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# POLYNOMIAL MATRIX SOLUTION TO THE DISCRETE FIXED-LAG SMOOTHING PROBLEM

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The solution of the optimal linear fixed lag smoothing problem is considered using a polynomial matrix description for the discrete systems. The smoother is given by the solution of a diophantine equation and is equivalent to the transfer-function matrix form of the Wiener smoother. The pole-zero properties of the optimal smoother are more obvious in the polynomial representation and new insights into the measurement noise rejection properties of the smoother are obtained. The signal model is assumed to be stable and allowance is made for both dynamic cost weighting and coloured measurement noise. The model structure was determined by the needs of industrial smoothing problems. The polynomial form of filter may easily be included in a self-tuning algorithm and a simple adaptive smoother is described.

## 1. INTRODUCTION

The linear fixed lag smoothing problem is considered where signals are to be estimated from noisy measurements and given causal signal and noise descriptions. The measurements noise or output disturbance can be coloured and the signal and noise models are represented by polynomial matrices. It is assumed that the observations are available up to time $t$ but that the estimate is required at some time $\tau \leqq t$. In this problem $t$ normally denotes real time and the smoothing lag $l := t - \tau$ is a fixed interval.

The advantage of a fixed lag smoother over a normal optimal filter, is that as the lag $l$ increases, the estimation error variance decreases due to the information provided by the additional data. For very large lag length the performance will approach that of a non-causal optimal estimator. In practice a lag of two or three times the dominant system time constant will guarantee near optimal performance (cf. Goodwin and Sin [1]).

Moir [2] considered a special form of optimal smoothing problem for use in reflection seismology where the signal to be estimated is white (the primary reflectogram) and the distorting system is the seismic wavelet. He developed a fixed-lag deconvolution smoother using a Wiener type of optimization argument.

The solution presented here follows a Kučera [3] type of argument and deals with the general smoothing problem which includes both white and coloured noise and dynamic cost weighting. The relationship to the non-causal Wiener smoothing problem is established providing a useful lower bound on the minimum estimation error which can be achieved with increasing lag $l$.

The $z$-dominant expression for the smoother is simple to implement and in many signal processing applications a transfer-function form of smoother is more appropriate than a state space based design. There is no need for backwards and forwards integration and the smoother may be implemented on-line in recursive form. Moreover, the polynomial matrix expressions for the smoother enable self-tuning smoothers to be defined. The development of self-tuning filters, predictors and smoothers was considered previously by Moir and Grimble [4].

The system and noise models are introduced in §2. The output smoothing problem is first considered where a signal is to be estimated in the presence of (possibly coloured) measurement noise. The main theorem in §3 gives the optimal smoother in terms of the solution of a diophantine and a spectral factorisation. Frequency domain aspects of the smoothing problem are considered in §4. The adaptive smoothing problem is discussed in §5 and conclusions are drawn in §6.

## 2. SIGNAL MODEL

The system model shown in Figure 1 can represent either an industrial plant or a message generationg process. The system is assumed to be linear and time-invariant and the noise sources are stationary. These noise signals $\xi(t)$, $\omega(t)$, $v(t)$ are

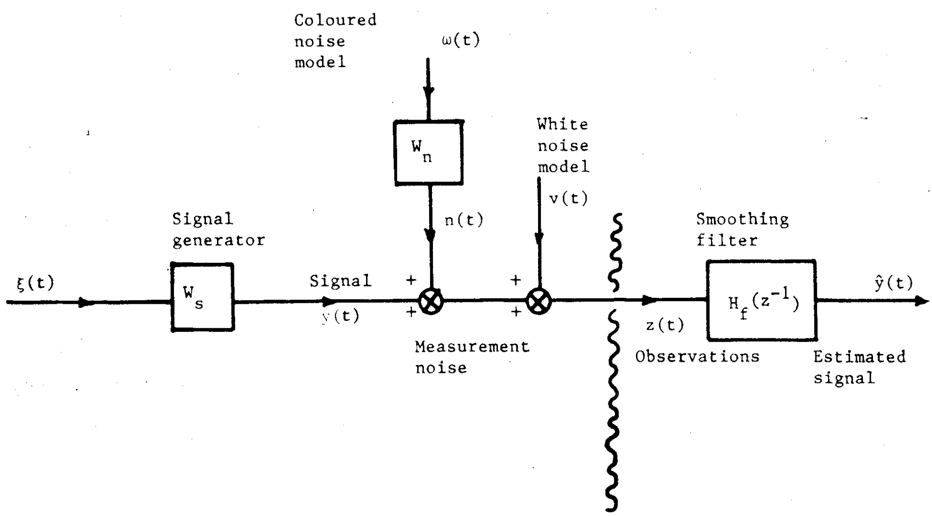


**Fig.1.** Canonical $H_2$ Smoothing Filter.

mutually independent with zero means, and the covariances are defined as

$$\operatorname{cov}[\xi(t), \xi(\tau)] = Q_s\delta_{t\tau}, \quad \operatorname{cov}[\omega(t), \omega(\tau)] = Q_n\delta_{t\tau} \quad \text{and} \quad \operatorname{cov}[v(t), v(\tau)] = R\delta_{t\tau},$$

respectively. Here $\delta_{t\tau}$ denotes the Kronecker delta-function and the assumption is normally made that $R = R^{\mathrm{T}} > 0$.

The system is assumed to be in the steady-state, that is $t_0 \to -\infty$. The various subsystems are necessarily taken to be free of unstable hidden modes and are defined as:

*Signal*

$$y(t) = W_s(z^{-1})\,\xi(t) \tag{1}$$

*Coloured measurement noise*

$$n(t) = W_n(z^{-1})\,\omega(t) \tag{2}$$

The observations signal $z(t) \in \mathbb{R}^r$ is given as:

$$z(t) = v(t) + n(t) + y(t) \tag{3}$$

where $v(t)$ denotes white measurement noise and $n(t)$ represents a coloured measurement noise or output disturbance.

The subsystem transfer-functions can be represented by the following coprime polynomial matrix representations

$$W_s(z^{-1}) = A_s(z^{-1})^{-1}\,C_s(z^{-1}), \quad W_n(z^{-1}) = A_n(z^{-1})^{-1}\,C_n(z^{-1}) \tag{4}$$

(The dependence upon $z^{-1}$ is often suppressed to simplify notation.) The measurement noise of output disturbance model $W_n$ is assumed to be asymptotically stable.

### 2.1. Spectral factorization

Let $A_f$ be defined from the following left-coprime decomposition:

$$A_f^{-1}[\tilde{C}_n, \tilde{C}_s] = [A_n^{-1}C_n, A_s^{-1}C_s] \tag{5}$$

The generalised spectral factor (cf. Shaked [9]) which is needed in the following solution is defined using:

$$\begin{aligned} Y_f Y_f^* &= R + W_n Q_n W_n^* + W_s Q_s W_s^* = R + A_n^{-1} C_n Q_n C_n^* A^{*-1} + \\ &\quad + A_s^{-1} C_s Q_s^* C_s^* A_s^{*-1} \end{aligned} \tag{6}$$

where the asterisk denotes conjugate transpose, $Y_f(z^{-1}) = Y_f^{\mathrm{T}}(z)$. The spectral factor $Y_f$ can be represented as: $Y_f = A_f^{-1} D_f$ where $D_f(0)$ is full rank, $D_f(z^{-1}) \in \mathbb{R}^{r \times r}(z^{-1})$, and the definition of the noise sources ensures $D_f$ is strictly Hurwitz. From (5):

$$Y_f Y_f^* = A_f^{-1}(A_f R A_f^* + \tilde{C}_n Q_n \tilde{C}_n^* + \tilde{C}_s Q_s \tilde{C}_s^*)\,A_f^{*-1} \tag{7}$$

and hence $D_f$ satisfies:

$$D_f D_f^* = A_f R A_f^* + \tilde{C}_n Q_n \tilde{C}_n + \tilde{C}_s Q_s \tilde{C}_s^* \tag{8}$$

## 3. FIXED LAG SMOOTHING PROBLEM

The smoothing problem is concerned with finding the best estimate of the signal $S(t)$ in the presence of the noise terms $v(t)$ and $n(t)$.

### 3.1. Estimation error variance criterion

Let the estimation error be defined as:

$$\tilde{y}(\tau \mid t) := y(\tau) - \hat{y}(\tau \mid t) \tag{9}$$

where $\hat{y}(\tau \mid t)$ is the estimate of $y(\tau)$, given observations $\{z(\tau_1), \tau_1 \in (-\infty, t]\}$, up to time $t$. The variance to be minimized is given, in terms of the trace function, as

$$J = \mathsf{E}\{\tilde{y}^{\mathrm{T}}(\tau \mid t)\, \tilde{y}(\tau \mid t)\} = \mathrm{Trace}\,\{\mathsf{E}\{\tilde{y}(\tau \mid t)\, \tilde{y}(\tau \mid t)^{\mathrm{T}}\}\} \tag{10}$$

### 3.2. Solution of the output smoothing problem

The signal model was defined in §2 and the variance to be minimised was given in (10). It remains only to introduce the smoother before proceeding with the solution. The smoothed estimate is assumed to be generated from a linear estimator of the form

$$\hat{y}(\tau \mid t) = H_f(z^{-1})\, z(t) \tag{11}$$

where $H_f$ is realised in a form free of unstable hidden modes.

To obtain an expression for $\tilde{y}(\tau \mid t)$, note from (1) and (3):

$$z(\tau) = v(\tau) + A_n^{-1} C_n\, \omega(\tau) + A_s^{-1} C_s\, \xi(\tau) \tag{12}$$

and hence from (9):

$$\begin{aligned} \tilde{y}(\tau \mid t) &= y(\tau) - \hat{y}(\tau \mid t) = A_s^{-1} C_s\, \xi(\tau) - H_f(z^{-1})\, z(t) = \\ &= A_s^{-1} C_s\, \xi(\tau) - H_f(z^{-1})\, z^l\, z(\tau) = (I_r - H_f z^l)\, A_s^{-1} C_s\, \xi(\tau) - \\ &\quad - H_f z^l (v(\tau) + A_n^{-1} C_n\, \omega(\tau)) \end{aligned} \tag{13}$$

Cost-function expansion

Let the unit-circle contour $|z| = 1$ be denoted by $U$ then the following contour integrals are computed around $U$. The covariance of the estimation error can be expressed, using (13) as:

$$\begin{aligned} &\frac{1}{2\pi \mathrm{j}} \oint_U \mathrm{Trace}\,\{(I - H_f z^l)\, A_s^{-1} C_s Q_s C_s^* A_s^{*-1}\, (I - H_f z^l)^* + \\ &+ H_f (R + A_n^{-1} C_n Q_n C_n^* A_n^{*-1})\, H_f^*\}\, \frac{\mathrm{d}z}{z} = \\ &\frac{1}{2\pi \mathrm{j}} \oint_U \mathrm{Trace}\,\{H_f A_f^{-1} D_f D_f^* A_f^{*-1} H_f^* + A_s^{-1} C_s C_s^* A_s^{*-1} - \\ &- H_f z^{-l} A_s^{-1} C_s Q_s C_s^* A_s^{*-1} - A_s^{-1} C_s Q_s C_s^* A_s^{*-1} z^{-l} H_f^*\}\, \frac{\mathrm{d}z}{z}\,. \end{aligned}$$

Completing the squares gives:

$$\frac{1}{2\pi j}\oint_U \text{Trace}\,\{(H_f A_f^{-1} D_f - A_s^{-1} C_s Q_s C_s^* A_s^{*-1} A_f^* D_f^{*-1} z^{-l}) .$$
$$.(H_f A_f^{-1} D_f - A_s^{-1} C_s Q_s C_s^* A_s^{*-1} A_f^* D_f^{*-1} z^{-l})^* + \tag{14}$$
$$+ [A_s^{-1} C_s Q_s C_s^* A_s^{*-1} - A_s^{-1} C_s Q_s C_s^* A_s^{*-1} A_f^* D_f^{*-1} D_f^{-1} A_f A_s^{-1} C_2 Q_c C_s^* A_s^{*-1}]\}\frac{dz}{z}$$

The final term in (14), within the square brackets, can be written, using (7), as:

$$[\cdot] = \Phi_{ss} - \Phi_{ss}\Phi_{ff}^{-1}\Phi_{ss} = \Phi_{ss}\Phi_{ff}^{-1}\Phi_{nn}$$

where

$$\Phi_{ss} := A_s^{-1} C_s Q_s C_s^* A_s^{*-1}\,, \quad \Phi_{nn} := A_n^{-1} C_n Q_n C_n^* A_n^{*-1}$$

and

$$\Phi_{ff} := Y_f Y_f^* \,.$$

## Non-causal solution

Since the final term in (14) is independent of $H_f$ it is clear that the minimum is achieved when the first term is null, giving:

$$H_f(z^{-1}) = \Phi_{ss}(z^{-1})\,\Phi_{ff}(z^{-1})^{-1}\,z^{-l}$$

where $\Phi_{ss}$ and $\Phi_{ff}$ denote the power spectral density matrices of the signal and the total noise spectrum. This solution clearly provides a non-causal smoother which is not feasible in real-time applications. However, the minimum cost in this case is as useful lower bound, obtained using (7) and (14) as:

$$J_{\text{non-causal}} = \frac{1}{2\pi j}\oint_U \text{Trace}\,\{\Phi_{ss}\Phi_{ff}^{-1}\Phi_{nn}\}\,\frac{dz}{z}$$

where $\Phi_{nn}$ denotes the spectral-density for the measurement noise. Note that the absolute minimum of the cost, in this case, is not dependent upon the smoothing lag $l$.

To minimise the cost-function, when the smoother is restricted to being causal, the first term in (14) must be expanded and simplified using a diophantine equation.

## Diophantine equation

First define the scalar $g = \deg(D_f(z^{-1}))$. Then the diophantine equation (cf. Kučera [3]) required follows as:

$$A_s F_0 + G_0 D_f^* z^{-g} = C_s Q_s \tilde{C}_s^* z^{-g-l}$$

This equation may be written, using (5), in the form:

$$F_0 D_f^{*-1} z^g + A_s^{-1} G_0 = A_s^{-1} C_s Q_s C_s^* A_s^{*-1} A_f^* D_f^{*-1} z^{-l}$$

Stable/unstable decomposition

Since $\deg F_0 < g$ the first term has all its poles strictly outside the unit-circle in the $z$-plane. Thus, the *squared* term in the cost-function (14) can be decomposed:

$$(H_f A_f^{-1} D_f - A_s^{-1} C_s Q_s C_s^* A_s^{*-1} A_f^* D_f^{*-1} z^{-l}) =$$
$$= [H_f A_f^{-1} D_f - A_s^{-1} G_0] - F_0 z^g D_f^{*-1} \tag{15}$$

The term within the square brackets in (15) has all its poles strictly within the unit-circle. The final term in (15) is strictly unstable with all poles outside the unit-circle. By the residue theorem the integral of the cross terms in (14) is zero and the cost-function can be written as:

$$J = \frac{1}{2\pi j} \oint_U \text{Trace}\,(H_f A_f^{-1} D_f - A_s^{-1} G_0)(H_f A_f^{-1} D_f - A_s^{-1} G_0)^* +$$
$$+ F_0 D_f^{*-1} D_f^{-1} F_0^* + [\Phi_{ss} \Phi_{ff}^{-1} \Phi_{nn}]\} \frac{dz}{z} \tag{16}$$

The only term in the cost-function which depends upon the smoothing filter is the first term and hence the cost is minimized by setting this term to zero. The optimal smoother follows.

### 3.3. Summary of the solution

There follows a summary of the results obtained:

**Theorem 3.1.** (*Fixed Lag Smoothing Filter.*) The optimal smoothed estimate for the signal and noise models, shown in Figure 1, at the time $\tau$, is obtained as:

$$\hat{y}(\tau \mid t) = H_f(z^{-1})\, z(t) \tag{17}$$

given the observations $\{z(t_1)\}$, for $t_1 \in (-\infty, t]$, and $t \geqq \tau$. The optimal smoothing filter, to minimize the cost (9), is given by:

$$H_f = A_s^{-1} G_o D_f^{-1} A_f \tag{18}$$

The polynomial matrix $G_o$ is obtained from the minimal-degree solution $(G_o, F_o)$, with respect to $F_o$, of the diophantine equation:

$$A_s F_o + G_o D_f^* z^{-g} = C_s Q_s \tilde{C}_s^* z^{-g-l} \tag{19}$$

where

$$g := \deg(D_f) \quad \text{and} \quad \deg(F_0) := g - 1\,.$$

Proof. By equating the first term in (16) to zero. □

The existence and uniqueness of the solution to this type of diophantine equation has been established by Kučera [3]. Note that to ensure solvability of the smoothing problem, $\det[D_f(z^{-1})] \neq 0$ on $|z| = 1$.

**Lemma 3.1.** (*Minimum-cost expression.*)

$$J_{\min} = \frac{1}{2\pi j}\oint_U \text{Trace}\{F_o D_f^{*-1} D_f^{-1} F_0^* + \Phi_{ss}\Phi_{ff}^{-1}\Phi_{nn}\}\frac{dz}{z} \tag{20}$$

Proof. Follows directly from (16) and (18). □

Note that if the measurement noise tends to zero ($R = 0$, $Q_n = 0$) then $D_f D_f^* = \tilde{C}_s Q_s \tilde{C}_s$ and (19) gives $F_o = 0$, $G_o = D_f z^{-l}$. The minimum cost then follows from (20) as $J_{\min} = 0$. The smoother is stable from the definition of the polynomial matrices in (18).

**Example 3.1.** (*Coloured measurement noise fixed lag smoothing problem.*)
Consider the smoothing problem illustrated in Figure 2 where the signal and noise models are defined as:

$$W_s = \frac{2 - 1{\cdot}5z^{-1}}{(1 - z^{-1})(1 - 0{\cdot}5z^{-1})}, \quad W_n = \frac{1 - 0{\cdot}2z^{-1}}{1 - 0{\cdot}5z^{-1}},$$

where $R = 0$, $Q_s = 1$ and $Q_n = 1$.

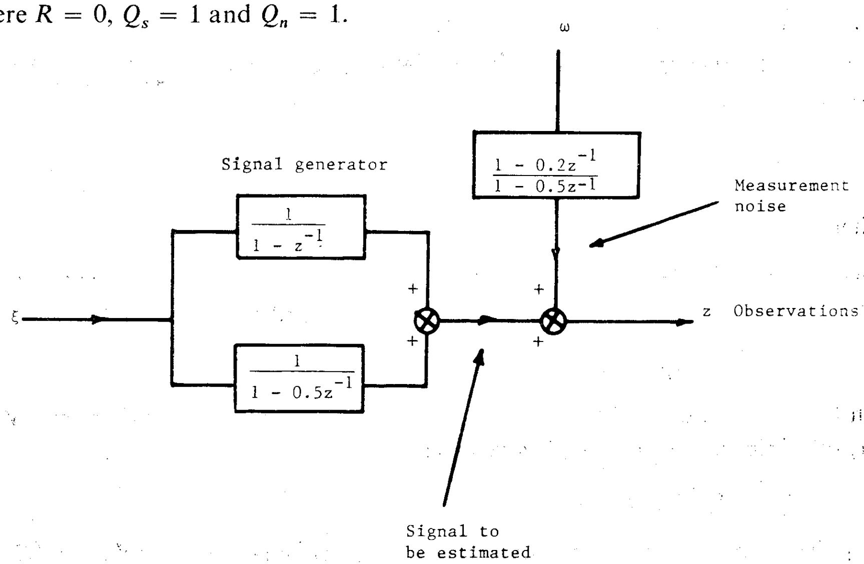


**Fig. 2.** Smoothing Problem.

Solution: In polynomial form:

$$A_s = (1 - z^{-1})(1 - 0{\cdot}5z^{-1}), \quad C_s = 2 - 1{\cdot}5z^{-1}, \quad A_n = 1 - 0{\cdot}5z^{-1},$$
$$C_n = 1 - 0{\cdot}2z^{-1}$$

Thus from (5):

$$\hat{A}_f^{-1}[\tilde{C}_n, \tilde{C}_s] = \frac{1}{(1 - z^{-1})(1 - 0{\cdot}5z^{-1})}[(1 - z^{-1})(1 - 0{\cdot}2z^{-1}), 2 - 1{\cdot}5z^{-1}]$$

and the spectral factor equation (8) gives:

$$D_f^* D_f^* = 8{\cdot}73 - 4{\cdot}44z^{-1} - 4{\cdot}44z + 0{\cdot}2z^{-2} + 0{\cdot}2z^2$$

hence

$$D_f = 0{\cdot}088(z^{-1} - 20{\cdot}08)(z^{-1} - 1{\cdot}3)$$

$$A_f = (1 - z^{-1})(1 - 0{\cdot}5z^{-1}).$$

The diophantine equation (19) becomes:

$$A_s F_o + G_0 D_f^* z^{-g} = C_s Q_s \tilde{C}_s^* z^{-g-l}$$

$$(1 - z^{-1})(1 - 0{\cdot}5z^{-1}) F_o + G_o\, 0{\cdot}088(z - 20{\cdot}08)(z - 1{\cdot}3) z^{-2} =$$

$$= (2 - 1{\cdot}5z^{-1})(2 - 1{\cdot}5z) z^{-2-l}$$

$$(1 - 1{\cdot}5z^{-1} + 0{\cdot}5z^{-2}) F_o + G_o(0{\cdot}088 - 1{\cdot}8814z^{-1} + 2{\cdot}2972z^{-2}) =$$

$$= (-3 + 6{\cdot}25z^{-1} - 3z^{-2}) z^{-1-l}$$

For optimality $\deg(F_0) = 1$ and $\deg(G_0) = 1 + l$.

From the expression (18) for the smoothing filter obtain:

$$H_f(z^{-1}) = 11{\cdot}3636G_o/((20{\cdot}08 - z^{-1})(1{\cdot}3 - z^{-1})). \tag{21}$$

The $G_o(z^{-1})$ polynomial was evaluated for different values of the smoothing lag $l$, and these are shown in Table 3.1. Note that for large delay the highest order coeffi-

**Table 3.1.** Fixed Lag Smoothing Filter Polynomials.

| | |
|---|---|
| $l = 0$: | $G^0 = 1{\cdot}9122 - 1{\cdot}4122z^{-1}$ |
| $l = 1$: | $G_0 = 0{\cdot}1979 + 1{\cdot}6154z^{-1} - 1{\cdot}3133z^{-2}$ |
| $l = 2$: | $G_0 = 0{\cdot}0889 + 0{\cdot}0646z^{-1} + 1{\cdot}6598z^{-2} - 1{\cdot}3133z^{-3}$ |
| $l = 3$: | $G_0 = 0{\cdot}0652 - 0{\cdot}009z^{-1} + 0{\cdot}0972z^{-2} + 1{\cdot}6598z^{-3} - 1{\cdot}3133z^{-4}$ |
| $l = 4$: | $G_0 = 0{\cdot}05 - 0{\cdot}0098z^{-1} + 0{\cdot}016z^{-2} + 0{\cdot}0972z^{-3} + 1{\cdot}6598z^{-4} - 1{\cdot}3133z^{-5}$ |
| $l = 5$: | $G_0 = 0{\cdot}0385 - 0{\cdot}0077z^{-1} + 0{\cdot}0094z^{-2} + 0{\cdot}016z^{-3} + 0{\cdot}0972z^{-4} + 1{\cdot}6598z^{-5} - 1{\cdot}3133z^{-6}$ |
| $l = 6$: | $G_0 = 0{\cdot}0296 - 0{\cdot}0059z^{-1} + 0{\cdot}0071z^{-2} + 0{\cdot}0094z^{-3} + 0{\cdot}016z^{-4} + 0{\cdot}0972z^{-5} + 1{\cdot}6598z^{-6} - 1{\cdot}3133z^{-7}$ |
| $l = 20$: | $G_0 = 0 - 0{\cdot}0000z^{-1} + 0{\cdot}0002z^{-2} + 0{\cdot}0002z^{-3} + 0{\cdot}0003z^{-4} + 0{\cdot}0004z^{-5} + 0{\cdot}0005z^{-6} + 0{\cdot}0007z^{-7} + 0{\cdot}0009z^{-8} + 0{\cdot}0011z^{-9} + 0{\cdot}0015z^{-10} + 0{\cdot}0019z^{-11} + 0{\cdot}0025z^{-12} + 0{\cdot}0032z^{-13} + 0{\cdot}0042z^{-14} + 0{\cdot}0055z^{-15} + 0{\cdot}0071z^{-16} + 0{\cdot}0094z^{-17} + 0{\cdot}016z^{-18} + 0{\cdot}0971z^{-19} + 1{\cdot}6598z^{-20} - 1{\cdot}3133z^{-21}$ |
| $l = 0$: | $F_0 = 0{\cdot}0837 - 0{\cdot}226z^{-1}$ |
| $l = 1$: | $F_0 = 0{\cdot}0087 - 0{\cdot}1015z^{-1}$ |
| $l = 2$: | $F_0 = 0{\cdot}0039 - 0{\cdot}0745z^{-1}$ |
| $l = 3$: | $F_0 = 0{\cdot}0029 - 0{\cdot}05715z^{-1}$ |
| $l = 4$: | $F_0 = 0{\cdot}0022 - 0{\cdot}044z^{-1}$ |
| $l = 5$: | $F_0 = 0{\cdot}0017 - 0{\cdot}03385z^{-1}$ |
| $l = 6$: | $F_0 = 0{\cdot}0013 - 0{\cdot}02605z^{-1}$ |
| $l = 20$: | $F_0 = 0{\cdot}0000 - 0{\cdot}00065z^{-1}$ |

cients converge to the same values and the lowest order coefficients converge to zero. These correspond to impulse-response coefficients and the shape is clearly two-sided with a peak at the coefficient of $z^{-l}$. The largest contribution to the smoother output is therefore due to the signal: $z(t-l) = z(\tau)$ which is reasonable since the estimate is required at this particular time.

The polynomial $F_o$ is also shown in Table 3.1 and clearly this becomes smaller as the lag $l$ increases. Recall from (20) that $J_{\min}$ has two basic terms an the first depends upon $F_o$. Clearly as the coefficients of $F_0$ become small $J_{\min} \to J_{\text{non-causal}}$. That is, the performance approaches that of the non-causal optimal fixed-lag smoother. Note that most of the decrease in the size of the $F_o$ coefficients occurs for a relatively small lag. This suggests that a relatively short smoothing lag is all that is required in some applications to obtain a reasonable cost implement relative to the filtering case ($l = 0$).

It is interesting to note that by implementing a sequence of the $G_o$ polynomials, for different values of $l$, a fixed point smoothing action can be achieved (computing $\hat{y}(\tau \mid t)$ for fixed $\tau$). This particular application is not so convenient because of the need to store data and polynomial coefficients. This fixed-point problem was previously considered by Shaked [5] using a transfer-function approach.

### 3.4. Smoothing with dynamic cost weighting

The previous results can be generalized by introducing a dynamic cost weighting term $W_p$ into the cost-function. This enables the estimation error in a particular frequency range to be heavily penalized. The results are summarized below.

**Theorem 3.2.** (*Smoothing filter with dynamic cost-function.*) Assume that the weighted estimation error $\tilde{y}_o(\tau \mid t) := W_p\, \tilde{y}(\tau \mid t)$ is to be minimized, where the cost: $J_0 := \mathsf{E}\{\tilde{y}_0^{\mathrm{T}}(\tau \mid t)\, \tilde{y}_0(\tau \mid t)\}$, and the signal model and noise are as shown in Figure 1. The optimal fixed-lag smoother becomes:

$$H_f = B_p^{-1} A_0^{-1} G_o D_f^{-1} A_f \,. \tag{22}$$

where $B_p$ is defined in Appendix 1. The polynomial matrix $G_o$ is obtained from the minimal-degree solution $(G_o, F_o)$, with respect to $F$, of the diophantine equation:

$$A_o A_p F_o + G_o D_f^{*} z^{-g} = B_o C_s Q_s \tilde{C}_s^{*} z^{-g-l} \tag{23}$$

where $g := \deg(D_f)$, $\deg(F_o) = g - 1$ and $A_o$, $B_o$ are obtained from the left-coprime decomposition: $A_0^{-1} B_o = B_p A_s^{-1}$. The minimum variance of the smoothing error can be computed as:

$$J_{\min} = \frac{1}{2\pi \mathrm{j}} \oint_U \operatorname{Trace}\left(\{F_o D_f^{*-1} D_f^{-1} F_0^{*} + W_p \Phi_{ss} \Phi_{ff}^{-1} \Phi_{nn} W_p^{*}\}\, \frac{\mathrm{d}z}{z}\right. \tag{24}$$

Proof. Presented in Appendix 1. □

This form of the main theorem is needed when deriving an $H_\infty$ optimal linear smoothing filter.

## 4. FREQUENCY DOMAIN PROPERTIES

The frequency domain and pole-zero properties of the optimal linear smoothing filter which equivalent to the Wiener smoother, are easy to establish in this polynomial a matrix representation (cf. Grimble and Johnson [12]).

**Lemma 4.1.** The optimal smoother $H_f$ has zeros (unless cancellations occur) at the poles of the measurement noise or output disturbance model $W_n$.

Proof. From (5) the zeros of $\det(A_f)$ include those of $\det(A_n)$ and $\det(A_s)$, and since from (18) $H_f = A_s^{-1} G_o D_f^{-1} A_f$, and therefore, unless cancellations occur, the zeros of $H_f$ include those of $\det(A_n)$. □

The fact that the filter includes blocking zeros, at the poles of the measurement noise model is necessary if the signal estimate is to be uncorrupted.

**Lemma 4.2.** If the measurement noise source is minimum phase and $C_n Q_n^{1/2}$ is square and full rank, then as the covariances of the noise sources $\xi$ and $v$ tend to zero, some of the poles of the smoother tend (unless cancellations occur) to the zeros of $C_n$.

Proof. From (6) $Y_f = A^{-1} C_n Q_n^{1/2}$ or $D_f = C_n Q_n^{1/2}$ and the results follow from (18). □

These results suggest that in the case of low signal to noise ratios the smoothing filter tends to the inverse of the measurement noise or output disturbance model $W_n$. Thus, as in classical design, the optimal smoothing filter tends to a notch frequency response characteristic.

**Lemma 4.3.** The gain of the optimal smoothing filter $H_f$ depends upon the signal $\Phi_{ss}$ and signal plus noise $\Phi_{mm}$ spectral densities, and for a scalar problem the gain is proportional to the square root of the signal to signal plus noise ratio.

Proof. Now $\Phi_{ss} := A_s^{-1} C_s Q_s C_s^* A_s^{*-1}$ and $\Phi_{mm} := A_f^{-1} D_f D_f^* A_f^{*-1}$ and from (18):

$$H_f = (A_s^{-1} C_s Q_s^{1/2})(C_s Q_s^{1/2})^{-1} G_o D_f^{-1} A_f \qquad \square$$

If in a particular frequency range the measurement noise is relatively large it is reasonable for the gain of the smoother to be reduced, as the above result confirms.

## 5. ADAPTIVE SMOOTHING

An adaptive smoothing filter has the advantage of accommodating slowly varying noise or signal model changes by on-line calculations. Consider the class of linear noise or signal models with unknown parameters which are slowly time-varying or time-invariant. *Explicit* self-tuning smoothers output estimation can be constructed by using an extended least squares (ELS) identifier (cf. Panuska [10]) in conjunction with the results of Theorem 3.1. *Implicit* self-tuning estimators where the estimator polynomials are identified directly from the ELS algorithm can be obtained as described by Moir and Grimble [4] and by Fung and Grimble [11].

Consider the output smoothing problem, and using (3) write the observations signal in the innovations form:

$$z(t) = v(t) + A_n^{-1} C_n \,\omega(t) + A_s^{-1} C_s \,\xi(t) = A_f^{-1} D_f \,\varepsilon(t)$$

where $\{\varepsilon(t)\}$ denotes white noise of zero mean and unity covariance. Assume first that $A_f$ and $D_f$ are estimated using an ELS algorithm. Also assume that the signal model spectrum $A_s^{-1} C_s Q_s C_s^* A_s^{*-1}$ is known, so that $\tilde{C}_s = A_f A_s^{-1} C_s$ can be computed. All the terms in (18) and (19) are then known and the smoothing filter can be computed at time $t$. Clearly as the noise model varies the smoothing filter will adapt accordingly.

## 6. CONCLUSIONS

The Kalman filter and related smoother has been very successful in many applications but being state equation based, it is not so convenient for adaptive smoothing problems where polynomial system models are used in the identification algorithms. The preceding polynomial matrix form of multivariable smoother is simple to implement in recursive discrete form and is also suitable for use in self-tuning schemes.

The smoothing problem considered was more general than the usual Kalman problem, allowing for coloured measurement noise, or even the singular case of white noise. It was shown that the minimal-cost which could be obtained, for large smoothing lag $l$, was determined by the minimum cost for the non-causal Wiener smoother. The frequency-domain properties of the smoother were easily established in this polynomial form, including the limiting pole-zero positions.

There are some signal processing applications where a discrete Wiener smoother is more appropriate than a Kalman smoother, since it is transfer function based. However, Wiener smoothers are difficult to compute using standard routines on a digital computer. The polynomial method overcomes this problem. Further research is necessary to establish the relative computational advantages of calculating either the polynomial or Kalman smoothers. The approach might also be applied to the evaluation of finite impulse-response smoothers (see Grimble [6], [7]) and to calculating an $H_\infty$ smoothing filter (cf. Grimble [8]).


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## APPENDIX 1: SOLUTION FOR DYNAMIC WEIGHTING CASE

If a dynamic weighting term $W_p = A_p^{-1}B_p$ is introduced into the cost-function the solution follows the same steps as previously but with the following changes.

Write $A_0^{-1}B_0 = B_pA_s^{-1}$ and note that introducing the weighting $W_p$ results in the integrand in (14) being left multiplied by $W_p$ and right multiplied by $W_p^*$. Thus, the new integrand includes a term:

$$\begin{aligned}&(W_pH_fA_f^{-1}D_f - A_p^{-1}A_0^{-1}B_oC_sQ_sC_s^*A_s^{*-1}A_f^*D_f^{*-1}z^{-l}) = \\ &= W_pH_fA_f^{-1}D_f - (A_oA_p)^{-1}\, B_oC_sQ_s\tilde{C}_s^*D_f^{*-1}z^{-l}.\end{aligned}$$

The diophantine equation needed to expand the second term is the same as (19) but with $A_oA_p$ replacing $A_s$ and $B_oC_s$ replacing $C_s$. The term within (16) then follow as:

$$(W_pH_fA_f^{-1}D_f - (A_oA_p)^{-1}\, G_o)$$

and setting this term to zero gives:

$$H_f = B_p^{-1}A_0^{-1}G_oD_f^{-1}A_f\,.$$

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