavioral approach to 2-D systems essentially consists in viewing a system as a family of admissible signals (the system behavior) instead of as a set of describing equations (which constitute a system representation). From this stand point, it is natural to define properties of systems in terms of their signals and not of their representations. The translation of the system structure into properties of the representation parameters is, of course, a crucial issue, but in our opinion should not be taken as a starting point.

This paper is mainly concerned with the study of controllability properties in the above framework. Our notions differ from the classical ones as they are introduced as properties of the external signals, without making appeal to state space realizations. Controllabilite of state space systems can however be considered as a particular case of our definitions if the state is regarded as an external variable.

2. PRELIMINARIES

We will start by defining a 2-D *system* Σ as a triple $\Sigma = (T, W, \mathfrak{B})$, where T is a two-dimensional index set, W the signal space and $\mathfrak{B} \subseteq W^{T}$ the behavior of the system, i.e., the set of all signals which are compatible with the system laws. Here, we consider in particular systems in q real valued variables — $W = \boldsymbol{R}^q$ — defined over the grid $T = \boldsymbol{Z}^2$.

Denote by σ_1 and σ_2 the following shift-operators: σ_i: $(\mathbf{R}^q)^{\mathbf{Z}^2} \to (\mathbf{R}^q)^{\mathbf{Z}^2}$, $i = 1, 2$, with $\sigma_1 w(t_1, t_2) = w(t_1 + 1, t_2)$ and $\sigma_2 w(t_1, t_2) = w(t_1, t_2 + 1)$ for all $w \in (\mathbf{R}^q)^{\mathbf{Z}^2}$. A subset $\mathfrak{B} \subseteq (\mathbf{R}^q)^{\mathbf{Z}^2}$ is said to be *shift-invariant* if $\sigma_i \mathfrak{B} = \mathfrak{B}$ $(i = 1, 2)$.

The class of systems studied in the sequel consists of all those systems whose behavior $\mathfrak{B}$ is a linear, shift-invariant and closed subspace of $(\mathbf{R}^q)^{\mathbf{Z}^2}$ (for some positive integer q), equipped with the topology of pointwise convergence. This behavioral characterization can be translated as follows.

Theorem 1 [2]. $\mathfrak{B} \subseteq (\mathbf{R}^q)^{\mathbf{Z}^2}$ is a linear, shift-invariant and closed subspace (in the topology of pointwise convergence) if and only if it can be described as $\mathfrak{B} = \{w \in (\mathbf{R}^q)^{\mathbf{Z}^2} \mid R(\sigma_1, \sigma_2) w = 0\}$, where $R(s_1, s_2)$ is a real polynomial matrix in in the indeterminates s_1 and s_2.

The equation $R(\sigma_1, \sigma_2) w = 0$ in the theorem above is said to be an *AR-representation* for $\mathfrak{B}$. We will call $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathfrak{B})$ and *AR-system* if $\mathfrak{B}$ admits such a representation.

3. CONTROLLABILITY PROPERTIES

In this section we introduce two notions of controllability for 2-D systems. The main aspects of our approach are the following. First, we do not restrict to state space realizations, and will view controllability as an external property of general 2-D systems $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathfrak{B})$. Second, our definitions are stated in a set theoretic way, in terms of the system signals, and not in terms of representations.

Intuitively, a 2-D system Σ is controllable if its memory has a limited range, i.e., no matter what system signal is given in $T' \subseteq T$, at sufficiently large distance from T' every other system signal can occur. Formally:

Definition 1. A 2-D system $\Sigma = (\mathbf{Z}^2, W, \mathfrak{B})$ is said to be *controllable* if the following condition holds. There exists a positive real number ϱ such that $\{w_1, w_2 \in \mathfrak{B}; I_2, I_2 \subseteq \mathbf{Z}^2;\ d(I_1, I_2) \geqq \varrho\} \Rightarrow \{w_1|_{I_1} \wedge w_2|_{I_2} \in \mathfrak{B}|_{I_1 \cup I_2}\}$. Here $d(I_1, I_2)$ stands for the Euclidean distance between I_1, and I_2; $w_1|_{I_1} \wedge w_2|_{I_2}$ denotes the signal w: $I_1 \cup I_2 \to W$ such that $w|_{I_1} = w_1|_{I_2}$ and $w|_{I_2} = w_2|_{I_2}$, and is called the concatenation of $w_1|_{I_1}$ and $w_2|_{I_2}$.

Thus, I_1 is surrounded by a band of width ϱ beyond which all the information about the phenomenon $w_1|_{I_1}$ occurring in I_1 is lost. The same happens for I_2 with respect to w_2. In this sense ϱ measures the memory range of the system.

In order to state our second notion of controllability we need to introduce the concept of local behavior. Given the system $\Sigma = (\mathbf{Z}^2, W, \mathfrak{B})$, the behavior $\mathfrak{B}$ is said the be *local* if these exist a positive integer N and a subset Δ_N of $W^{(2N+1)^2}$ such that $\{w \in \mathfrak{B}\} \Leftrightarrow \{w|_{I_N(t_1,t_2)} \in \Delta_N$, for all $(t_1, t_2) \in \mathbf{Z}^2\}$ (where $I_N(t_1, t_2) := [t_1 - N, t_1 + N] \times [t_2 - N, t_2 + N] \cap \mathbf{Z}^2$). This means that in order to check whether or not w is an admissible system signal it suffices to verify if it satisfies the system laws locally, on the finite windows $I_N(t_1, t_2)$. If N^* is the smallest value of N for which the condition of the definition of local behavior is satisfied we will say that $\mathfrak{B}$ is N^*-local. Behaviors which can be described by means of an AR-representation (as in Section 1) constitute an example of local behaviors. Further, given a system Σ with N^*-local behavior, we will say that a signal $w \in W^{\mathbf{Z}^2}$ *satisfies the laws of* Σ *in* $I \subseteq \mathbf{Z}^2$ if $w|_{I_{N^*}(t_1,t_2)} \in \Delta_{N^*}$ for all $(t_1, t_2) \in I$.

Definition 2. A 2-D system $\Sigma = (\mathbf{Z}^2, W, \mathfrak{B})$ with N^*-local behavior is said to be *strongly controllable* if there exists a positive real number ϱ such that $\{w_1, w_2 \in W^{\mathbf{Z}^2};\ I_1, I_2 \subseteq \mathbf{Z}^2;\ d(I_1, I_2) \geqq \varrho;\ w_i$ satisfies the laws of Σ in I_i $(i = 1, 2)\} \Rightarrow \{w_1|_{I_1} \wedge \wedge\, w_2|_{I_2} \in \mathfrak{B}|_{I_1 \cup I_2}\}$.

Note that while controllability deals exclusively with admissible system signals, the signals w_1, w_2 involved in the definition of strong controllability do not a priori necessarily belong to the system behavior as they are required to satisfy the system laws only in some subsets of $\mathbf{Z}^2$ and not in the whole grid. It follows immediately from the definitions that strong controllability implies controllability. We shall later show that the converse is not true.

For AR-systems, controllability and strong controllability can be characterized in terms of the corresponding AR descriptions by means of primeness conditions. A full rank 2-D polynomial matrix $R(s_1, s_2)$ is said to be *factor-left-prime* if all its left-divisors are unimodular (i.e., invertible as polynomial matrices in s_1 and s_2). If, for all $(\lambda_1, \lambda_2) \in \mathbf{C}\backslash\{0\} \times \mathbf{C}\backslash\{0\}$, rank $R(\lambda_1, \lambda_2)$ is constant $R(s_1, s_2)$ we will say that *zero-left-prime.*

Theorem 2 [3]. Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathfrak{B})$ be an AR 2-D system. Then Σ is controllable if and only is there exists a factor-left-prime polynomial matrix $R(s_1, s_2)$ such that $R(\sigma_1, \sigma_2)\, w = 0$ is an AR-representation of $\mathfrak{B}$.

Theorem 3. $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathfrak{B})$ is a strongly controllable AR 2-D system if and only if $\mathfrak{B}$ admits an AR representation $R(\sigma_1, \sigma_2)\, w = 0$ with $R(s_1, s_2)$ zero-left-prime.

Proof. See the Appendix.

Remark. The conditions obtained in Theorems 2 and 3 are similar to the ones given in [1] for approximate and exact modal controllability of state space systems. In fact, Definitions 1 and 2 can also be applied to this class of systems and provide a new (system theoretic) interpretation of modal controllability.

Given the above characterizations, it is not difficult to construct examples of controllable systems which are not strongly controllable.

Example. Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^2, \mathfrak{B})$ be the system described by:

$$(\sigma_1 - 1)\, w_1 = (\sigma_2 - 1)\, w_2 \tag{1}$$

Thus $\mathfrak{B}$ is represented by $R(\sigma_1, \sigma_2)\, w = 0$ with $R(s_1, s_2) := [s_1 - 1 | -(s_1 - 1)]$. Clearly, the matrix $R(s_1, s_2)$ is factor-left-prime and hence (by Theorem 2) Σ is controllable. However, since $R(1, 1) = [0\ 0]$, $R(s_1, s_2)$ is not zero-left-prime implying (by Theorem 3) that Σ is not strongly controllable. This can also be shown using Definition 2 in the following way.

For a given positive integer N define $T_N := \mathbf{Z}^2 \backslash [-N, N] \times [-N, N]$ and consider the following trajectories $w^{(N)}$ and $w^{(0)}$ in $(\mathbf{R}^2)^{\mathbf{Z}^2}$. The trajectory $w^{(N)} := \mathrm{col}\,(w_1^{(N)}, w_2^{(N)})$ is such that $w_2^{(N)}(k, l) = 0$ for all $(k, l) \in \mathbf{Z}^2$ and $w_1^{(N)}(k, 0) = 1$ if $k \geqq N + 1$ and otherwise $w_1^{(N)}(k, l) = 0$ (see Figure 1). The trajectory $w^{(0)}$ is simply defined as the zero trajectory.

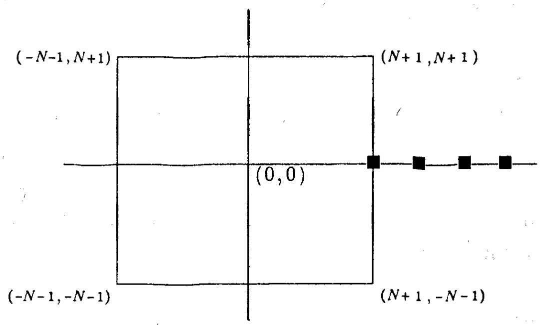

Fig. 1. On the points indicated by ■ $w_1 = 1$, elsewhere $w_1 = 0$.

It is not difficult to check that $w^{(N)}$ satisfies the laws of Σ in T_N. On the other hand, $w^{(0)}$ obviously satisfies the laws of Σ in $T_0 = \{(0, 0)\}$. Moreover, $d(T_N, T_0) = N + 1$. Thus, if Σ is strongly controllable, it follows from Definition 2 that the condition $w^{(0)}|_{T_0} \wedge w^{(N)}|_{T_N} \in \mathfrak{B}|_{T_0 \cup T_N}$ holds for some N. In particular this implies that $w^{(N)}|_{T_N} \in \mathfrak{B}|_{T_N}$, i.e., that $w^{(N)}|_{T_N}$ can be extended to the plane as an admissible system trajectory. We next show that this is not the case.

For a given positive integer N, let $\mathrm{col}\,(w_1, w_2) = w \in \mathfrak{B}$ be an admissible system trajectory such that $w(k, l) = 0$ if $|l| \geqq N + 1$ or $k \leqq -(N + 1)$. Further, define the 1-D trajectories $w_1^{N+1} := w_1(N + 1, \cdot)$ and $v^{N+1} := \sum_{j=-N}^{N} w_2(j, \cdot)$, and let σ denote the 1-D shift. Then, it is easily seen from Equation (1) that:

$$w_1^{N+1} = (\sigma - 1)\, v^{N+1} \tag{2}$$

Note that v^{N+1} has compact support. Hence, Equation (2) implies that w_1^{N+1} is contained in the image under $\sigma - 1$ of the compact support sequences in $\mathbf{R}^{\mathbf{Z}}$. This condition is clearly not satisfied by the trajectory $w^{(N)}$. Consequently $w^{(N)}|_{T_N}$ can not be extended to $\mathbf{Z}^2$ as an element of the system behavior $\mathfrak{B}$. This shows that Σ is not strongly controllable.

4. CONCLUSION

The main aspects of this note can be summarized as follows. A distinction was made between *systems* and their *representations*. Structural properties of 2-D behaviors (such as linearity, shift-invariance and closedness, controllability and strong controllability) were first defined in terms of the system signals and then expressed in terms of the system representations. It turns out that our definitions of controllability and strong controllability provide system theoretic interpretation of the notions of approximate and exact modal controllability introduced in [1].

APPENDIX: PROOF OF THEOREM 3

"only if"

Suppose that $\Sigma = (\boldsymbol{Z}^2, \boldsymbol{R}^q, \mathfrak{B})$ is strongly controllable. Then, it will obviously be controllable and, by Theorem 2, there exists a full rank, factor-left-prime polynomial matrix $R(s_1, s_2)$ such that $\mathfrak{B}$ is represented by $R(\sigma_1, \sigma_2)\, w = 0$. We will prove that R is also zero-left-prime. This is equivalent to say that, if R is a $g \times q$ matrix, the operator $R(\sigma_1, \sigma_2)\colon K_q \to K_g$ (where K_l denotes the set of compact support signals in $(\boldsymbol{R}^l)^{\boldsymbol{Z}^2}$) is surjective (cf. [1]). Note that as $R(s_1, s_2)$ is full row rank, $R(\sigma_1, \sigma_2)$ is surjective as an operator $(\boldsymbol{R}^q)^{\boldsymbol{Z}^2} \to K_g$. Thus, for every element $a \in K_g$, there exists $w \in (\boldsymbol{R}^q)^{\boldsymbol{Z}^2}$ such that $R(\sigma_1, \sigma_2)\, w = a$. Consequently, outside the support S of a, w satisfies the laws of Σ, once $R(\sigma_1, \sigma_2)\, w(t_1, t_2) = 0$ for all $(t_1, t_2) \in \boldsymbol{Z}^2 \setminus S$. It is not difficult to check that strong controllability implies that there exist $\bar{w} \in \mathfrak{B}$ and a finite subset $I \subseteq \boldsymbol{Z}^2$ such $\bar{w}|_{\boldsymbol{Z}^2 \setminus I} = w|_{\boldsymbol{Z}^2 \setminus I}$. Finally, defining $w^* = w - \bar{w}$, we will have that $w^* \in K_g$ and $R(\sigma_1, \sigma_2)\, w^* = R(\sigma_1, \sigma_2)\, w - R(\sigma_1, \sigma_2)\, \bar{w} = a - 0 = a$. This shows that $R(\sigma_1, \sigma_2)\colon K_q \to K_g$ is surjective as desired.

"if"

Let Σ admit a representation $R(\sigma_1, \sigma_2)\, w = 0$, with $R(s_1, s_2)$ zero-left-prime. Then, as the ring of polynomials in two variables is a Hermite ring, R admits a unimodular completion, i.e., there exists a polynomial matrix $S(s_1, s_2)$ such that $U(s_1, s_2) := \operatorname{col}(R(s_1, s_2), S(s_1, s_2))$ is square, and invertible as polynomial matrix. Thus, denoting $U^{-1}(s_1, s_2) =: [N(s_1, s_2) \mid M(s_1, s_2)] =: V(s_1, s_2)$, $\bar{w} := \operatorname{col}(\bar{w}_1, a)$, $\bar{w}_1 := R(\sigma_1, \sigma_2)\, w$ and $a := S(\sigma_1, \sigma_2)\, w$, there holds that $\{Rw = 0\} \Leftrightarrow \{RVUw = 0\} \Leftrightarrow \{[I\ 0]\, \bar{w} = 0 \text{ and } w = V\bar{w}\} \Leftrightarrow \{w = M(\sigma_1, \sigma_2)\, a,\ a \text{ free}\}$. Suppose further that S can be written as $S(\sigma_1, \sigma_2) = \sum_{(i,j)\in J} \bar{S}_{ij} \sigma_1^i \sigma_2^j$, for a finite index set J and suitable nonzero real matrices $\bar{S}_{ij}$, and define the radius of S as $r(S) := \max\{i^2 + j^2 \mid (i, j) \in J\}$.

Now, take $\varrho > 2r(S)$, $I_1, I_2 \subseteq \boldsymbol{Z}^2$ such that $d(I_1, I_2) \geqq \varrho$, and consider two signals $w_1, w_2 \in (\boldsymbol{R}^q)^{\boldsymbol{Z}^2}$ satisfying the laws of Σ in I_1 and I_2 respectively. It follows from the above that there exist a_i $(i = 1, 2)$ such that $w_i(t_1, t_2) = M(\sigma_1, \sigma_2)\, a_i(t_1, t_2)$ for all $(t_1, t_2) \in I_i$, namely $a_i := S(\sigma_1, \sigma_2)\, w_i$. With basis on a_1 and a_2, construct a new signal a^* such that $a^*(t_1, t_2) = a_1(t_1, t_2)$ if $d((t_1, t_2), I_1) \leqq r(S)$ and $a^*(t_1, t_2) = a_2(t_1, t_2)$ if $d((t_1, t_2), I_2) \leqq r(S)$, and define $w^* = Ma^*$. Note that w^* is an admissible signal of Σ, i.e., $w^* \in \mathfrak{B}$. Moreover, is not difficult to check that $w^*|_{I_1} = w_1|_{I_1}$ and $w_2^*|_{I_2} = w_2|_{I_2}$. This means that $w_1|_{I_1} \wedge w_2|_{I_2} \in \mathfrak{B}|_{I_1 \cup I_2}$, and hence Σ is strongly controllable. □

(Received November 14, 1990.)

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