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INPUT AND OUTPUT DECOUPLING ZEROS OF LINEAR PERIODIC DISCRETE-TIME SYSTEMS*

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The notions of input and output decoupling zeros are extended to a linear periodic discrete-time system. The ordered sets of structural indices are also analyzed for these notions and for the notions of invariant zero, transmission zero, eigenvalue and pole of such a system. For any non-zero zero, eigenvalue and pole, the corresponding ordered set of structural indices is time-invariant. The input decoupling zeros, the invariant zeros and their ordered sets of structural indices are not altered by a linear periodic state feedback. New characterizations of the zeros, eigenvalues and poles are introduced through a time-invariant matrix mechanism, which is related with the periodic matrices describing the system more directly than the associated system.

1. INTRODUCTION

The increasing attention devoted in the last years to linear periodic systems [1 – 28] is motivated by the large amount of processes which can be modeled through a linear periodic system, such as periodically time-varying networks and filters, chemical processes, and multirate sampled-data systems (e.g. [5, 29 – 31]), as well as by the relevance of periodic control to a wide range of applications (see for example [6]), and even for the stabilization and control of time-invariant linear systems [32 – 36] and bilinear ones [37 – 39].

For discrete-time linear periodic systems a fairly satisfactory structure and control theory has been developed [6 – 26], partly based on geometric methods, while through polynomial methods it has been possible to introduce time-varying notions of pole, invariant zero and transmission zero for such systems [27] (see [28] for a similar less general definition), with a meaning wholly similar to the time-invariant corresponding notions [40 – 43], and with a property of quasi time-independence. Namely, it was shown that the non-zero poles, transmission zeros and invariant zeros are

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independent of time, together with their algebraic multiplicities. This result, as far as invariant zeros are concerned, was based on their geometric interpretation, which was extended to the periodic case.

In this paper the notion of input decoupling zero and output decoupling zero, introduced by Rosenbrock [44] are extended to a linear periodic discrete-time system. For these notions and for those of invariant zero, transmission zero, eigenvalue and pole of such a system, the ordered sets of structural indices are also analyzed. It is pointed out that the ordered sets of structural indices are time-invariant, except for the null input (output) decoupling zero, invariant zero, transmission zero, eigenvalue and pole. The output (input) decoupling zeros at time k are defined through the time-invariant associated system at time k, and characterize modes of the state free response of the periodic system starting at time k whose corresponding output response is zero (the meaning of input decoupling zeros is dual). As in the time invariant case, the input decoupling zeros, the invariant zeros and their structural indices are not altered by a linear periodic state feedback. The input (output) decoupling zeros of a periodic system are related with the standard structural properties of such a system and with its invariant zeros, like in time-invariant systems. Moreover, for the eigenvalues, poles and all types of zeros of a periodic system, different characterizations are obtained through time-invariant matrix mechanisms, which are related with the periodic matrices describing the system more directly then the associated system.

2. ZEROS, POLES AND THEIR ORDERED SETS OF STRUCTURAL INDICES

Consider the linear periodic discrete-time system Σ described by:

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k) \tag{1.a}$$

$$y(k) = C(k)\,x(k) \tag{1.b}$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{C}^n =: \boldsymbol{X}$ is the state, $u(k) \in \mathbb{C}^p =: \boldsymbol{U}$ is the control input, $y(k) \in \mathbb{C}^q =: \boldsymbol{Y}$ is the output, and $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are periodic matrices of period ω (briefly, ω-periodic) with entries in $\mathbb{C}$. The state transition matrix of Σ is expressed by $\Phi(k, k_0) := A(k-1) \ldots A(k_0)$, with $k > k_0$, $k, k_0 \in \mathbb{Z}$, and $\Phi(k, k) := I_n$ for all $k \in \mathbb{Z}$, where I_n is the identity matrix of dimension n. For any initial time $k_0 \in \mathbb{Z}$, the output response of system Σ for $k \geqq k_0$, to given initial state $x(k_0)$ and input function $u(\cdot)$, can be obtained through the time-invariant *associated system of Σ at time k_0*, denoted by $\Sigma^a(k_0)$ [31]. System $\Sigma^a(k)$ is represented by:

$$x_k(h+1) = E_k\,x_k(h) + J_k\,u_k(h) \tag{2.a}$$

$$y_k(h) = L_k\,x_k(h) + M_k\,u_k(h) \tag{2.b}$$

where $h \in \mathbb{Z}$, $x_k(h) \in \mathbb{C}^n$, $u_k(h) \in \mathbb{C}^{p\omega}$, $y_k(h) \in \mathbb{C}^{q\omega}$, $E_k := \Phi(k+\omega, k)$, $J_k := [\Delta_k(0) \ldots$

$\ldots \Delta_k(\omega - 1)]$ with $\Delta_k(j) := \Phi(k + \omega, k + j + 1)\, B(k + j)$, $j \in \{0, 1, \ldots, \omega - 1\}$, $L_k := [\Gamma_k'(0) \ldots \Gamma_k'(\omega - 1)]'$ with $\Gamma_k(j) := C(k + j)\, \Phi(k + j, k), j \in \{0, 1, \ldots, \omega - 1\}$, and M_k is a block matrix $\{\Theta_k(i, j)\}$ with $\Theta_k(i, j) := 0$, $i \leqq j$, $\Theta_k(\mathrm{i}, \mathrm{j}) := C(k + i)\,.$ $.\, \Phi(k + \mathrm{i}, k + j + 1)\, B(k + j)$ $i > j$, $i, j \in \{0, 1, \ldots, \omega - 1\}$. In fact, it is easy to see that, if $x_k(0) = x(k)$ and $u_k(h) = [u'(k + h\omega) \ldots u'(k + \omega - 1 + h\omega)]'$ for all $h \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of non-negative integers, then $x_k(h) = x(k + h\omega)$ and $y_k(h) = [y'(k + h\omega) \ldots y'(k + \omega - 1 + h\omega)]'$ for all $h \in \mathbb{Z}^+$.

Remark 1. The subspaces of reachable (unobservable) and controllable (unreconstructible) states of system Σ at time k are readily seen to coincide with those of systems $\Sigma^a(k)$ if they are expressed in terms of matrices E_k, J_k and L_k [12, 13, 27].

The characteristic polynomial of E_k is independent of k, whence it characterizes the stability of Σ [45]. The eigenvalues of E_k are called here *the eigenvalues* of Σ.

The transfer matrix of $\Sigma^a(k)$, expressed by $W_k(z) := L_k(zI_n - E_k)^{-1} J_k + M_k$, was called the *associated transfer matrix of Σ at time k* [27].

Lemma 1. [27] The rank of the rational matrix $W_k(z)$ is independent of k.

The rank of $W_k(z)$ will be denoted simply by r. Then the symbols $\varepsilon_i^k(z)$ $(\psi_i^k(z))$, $i = 1, 2, \ldots, r$, will denote the r numerator (denominator) polynomials in the Smith-McMillan form of $W_k(z)$ in such an order that $\varepsilon_i^k(z)$ $(\psi_{i+1}^k(z))$ divides $\varepsilon_{i+1}^k(z)$ $(\psi_i^k(z))$. The zeros of the polynomial $\eta_k(z) := \prod_{i=1}^{r} \varepsilon_i^k(z)$ $(\chi_k(z) := \prod_{i=1}^{r} \psi_i^k(z))$ were called the *transmission zeros of Σ (the poles of Σ) at time k* [27]. The *associated system matrix of Σ at time k* was defined by:

$$P_k(z) := \begin{bmatrix} E_k - zI_n & J_k \\ L_k & M_k \end{bmatrix} \tag{3}$$

and has a rank of $n + r$ [44]. Then the symbols $\xi_i^k(z)$, $i = 1, 2, \ldots, n + r$, will denote the $n + r$ invariant polynomials in the Smith form of $P_k(z)$, in such an order that $\xi_i^k(z)$ divides $\xi_{i+1}^k(z)$. The zeros of the polynomial $\zeta_k(z) := \prod_{i=1}^{n+r} \xi_i^k(z)$ were called the *invariant zeros of Σ at time k*. The meaning of transmission zeros and invariant zeros [27] (as well as that of poles), is a straightforward extension of the time-invariant one [40, 46], even when they coincide with the poles.

A notion of zero can be obtained also through the geometric approach, like in the time invariant case [41]. It is based on the concepts of *$(A(\cdot), B(\cdot))$-invariant subspace* [13] and *reachability subspace* [27]. The former is an ω-periodic subspace $V(k) \subset X$ such that $A(k)\, V(k) \subset V(k + 1) + \operatorname{Im} B(k)$ for all $k \in \mathbb{Z}$, that is equivalent to the existence of an ω-periodic linear map $F(\cdot) : X \to U$ such that $A^F(k)\, V(k) \subset V(k+1)$ (or, briefly, s.t. $V(\cdot)$ is $A^F(\cdot)$-invariant), where $A^F(k) := A(k) + B(k)\, F(k)$. The latter is an $(A(\cdot), B(\cdot))$-invariant subspace $V(k)$ whose states are reachable at time k from the null state with a trajectory $x(h)$ contained in $V(h)$ for every

$h < k$. For a given $(A(\cdot), B(\cdot))$-invariant subspace $V(\cdot)$, denote by $\mathbb{F}(V(\cdot))$ the class of ω-periodic linear maps $F(\cdot)$ such that $V(\cdot)$ is $A^F(\cdot)$-invariant; denote by E_k^F the matrix defined like E_k but with $A^F(\cdot)$ instead of $A(\cdot)$; then, for any $F(\cdot) \in \mathbb{F}(V(\cdot))$, $E_k^F V(k) \subset V(k + \omega) = V(k)$ for all $k \in \mathbb{Z}$, and the standard notations for restriction and induced map in the quotient spaces can be applied to E_k^F and $V(k)$. The following lemma allows one to define the geometric notion of zero.

Lemma 2. [27] For a given ω-periodic subspace $K(k) \subset X$, for the largest $(A(\cdot), B(\cdot))$-invariant subspace $V^*(k)$ contained in $K(k)$ for all $k \in \mathbb{Z}$, for the largest reachability subspace $V_{re}^*(k)$ contained in $K(k)$ for all $k \in Z$, and for $F(\cdot) \in \mathbb{F}(V^*(\cdot))$ the linear map $H_k := E_k^F \mid V^*(k) \,(\mathrm{mod}\, V_{re}^*(k))$ is independent of $F(\cdot)$ for all $k \in \mathbb{Z}$.

The subspaces $V^*(\cdot)$ and $V_{re}^*(\cdot)$ can be computed through recursive algorithms [13, 15, 27]. For $K(\cdot) = \mathrm{Ker}\, C(\cdot)$, the eigenvalues of H_k are called here *geometric zeros* ("structural zeros" in [27]) *of Σ at time k*. By Lemma 2, they are independent of $F(\cdot) \in \mathbb{F}(V^*(\cdot))$. The dimension of H_k, which in general depends on k, will be denoted by $m(k)$.

For the above recalled notions of zeros, poles and eigenvalues, possibly time-varying structural indices can be defined as follows.

For a given transmission zero (pole) α of Σ at time k and for each $i = 1, \dots, r$, let $\mu_i(k)$ ($\nu_i(k)$) be the multiplicity of α as zero of $\varepsilon_i^k(z)$ ($\psi_i^k(z)$). The non-decreasing finite sequence $\{\mu_1(k), \mu_2(k), \dots, \mu_r(k)\}$ ($\{\nu_r(k), \nu_{r-1}(k), \dots, \nu_1(k)\}$) will be called the *ordered set of structural indices at time k* of the transmission zero (pole) α. In a similar way, *ordered sets* of $n + r$, $m(k)$ *and* n *structural indices at time k* of an invariant zero α of Σ at time k, a geometric zero α of Σ at time k and an eigenvalue α of Σ, respectively, are defined as the non-decreasing finite sequences of multiplicities of α as zero of the invariant polynomials $\xi_i^k(z)$ of $P_k(z)$, of those of $zI_{m(k)} - H_k$ and of those of $zI_n - E_k$, respectively.

Remark 2. A transmission zero (pole) of Σ at time k is an invariant zero of Σ at time k (an eigenvalue of Σ) and the two sets of structural indices at time k are related like in the time-invariant case [47]. In particular, when Σ is reachable and observable at time k, an invariant zero at time k (an eigenvalue) is a transmission zero (pole) at time k, with the same set of structural indices, except for some null structural indices [44].

As in the time-invariant case, geometric zeros coincide with invariant zeros.

Theorem 1. [48] For each integer k, the invariant zeros of Σ at time k and their ordered sets of structural indices coincide with its geometric zeros at time k and their ordered sets of structural indices except for $n + r - m(k)$ null structural indices of invariant zeros, and are not altered by a linear periodic state feedback, i.e. by substituting $A(\cdot)$ with $A^F(\cdot)$ (with $F(\cdot)$ ω-periodic).

Evans [45] stated the time-invariance of the structural indices of the non-zero eigenvalues of Σ, but his proof was incomplete. The following theorem states a similar property also for invariant zeros, transmission zeros, and pole.

Theorem 2. [48] The non-zero invariant zeros, the non-zero transmission zeros, the non-zero poles, the non-zero eigenvalues of Σ at time k and their ordered sets of structural indices, are independent of k.

Theorems similar to Theorems 1 and 2, but restricted to multiplicities, defined as the sum of structural indices, were proved in [27]. The existence and the ordered set of structural indices of the null invariant zero, transmission zero and pole, as well as the ordered set of structural indices of the null eigenvalue, can depend on k, as shown through counter-examples in [27, 48].

Two equivalent characterizations of invariant zeros, transmission zeros, poles and eigenvalues of Σ and their structural indices can be given through simpler, block-diagonal matrices, defined by:

$$\mathscr{A}_k := \operatorname{diag}\{A(k), A(k+1), \ldots, A(k+\omega-1)\} \tag{4.a}$$

$$\mathscr{B}_k := \operatorname{diag}\{B(k), B(k+1), \ldots, B(k+\omega-1)\} \tag{4.b}$$

$$\mathscr{C}_k := \operatorname{diag}\{C(k), C(k+1), \ldots, C(k+\omega-1)\} \tag{4.c}$$

and through the following matrices:

$$D(z) := \begin{bmatrix} 0 & I_{(\omega-1)n} \\ zI_n & 0 \end{bmatrix} \tag{5.a}$$

$$\tilde{\mathscr{A}}_k := [D(1)]^{-1} \mathscr{A}_k \tag{5.b}$$

$$\tilde{\mathscr{B}}_k := [D(1)]^{-1} \mathscr{B}_k \,. \tag{5.c}$$

Write $Q_1(z) \sim Q_2(z)$ for polynomial matrices $Q_1(z)$, $Q_2(z)$ with the same dimensions and the same Smith form (i.e. $Q_1(z) = U(z)\, Q_2(z)\, V(z)$, with $U(z)$ and $V(z)$ unimodular).

Theorem 3. For each integer k

$$\mathscr{W}_k(z) := \mathscr{C}_k(D(z) - \mathscr{A}_k)^{-1} \mathscr{B}_k = W_k(z) \qquad \forall z \in \mathbb{C} \tag{6.a}$$

$$W_k(z) = \mathscr{C}_k(\operatorname{diag}\{zI_n, I_{(\omega-1)n}\} - \tilde{\mathscr{A}}_k)^{-1} \tilde{\mathscr{B}}_k \qquad \forall z \in \mathbb{C} \tag{6.b}$$

$$(D(z) - \mathscr{A}_k) \sim \begin{bmatrix} zI_n - E_k & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \tag{7a}$$

$$\begin{bmatrix} zI_n - E_k & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \sim \operatorname{diag}\{zI_n, I_{(\omega-1)n}\} - \tilde{\mathscr{A}}_k \tag{7.b}$$

$$\mathscr{P}_k(z) := \begin{bmatrix} \mathscr{A}_k - D(z) & \mathscr{B}_k \\ \mathscr{C}_k & 0 \end{bmatrix} \sim \begin{bmatrix} P_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \tag{8.a}$$

$$\begin{bmatrix} P_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \sim \begin{bmatrix} \tilde{\mathscr{A}}_k - \operatorname{diag}\{zI_n, I_{(\omega-1)n}\} & \tilde{\mathscr{B}}_k \\ \mathscr{C}_k & 0 \end{bmatrix}. \tag{8.b}$$

Proof. For (6.a), (7.a) and (8.a) see [48]. From these relations and from (5), (6.b), (7.b) and (8.b) follow directly. □

By Theorem 3, the invariant zeros, transmission zeros, poles and eigenvalues of Σ and their ordered sets of structural indices at time k can be computed from $\mathscr{P}_k(z)$, $\mathscr{W}_k(z)$ and $D(z) - \mathscr{A}_k$, or from the matrices in the right-hand side of (6.b), (7.b) and (8.b). They are more simply related with $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ than $P_k(z)$, $W_k(z)$ and $zI_n - E_k$ are.

Remark 3. Denote with δ the one-step forward shift operator, satisfying $\delta x(h) = x(h + 1)$. For a given initial state $x(k)$ at time k, the matrices $[\mathscr{A}_k - D(\delta^\omega), \mathscr{B}_k]$ and $[\tilde{\mathscr{A}}_k - \text{diag}\{\delta^\omega I_n, I_{(\omega-1)n}\}, \tilde{\mathscr{B}}_k]$ allow to compute the state response of Σ at any time $k_1 > k$ just like the matrix $[A - \delta I_n, B]$ does for time-invariant systems (i.e. for $\omega = 1$), while $\Sigma^a(k)$ does not provide the state response at time instants different from $k + i\omega$. For this reason, the triplets $(\mathscr{A}_k, \mathscr{B}_k, \mathscr{C}_k)$ and $(\tilde{\mathscr{A}}_k, \tilde{\mathscr{B}}_k, \mathscr{C}_k)$ provide time-invariant characterizations of the ω-periodic geometric notions, such as that of $(A(\cdot), B(\cdot))$-invariant subspace [18, 19].

Verriest [25] and Park and Verriest [23] introduced time-invariant representations of Σ, similar to these triplets.

Note that, if $p = q$ and $r = \omega p$, the structure of $\mathscr{P}_k(z)$ easily shows that the existence of a null invariant zero at time k implies the existence of the same zero at all times, while the corresponding ordered set of structural indices can depend on k, as shown through a counter-example in [48].

3. DECOUPLING ZEROS

The notion of input (output) decoupling zero, introduced by Rosenbrock [44], can be extended to system Σ as follows. Define

$$ {}^iP_k(z) := [E_k - zI_n \quad J_k] , \quad {}^0P_k(z) := [E_k' - zI_n \quad L_k']' . \tag{9} $$

and denote with $\beta_j^k(z)$ ($\gamma_j^k(z)$) the n invariant polynomials of ${}^iP_k(z)$ (${}^0P_k(z)$), in the same order as the $\xi_j^k(z)$'s. The zeros of the polynomial $\prod_{i=1}^{n} \beta_i^k(z)$ ($\prod_{i=1}^{n} \gamma_i^k(z)$) are said to be the *input decoupling zeros* (*output decoupling zeros*) *of Σ at time k*. The *ordered set of n structural indices* of one of these zeros *at time k* is defined as the non-decreasing sequence of the n multiplicities of it as zero of the polynomials $\beta_j^k(z)$ ($\gamma_j^k(z)$). These notions are dual. The meaning of input or output decoupling zeros is easily obtained from the meaning of invariant zeros, on the basis of Lemma 2.2b in [27]: specifically, the output decoupling zeros of Σ at time k characterize modes of the state free response of the periodic system Σ starting at time k, whose corresponding output response is zero (the meaning of input decoupling zeros is dual).

Now, for any ω-periodic linear map $G(k)\colon Y \to X$, define ${}^GA(k) := A(k) + G(k)\,C(k)$, and denote by ${}^GE_k$ the matrix defined like E_k but with ${}^GA(\cdot)$ instead of $A(\cdot)$. Theorems 1, 2 and 3 for $C(\cdot) = 0$ ($B(\cdot) = 0$) yield the following result.

Theorem 4. (a) The input (output) decoupling zeros of Σ at time k and their ordered sets of structural indices, are not altered by substituting $A(\cdot)$ by $A^F(\cdot)$ (${}^GA(\cdot)$), with $F(\cdot)$ ($G(\cdot)$) ω-periodic.

(b) The non-zero input (output) decoupling zeros of Σ at time k and their ordered sets of structural indices are independent of k.

(c) For each integer k

$$ {}^i\mathscr{P}_k(z) := [\mathscr{A}_k - D(z) \;\; \mathscr{B}_k] \sim \begin{bmatrix} {}^iP_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \tag{10a} $$

$$ \begin{bmatrix} {}^iP_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \sim [\tilde{\mathscr{A}}_k - \operatorname{diag}\{zI_n, I_{(\omega-1)n}\} \;\; \tilde{\mathscr{B}}_k] \tag{10b} $$

$$ {}^0\mathscr{P}_k(z) := \begin{bmatrix} \mathscr{A}_k - D(z) \\ \mathscr{C}_k \end{bmatrix} \sim \begin{bmatrix} {}^0P_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \tag{11a} $$

$$ \begin{bmatrix} {}^0P_k(z) & 0 \\ 0 & I_{(\omega-1)n} \end{bmatrix} \sim \begin{bmatrix} \tilde{\mathscr{A}}_k - \operatorname{diag}\{zI_n, I_{(\omega-1)n}\} \\ \tilde{\mathscr{C}}_k \end{bmatrix}. \tag{11b} $$

Note that the existence and the ordered set of structural indices of the null input (output) decoupling zero can depend on time k [48].

Remark 4. The relations of input and output decoupling zeros with the standard structural properties (see [9, 11–13] for algebraic conditions) are similar to the time-invariant case. Namely, system Σ is reachable (observable) at time k, controllable (reconstructible), stabilizable (detectable) if and only if, respectively, it has no input (output) decoupling zeros at time k, no non-zero input (output) decoupling zeros, all input (output) decoupling zeros in the open unit disk [6, 7]. In the same way, there exists an ω-periodic $F(\cdot)$ ($G(\cdot)$) such that all the eigenvalues of E_k^F (${}^GE_k$) lie in the open disk of radius ϱ, if and only if system Σ has all input (output) decoupling zeros in this open disk [48] (see also the following theorem). On this basis, the possible time-dependence of the only null input (output) decoupling zero justifies the possible time-dependence of the only reachability (observability) property among the above mentioned structural properties.

The meaning of the statements in Remark 4 is further clarified, and the relation between input decoupling zeros and invariant zeros pointed out, if the basis of X at each time k is taken to contain an ω-periodic basis of the subspace $X_r(k)$ of reachable states at time k. In such a basis:

$$ A(k) = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{bmatrix}, \quad B(k) = \begin{bmatrix} B_1(k) \\ 0 \end{bmatrix} \tag{12.a} $$

$$ {}^iP_k(z) = \begin{bmatrix} E_{11}(k) - zI_{\sigma(k)} & E_{12}(k) & J_1(k) \\ 0 & E_{22}(k) - zI_{n-\sigma(k)} & 0 \end{bmatrix} \tag{12.b} $$

where $\sigma(k)$ is the possibly time-varying dimension of $X_r(k)$, the square matrices $E_{11}(k)$ and $E_{22}(k)$ are defined like E_k but with $A(\cdot)$ substituted by $A_{11}(\cdot)$ and $A_{22}(\cdot)$, respectively, $J_1(k)$ is defined like J_k but with the pair $(A(\cdot), B(\cdot))$ substituted by the pair $(A_{11}(\cdot), B_1(\cdot))$, and the rank of $[E_{11}(k) - zI_{\sigma(k)}\ J_1(k)]$ is $\sigma(k)$ for all $z \in \mathbb{C}$ and for all $k \in \mathbb{Z}$ [16, 20]. In general $A_{11}(k)$ is not square, whence the pair $(A_{11}(\cdot), B_1(\cdot))$ does not represent a dynamical reachable system but only a "reachable pair" [16, 20]. Thus, the following theorem is yielded by time-invariant results [43, 44], and extends well-known time-invariant properties to the ω-periodic system Σ.

Theorem 5. (a) The input decoupling zeros of Σ at time k and their ordered sets of structural indices, coincide with the eigenvalues of $E_{22}(k)$ and the corresponding ordered sets of structural indices, except for $\sigma(k)$ null structural indices of input decoupling zeros.

(b) If $p \geqq q$ and $r = q\omega$, for each integer k the product of the polynomial $\eta_k(z)$ and the characteristic polynomial of $E_{22}(k)$ divides the polynomial $\zeta_k(z)$.

Dual statements concern output decoupling zeros.

4. CONCLUDING REMARKS

The meaning and the role of the input (output) decoupling zeros of Σ and their relations with the structural properties of it are similar to those of the time-invariant case, as well as the meaning and the role of invariant zeros, transmission zeros and poles.

In particular, the conditions for checking the structural properties of Σ which are expressed in terms of the input (output) decoupling zeros of Σ (see Remark 4) can clarify the mechanism of loss of one of such properties when two ω-periodic systems enjoying the same property are connected in series or feedback (e.g. as a "pole-zero cancellation"), just like for time-invariant systems. In fact, the time-invariant associated system of the connection of two or more linear ω-periodic discrete-time systems coincides with the same connection of the corresponding time-invariant associated systems [18], and this result allows to study any connection of periodic systems through the connection of their associated systems, and to apply time-invariant reachability, controllability, stabilizability conditions of composite systems and the dual ones in order to check these properties for periodic composite systems: e.g., to apply the reachability and observability conditions of composite systems in [46], or the conditions in [49, 50], the latters based just on the notions of input and output decoupling zeros.

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