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STRUCTURE OF LINEAR SYSTEMS: GEOMETRIC AND TRANSFER MATRIX APPROACHES

CHRISTIAN COMMAULT, JEAN F. LAFAY, MICHEL MALABRE

The aim of this communication is to show how, depending on the type of the control law (static or dynamic), some fine structures (internal or input-output ones) have to be known precisely, since they completely characterize the solvability of control problems like decoupling, disturbance decoupling or model matching, ... These structures mainly describe zeros (finite and at infinity) and kernel indices. Both geometric and transfer matrix approaches are used in accordance with internal and external points of view.

Up to the sixties, the analysis and the control design of linear systems was performed in the frame of the transfer matrix approach (Bode, Black, Nyquist, ...). Then, from 1960 to 1970, the notion of state, which gave rise to the famous break-through in the study of *multivariable* systems, became so popular that it often received the label "modern approach". From 1970 to 1980 all approaches have been developed giving rise to the transfer matrix approach, to the polynomial approach and to the geometric approach and many control problems have been solved within each approach but with specific tools and, most often, the given conditions were approach-dependent in the sense that these specific tools were explicitly used inside.

Then, it became more clear that, as far as linear systems only were concerned, all approaches were somewhat equivalent: each new result in one direction could almost systematically be obtained in another approach. After the description of some common bridges between the tools of one approach and another, a deeper global vision has finally been reached, since the beginning of the eighties, with the help of structural information. Indeed, some authors have tried to exploit more intensively the fine structure of linear systems, breaking free from particular tools such as invariant subspaces or specific factorizations and, thanks to this structural frame have provided structural solutions to control problems like model matching or decoupling. These structural informations are invariants as controllability and observability indices, finite and infinite zeros, Morse's invariants, Kernel indices, essential orders, ...

Now, this way of tackling systems, within a structural approach, appears to be particularly efficient. The aim of this communication is to give a short review of the structures that control people should precisely know and, depending on the type of control law (in essence: static or dynamic), to explain why internal or external structures play a key role.

The paper is organized as follows: Sections 1 (Introduction) and 2 (Basic concepts and notation) describe, in more details, the different control laws which are frequently used and the basic geometric tools. Section 3 is devoted to geometric and transfer matrix characterization of internal and external structures like zeros or kernel indices. Then, applications of this structural approach are given in Section 4 in the context of the decoupling and model matching problems. Section 5 is devoted to concluding remarks.

1. INTRODUCTION

We shall consider linear multivariable systems described either by a (C, A, B) state space representation:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) \end{cases} \tag{1}$$

where $x \in \mathscr{X} \approx \mathbb{R}^n$, $y \in \mathscr{Y} \approx R^p$, $u \in \mathscr{U} \approx \mathbb{R}^m$, or by their transfer matrix $T(s)$:

$$T(s) = C(s\mathbb{I} - A)^{-1}\,B \tag{2}$$

where s denotes the Laplace variable.

From a structural point of view, these descriptions are equivalent if and only if (A, B) is controllable and (C, A) is observable. In this case the state space representation is called minimal. Otherwise, the input-output structure deduced from $T(s)$ is only part of the internal structure which can be obtained from the state space representation (1).

These two types of structures have to be considered, depending on the type of control law which is chosen for the solution of a given problem.

Suppose that we consider a system described by a state space representation (C, A, B) and that (C, A) is observable: the more natural control law is the static state feedback $u(t) = F\,x(t) + G\,v(t)$, which is called regular if G is invertible. In a transfer matrix approach, this corresponds to a control law: $u(s) = F\,x(s) + G\,v(s)$.

If such a control law is not sufficient for the solution of the problem, it is possible, in a first step, to consider dynamic state feedbacks, that is control laws of the type:

$$u(s) = F(s)\,x(s) + G\,v(s) \quad (G \text{ regular or not}).$$

In the case of an internal description, this operation amounts to insert integrators in the feedback loops on some components of the state.

More generally, we can consider dynamic extensions [24], which amount to enlarging (C, A, B) with a bank of integrators.

$$\dot{x}_a(t) = u_a(t)$$

and the control law is

$$\begin{bmatrix} u(t) \\ u_a(t) \end{bmatrix} = F_e \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} + G_e\, v(t) .$$

Such laws correspond in fact to the addition of integrators on both the inputs and the feedback paths (dynamic compensator and dynamic state feedback).

When we consider a transfer matrix approach, such a compensator is given by:

$$u(s) = F(s)\, x(s) + G(s)\, v(s) ,$$

which is equivalent to:

$$u(s) = G'(s)\, v(s) .$$

When controlled by a dynamic state feedback law such as:

$$u(s) = F(s)\, x(s) + Gv(s) ,$$

the closed loop transfer of system (1) is:

$$T_{FG}(s) = C(s\mathbb{I} - (A + B\, F(s)))^{-1}\, BG ,$$

which can also be written:

$$\begin{aligned} T_{FG}(s) &= T(s)\, (\mathbb{I} - F(s)\, (s\mathbb{I} - A)^{-1} B)^{-1}\, G . \\ &= T(s)\, C(s) \end{aligned}$$

When the control law is regular (G invertible), $C(s)$ is an invertible matrix with the remarkable property that $C(s)$ and $C^{-1}(s)$ are both proper. Such matrices are called bicausal matrices, and play a fundamental role in the study of dynamic state feedback control. We have seen that a regular dynamic state feedback can be represented by a bicausal precompensation. The converse can be proved easily, then bicausal precompensation and dynamic state feedback are indeed equivalent.

As a particular case, static state feedback can be represented by a bicausal precompensator but the converse is no longer true in general. The following equivalence was stated by [8]: the precompensator $C(s)$ can be realized by a static state feedback on a minimal realization of $T(s) = (s\mathbb{I} - A)^{-1} B$ if and only if $C(s)$ is bicausal, and for any $u(s)$ such that $T(s)\, u(s)$ is polynomial $C^{-1}(s)\, u(s)$ is also polynomial.

Finally, let us precise that, to avoid trivialities, admissibility conditions must be guaranteed by the compensators according to the problem to be solved. In the transfer matrix approach, the classical admissibility condition for $C(s)$ is the following rank preservation:

$$\text{rank } T(s)\, C(s) = \text{rank } T(s)$$

In a geometric (internal) approach, we usually impose the preservation of the output controllability, that is:

$$C\langle A + BF|\{BG\}\rangle = C\langle A|\mathscr{B}\rangle,$$

where $\{B\}$ or $\mathscr{B}$ denotes the image of B, and

$$\langle A \mid \mathscr{B}\rangle = \mathscr{B} + A\mathscr{B} + \ldots + A^{n-1}\mathscr{B}$$

The use of regular feedbacks, (in the sense G is invertible), amounts to only applying group's actions on the system: some structure of the initial system is then unmodified, and it is thus important to characterize this structure. As shown in the sequel, this structure is usually described by some lists of invariants (finite zeros, zeros at infinity, ...).

Now, for a given control problem, it can happen that only non regular solutions exist: this usually means that some structure has to be modified to solve the problem. These considerations illustrate how important the description of these structures is. Let us then define the main tools and concepts which will be useful in the sequel for this description.

2. BASIC CONCEPTS AND NOTATIONS

As mentioned before, $\mathscr{B}$ or Im B or $\{B\}$ will denote the range of B, and $\mathscr{K}$ the kernel of C. Let $\langle A \mid \mathscr{B}\rangle = \mathscr{B} + A\mathscr{B} + \ldots + A^{n-1}\mathscr{B}$, and $\langle \mathscr{K} \mid A\rangle = \operatorname{Ker} C \cap \cap \operatorname{Ker} CA \cap \ldots \cap \operatorname{Ker} CA^{n-1}$, which respectively denote the reachable and unobservable subspaces.

For any space $\mathscr{J} \subset \mathscr{W}$, the quotient space $\mathscr{W}$ modulo $\mathscr{J}$ will be written $\mathscr{W}/\mathscr{J}$. A set of k elements will be represented by $\{\cdot\}_k$ and the number of elements $\{\cdot\}$ is noted by card $\{\cdot\}$. In what follows $\mathscr{V}_\mu$, $\mathscr{S}_\mu$ and $\mathscr{R}_\mu$ denote the different steps of the well-known algorithms ISA, CISA and CSA [24], [22], the limits of which are denoted by $\mathscr{V}^*$, $\mathscr{S}^*$ and $\mathscr{R}^*$. Let us recall these fundamental algorithms.

$$\text{ISA:}\quad \begin{cases} \mathscr{V}_0 = \mathscr{X} \\ \mathscr{V}_{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}_\mu) \end{cases}$$

$$\text{CISA:}\quad \begin{cases} \mathscr{S}_0 = 0 \\ \mathscr{S}_{\mu+1} = \mathscr{B} + A(\mathscr{K} \cap \mathscr{S}_\mu) \end{cases}$$

$$\text{CSA:}\quad \begin{cases} \mathscr{R}_0 = 0 \\ \mathscr{R}_{\mu+1} = \mathscr{V}^* \cap (A\mathscr{R}_\mu + \mathscr{B}) \end{cases}$$

$\mathscr{R}^*$ also satisfies $\mathscr{R}^* = \mathscr{V}^* \cap \mathscr{S}^*$.

When a space $\mathscr{E}$ is $(A - B)$ invariant, the set of maps F satisfying $(A + BF)\,\mathscr{E} \subset \mathscr{E}$ is noted $\mathscr{F}(\mathscr{E})$.

Let $\mathscr{R}$ be the supremal controllability subspace contained in $\mathscr{E}$, then, for any $F \in \mathscr{F}(\mathscr{E})$, $(A + BF)\,\mathscr{R} \subset \mathscr{R}$ and the map $A + BF \parallel \mathscr{E}/\mathscr{R}$ is the map induced by the double restriction of $(A + BF)$ to $\mathscr{E}$ in the quotient space $\mathscr{E}/\mathscr{R}$.

3. STRUCTURAL DESCRIPTIONS OF LINEAR SYSTEMS

In this main section, we shall alternatively consider descriptions of type (1) and (2), that is:

– An internal description, where the non minimality of the realization will take part, and from which a fine internal structure can be derived.
– An input-output description, which will generally lead to a more succinct structure.

The main mathematical tools used for the analysis of these two structures are the following:

– Internal structure, which depends upon the state space representation can easily be characterized with the help of geometric concepts [18], or equivalently, in terms of the structure of the system matrix defined by [19].
– Input-output structure can be characterized through the analysis of the transfer $T(s)$ above defined or, equivalently, using a geometric approach (see [1]), on a minimal part of the state space description.

These two types of structures naturally appear in the study of a system, and they usually correspond to finite and infinite zeros, and to right and left indices (in relation with the properties of observability and/or controllability).

In the following, it will be clear that the choice of the structural elements to be considered depends upon the design of the compensator to be implemented on the system. It will be seen in Section 4 that internal structure is closely related to static state feedback laws, while input-output structure is sufficient in the case of dynamic laws.

A global description of the main elements of these two structures is given in the

Pencil Description KRONECKER ROSENBROCK -1-		Geometric Approach MORSE -2-		Transfer Matrix Approach -3-		Geometric Approach ALING SCHUMACHER -4
column minimal indices	$\equiv$	list I_2	$\supset$	right Kernel indices	$\equiv$	Kernel indices
row minimal indices	$\equiv$	list I_3	$\supset$	left Kernel indices	$\equiv$	corange indices
finite elementary divisors	$\equiv$	list I_1	$\supset$	finite transmission zeros	$\equiv$	finite transmission zeros
infinite elementary divisors	$\equiv$	list I_4	$\equiv$	infinite zeros	$\equiv$	infinite zeros

Here $\equiv$ means that the structural information on both sides is equivalent, and $\supset$ means that the right hand side is a "substructure" of the left hand one.

following picture, where the different tools for characterizing them are:
1 – The analysis of the system matrix (Rosenbrock or Kronecker, [19]).
2 – The Morse's canonical decomposition in a geometric framework, [18].
3 – The analysis of the transfer $T(s)$.
4 – A geometric approach on a minimal part of the realization, [1].

Because of these equivalences **1** $\equiv$ **2** and **3** $\equiv$ **4**, we shall only consider in the sequel the geometric approach for the description of the internal structure, and the transfer matrix approach for the input-output structure: the equivalence between geometric structures (Morse) and Rosenbrock's or Kronecker's ones have been established in [20], [12] ... and will not be detailed here. On the other hand, some geometric characterizations of the input-output structures, [1], will be recalled since they allow to draw links between the two types of structures.

We shall now give a more precise description of the above mentioned structures.

3.1. Right Indices

Many control problems are in fact output nulling problems (decoupling, disturbance rejection, ...). It is clear that all the inputs of the system which are related to the right kernel of $T(s)$ have the property to cancel the outputs of $T(s)$. It is possible to associate with this kernel a minimal polynomial basis (see [7]). The degrees of these polynomials, arranged in non increasing order, give the list noted $\{r_i\}$. These integers are called "right kernel indices" of the transfer $T(s)$.

Example 1. Let

$$T(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

A minimal basis for its kernel is $[-1 \;\; s]^{\mathrm{T}}$ and the associated list $\{r_i\}$ is

$$\{r_i\} = \{1\} = \{r_1\} .$$

As these indices are defined through the transfer $T(s)$, they are related to a minimal system (1), and a geometric characterization of the r_i's has been established by [1] as follows:

Define γ_j, $j = 1, 2, \ldots$ such that $\gamma_j = \alpha_j - \alpha_{j-1}$, where:

$$\alpha_j = \dim\left(\langle \mathscr{K} \mid A \rangle + \mathscr{R}_j\right)$$

with $\langle \mathscr{K} \mid A \rangle$ and $\mathscr{R}_j$ defined in Section 2. Then: $r_i = \operatorname{card}\{j, / \gamma_j \geqq i\}$.

From an internal point of view, a similar structure can be obtained through the steps of CSA. For this, following the notation of [18], we define the list $\mathscr{I}_2$ as:

$$\mathscr{I}_2(i) = \operatorname{card}\{j, j \geqq 1 / \dim\left(\mathscr{R}_j / \mathscr{R}_{j-1}\right) \geqq i\} .$$

It is important to note that, as the state space representation (1) can be unobservable, the structure $\{\ell_2(i)\}$ is usually larger than $\{r_i\}$, as illustrated in the following example.

Example 2. Consider two representations of

$$T(s) = \left[\begin{array}{c|c} \frac{1}{s} & \frac{1}{s^2} \end{array}\right]$$

This first one is minimal:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = [1 \ 0]$$

An easy computation gives $\mathscr{V}^* = \operatorname{Ker} C = \operatorname{span} [0 \ 1]^T$, where ^{T} denotes the transpose, and the steps of CSA are:

$$\mathscr{R}_0 = 0, \quad \mathscr{R}_1 = \mathscr{V}^*$$

so, $\{\ell_2(\mathrm{i})\} = \{1\} = \{r_i\}$.

Consider now the following non minimal realization of $T(s)$:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = [-1 \ 1 \ 0]$$

$$\mathscr{V}^* = \operatorname{Ker} C = \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathscr{R}_0 = 0, \quad \mathscr{R}_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathscr{R}_2 = \mathscr{R}^* = \mathscr{V}^*$$

So, $\{\ell_2(\mathrm{i})\} = \{2\}$.

From the geometric descriptions of these right indices, we see that, when the state space representation is observable, (say $\langle \mathscr{K} \mid A \rangle = 0$), both definitions of Aling-Schumacher and Morse are the same. Otherwise, when $\langle \mathscr{K} \mid A \rangle \neq 0$, the two lists $\{\ell_2(\mathrm{i})\}$ and $\{r_i\}$ have the same number of terms, but $\ell_2(\mathrm{i})$ is greater than or equal to r_i for all i.

3.2. Left Indices

These indices are obviously defined by duality from the right indices using the left kernel of $T(s)$, or, in a geometric framework, considering the structure of the system in the factor space $\mathscr{X}/(\mathscr{V}^* + \mathscr{S}^*)$. These indices are Morse's list ℓ_3.

3.3. Finite Zeros

It is well known that, in the analysis of any control problem, the knowledge of the finite zeros of the system is fundamental, particularly if they are unstable. Indeed, cancellations between poles and zeros occur almost always in output nulling problems. Now, a geometric characterization of these zeros is very natural because $\mathscr{V}^*$ is associated with all the output nulling trajectories, while $\mathscr{R}^*$ is the supremal subset (subspace) of such trajectories with free dynamics. Hence, the factor space $\mathscr{V}^*/\mathscr{R}^*$ naturally depicts the set of output nulling trajectories with fixed dynamics. These fixed dynamics are called invariant zeros of the system. They are characterized by the invariant polynomials of the map induced by $(A + BF)$ in the factor space $\mathscr{V}^*/\mathscr{R}^*$ for any feedback $F \in \mathscr{F}(\mathscr{V}^*)$. These invariant polynomials are the components of Morse's list $\mathscr{I}_1$. Note that these zeros eventually include some unobservable modes (output decoupling zeros) as it can be shown in the following example.

Example 3. Consider the following state space representation and note $\{e_i\}$ the corresponding basis of $\mathscr{X}$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = [-1 \;\; 1 \;\; 0]$$

This realization of the transfer $T(s) = (s - 1)/(s + 1)^2$ is not minimal: $\langle A \mid \mathscr{B} \rangle = \mathscr{X}$ but $\langle \mathscr{K} \mid A \rangle \neq 0$. An easy computation gives $\mathscr{V}^* = \operatorname{Ker} C = (e_1 + e_2, e_3)$, and $\mathscr{R}^* = 0$. Any $F \in \mathscr{F}(\mathscr{V}^*)$, satisfies:

$$(A + BF)(e_1 + e_2) = (e_1 + e_2) + \lambda e_3$$

$$(A + BF)\, e_3 = 0$$

So, when choosing the basis $(e_1 + e_2, e_3)$ for $\mathscr{V}^*$, the application $(A + BF) \parallel (\mathscr{V}^*/\mathscr{R}^*)$ has the following form:

$$(A + BF) \parallel (\mathscr{V}^*/\mathscr{R}^*) = \begin{bmatrix} 1 & 0 \\ \lambda & 0 \end{bmatrix}$$

The invariant zeros of this realization are $s = 1$ and $s = 0$.

As mentioned before, the transfer corresponding to this realization is

$$T(s) = \frac{s - 1}{(s + 1)^2}$$

So $s = 1$ is obviously the input-output zero of this transfer. This zero is called a *transmission zero*, and the zero $s = 0$ of the previous non minimal realization is an output decoupling zero. In the most general case, (multi-input, multi-output systems), the transmission zeros can be characterized as the zeros of the transmission polynomials corresponding to the numerators of the Smith McMillan form of the transfer $T(s)$, and obtained as follows:

Let $T(s)$ be a rational matrix, $T(s)$ can be written as:

$$T(s) = \mathscr{U}_1(s)\, \Lambda(s)\, \mathscr{U}_2(s)\,,$$

where $\mathscr{U}_1(s)$ and $\mathscr{U}_2(s)$ are unimodular matrices (invertible polynomial matrices with constant determinant), and:

$$\Lambda(s) = \begin{bmatrix} \dfrac{p_1(s)}{q_1(s)} & 0 & . & 0 & \\ 0 & . & . & 0 & (0) \\ 0 & . & 0 & \dfrac{p_r(s)}{q_r(s)} & \\ & & (0) & & (0) \end{bmatrix}$$

$r = \operatorname{rank} T(s)$

$p_i(s)$ and $q_i(s)$ are monic polynomials such that q_1 divides q_2, q_2 divides $q_3, \ldots$, and p_r divides p_{r-1}, p_{r-1} divides $p_{r-2}, \ldots$. The $p_i(s)$ are the transmission polynomials and represent the finite zero structure of $T(s)$. The $q_i(s)$ represent the finite pole structure of $T(s)$ and, in particular, $\delta = \sum_{i=1}^{r} \deg(q_i)$ is the McMillan degree of $T(s)$, or the minimal order of a state realization of $T(s)$.

$\Lambda(s)$ is the Smith McMillan form of $T(s)$; it concentrates all the information about the finite pole-zero structure of $T(s)$, since unimodular matrices have no finite singularities.

Different equivalent geometric characterizations of these transmission zeros are given by [1]. One of them is the following:

Transmission polynomials correspond to the invariant polynomials of:

$$A + BF \,\|\, ((\mathscr{V}^* \cap \langle A \mid \mathscr{B}\rangle)/(\langle \mathscr{K} \mid A\rangle \cap \langle A \mid \mathscr{B}\rangle) + \mathscr{R}^*),$$

where $F \in \mathscr{F}(\mathscr{V}^*)$ and $\operatorname{Ker} F \supset \langle \mathscr{K} \mid A\rangle$.

As developed in Section 4, the knowledge of the invariant zeros is crucial for the study of the stability of the compensated system, and thus for the design of a stabilizing control law to be implemented.

Now, before examining the existence of stabilizing control laws achieving some given control problem, the first question to be asked is: does there exist any solution at all? As illustrated in Section 4, the answer usually goes through conditions in terms of zeros at infinity.

3.4. Infinite Zeros

A direct interpretation of the infinite zeros is available for a discrete time state space representation: they do correspond to the input-output delays. This notion is thus very important, since it is clear, for instance, that these delays can never be decreased with any physically implementable compensator.

Let us consider again the above mentioned Example 1: $T(s) = [1/s \mid 1/s^2]$. This transfer has one infinite zero, its order being 1, since $T(s)$ is equivalent to $1/s$ when $s \to \infty$. For multivariable systems, this structure at infinity exhibits the global structure of the internal "delays" of a given system, and can be obtained through the Smith McMillan form at infinity of the transfer, as described now.

Let $T(s)$ be a proper rational matrix, $T(s)$ can be written as: $T(s) = B_1(s)\, \Delta(s)\, B_2(s)$, where $B_1(s)$ and $B_2(s)$ are bicausal matrices (invertible proper matrices with a proper inverse), and:

$$\Delta(s) = \begin{bmatrix} s_1^{-n_1} & & & (0) \\ & \ddots & & \\ & & s_r^{-n_r} & \\ (0) & & 0 & \\ & & & \ddots \end{bmatrix}$$

$$n_1 \geqq n_2 \geqq \ldots \geqq n_r \quad \text{and} \quad r = \operatorname{rank} T(s)\,.$$

$\Delta(s)$ is called the Smith McMillan form at infinity of $T(s)$.

The non zero integers n_i are the infinite zero orders of $T(s)$ (note that when $T(s)$ is strictly proper all the n_i's are strictly positive). The bicausal matrices can be seen as "just proper" matrices having neither poles nor zeros at infinity.

From a geometric point of view, this structure at infinity does not depend upon the minimality of the realization, and can be characterized as follows:

Let

$$p_j = d((\mathscr{V}^* + \mathscr{S}_j)/\mathscr{V}^* + \mathscr{S}_{j-1}))\,, \quad j \geqq 1$$

The orders of the infinite zeros of (C, A, B) are given by the (ordered) list $\{n_i\}$, where:

$$n_i = \operatorname{card}(j/p_j \geqq i)\,, \quad i \geqq 1\,.$$

This list $\{n_i\}$, defined using geometric considerations, corresponds to Morse's list I_4.

Example 4. Let us consider both realizations of $T(s) = [1/s \mid 1/s^2]$ presented in Example 2.

For the first realization, which is minimal, $\mathscr{S}_0 = 0$, $\mathscr{S}_1 = \mathscr{B} = \mathscr{X}$.

So: $p_1 = 1$, and $n_1 = 1$.

For the second realization, $\mathscr{S}_0 = 0$, $\mathscr{S}_1 = \mathscr{B}$, $\mathscr{S}_2 = \mathscr{S}^* = \mathscr{X}$.

So: $p_1 = 1$, and $n_1 = 1$.

The structure of the zeros at infinity does not depend upon the realization of the transfer $T(s)$.

At this level, we can note that the definitions of infinite and finite zeros require, in the transfer matrix approach, the use of two canonical forms (Smith McMillan and Smith McMillan at infinity). In a geometric framework, there is one unique canonical form (Morse's canonical form, [19]), which provides all the four above presented structures, according to the following state space decomposition:

$$\underbrace{O \subset \mathscr{R}^*}_{I_2} \underbrace{\subset \mathscr{V}^*}_{I_1} \underbrace{\subset \mathscr{V}^* + \mathscr{S}^*}_{I_4} \underbrace{\subset \mathscr{X}}_{I_3}$$

This Morse's form is a canonical form under the group of transformations

$g = (T, F, G, R, S)$ with T, G, S invertible, and the action of which is described as:

$$(A, B, C) \xrightarrow{g} (T^{-1}(A + BF + RC)\,T,\, T^{-1}BG, SCT)$$

The action of (T, F, G, R, S) amounts to pre-multiplying and post-multiplying $T(s)$ by bicausal transformations:

$$T(s) \xrightarrow{g} B_1(s)\,T(s)\,B_2(s)$$

Indeed: $B_1(s) = S[I + C(sI - A - BF)^{-1}\,R]^{-1}$

and: $B_2(s) = [I - F(sI - A)^{-1}\,B]^{-1}\,G\,.$

This shows (see part 1) that any static state feedback law $u = Fx + Gv$ (with G regular) is only a particular case of a bicausal precompensation, which obviously cannot modify the infinite zero structure.

This property is still true when using dynamic feedbacks $u(s) = F(s)\,x(s) + G\,v(s)$, G regular, since this transformation is actually equivalent to a right multiplication of the transfer by a bicausal $B(s)$. In the study of control problems like decoupling, disturbance decoupling, ..., the use of output injections R or basis transformations S in the output space is not usually allowed. In that case, only the actions of the subgroup (T, F, G), or of right bicausal transformations on $T(s)$, are available. In a transfer matrix approach, the appropriate canonical form for this kind of right transformations is the well known Hermite form defined as follows:

Let $T(s)$ be a proper rational matrix, $T(s)$ can be written as $T(s) = H(s)\,B(s)$. $B(s)$ is bicausal, and $H(s)$, the Hermite form of $T(s)$, has a "pseudo triangular" form which depends on the dimension and rank of $T(s)$. For example, if $T(s)$ is invertible, $H(s)$ is a lower-triangular matrix with some other properties that we will not detail here [23]. From a geometric point of view, such right operations are equivalent to the elimination of $\mathscr{V}^*$.

We can now conclude this section devoted to a presentation of the main internal and input-output structural informations. As shown in the next section, depending on the type of the control law to be chosen, both structures have to be known: roughly speaking, for a given problem, the internal structure is convenient for static control laws, and input-output structure is in general sufficient for dynamics laws.

4. APPLICATIONS

The advances in the knowledge of structural properties of linear systems have allowed to solve some important control problems. For some other problems where solutions were known, the structural point of view has provided simpler formulations and deeper interpretations. Let us now examine some examples.

4.1. Model Following

Let us consider two systems, one is the process to be controlled, the other one is a model of the desired behaviour. Both systems are described either by a form (1) or by a form (2).

The problem is: *can we control the process in such a way that it behaves as the model*? If the model has a transfer matrix $T(s)$ and the process a transfer matrix $E(s)$, and if the control is a precompensator $X(s)$, the problem amounts to solve the linear equation,

$$E(s)\,X(s) = T(s)\,,$$

and to look for a proper solution.

The problem can be formulated in a state space framework and we get an exact equivalence with the previous formulation if we look for solutions using a precompensator. The previous rational matrix equation is a well known problem. This equation has a proper solution if and only if:

$$[E(s) \mid T(s)] \quad \text{and} \quad E(s) \quad \text{have the same structure at infinity}\,.$$

The problem, in its state space formulation, was solved by [14] for strictly proper systems and extended to proper systems in [15], and, of course, lead to the same solution.

4.2. Decoupling

The row by row decoupling problem, or Morgan's problem, can be stated as follows:

> Given a p outputs system, is it possible to define p sets of exogeneous inputs and a corresponding control law such that each set of inputs only influences one output?

From now on, we assume that the transfer matrix of the system, $T(s)$, is full-row rank, which is a necessary condition for row by row decoupling. Moreover we will require the controlled system to be also of rank p.

If we look for a general precompensator, the solution is simple. Let $T^+(s)$ be a right inverse of $T(s)$, and k be an integer large enough for $s^{-k}\,T^+(s)$ to be proper. Then a possible solution is:

$$C(s) = s^{-k}\,T^+(s)\,.$$

A more interesting (and more difficult) problem is to look for a static or dynamic state feedback (with input transformation G invertible). The following results were obtained:

- decoupling by regular static state feedback is possible if and only if decoupling by regular dynamic state feedback is possible, [9].
- decoupling by regular state feedback is possible if and only if the infinite structure

of $T(s)$ is the union of the infinite structures of its rows, [3]. In other words no singularity at infinity may appear from relation between some rows.

Let us consider the following example.

Example 4.1.

$$T(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ \frac{1}{s} & 0 \end{bmatrix}$$

This system has two infinite zeros of order 1. Each row has one infinite zero of order 1, then the system is decouplable.

Notice that the previous decoupling condition is a simpler formulation of the decoupling condition of [6], when $T(s)$ is invertible.

The regularity assumption for the feedback law was not due to practical design considerations, but rather to mathematical simplification. As an example, the system:

$$T(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & 0 \\ \frac{1}{s} & 0 & \frac{1}{s^2} \end{bmatrix}$$

is not decouplable via a regular feedback while the simple (non-regular) input transformation:

$$G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

makes it decoupled.

When the matrix G is no longer restricted to be invertible, the compensator equivalent to the feedback $u = Fx + Gv$ is no longer a bicausal matrix. Then the feedback will not, in general, preserve the infinite structure of the system.

The decoupling conditions involve a new set of feedback invariants which are called the *essential orders*. They correspond to the minimal infinite structure which can be achieved for the decoupled system.

These invariants were defined both in the geometric and transfer matrix approaches [2]. Let $\mathscr{V}^*$ be the maximal $(A - B)$ invariant subspace contained in Ker C and $\mathscr{T}_i^*$ the maximal $(A - B)$ invariant subspace contained in Ker C_i (C_i denotes C without the ith row). Then the ith essential order is defined as:

$$n_{ie} = d(\mathscr{T}_i^*/\mathscr{V}^*)$$

A physical interpretation of these invariants is the following. Let us precise that, for right invertible systems, $d(\mathscr{V}^*) = n - \sum_1^p n_i$ where $\{n_i\}_p$ is the infinite structure of (C, A, B). This comes from the geometric definition of the integers n_i (Section 3) and the fact that right invertibility is equivalent to $\mathscr{V}^* + \mathscr{S}^* = \mathscr{X}$. So the above relation is equivalent to: $n_{ie} = \sum_1^p n_i - \sum_1^{p-1} \bar{n}_i$, where $\{\bar{n}_i\}_{p-1}$ is the infinite structure of (C_i, A, B). Through this characterization, it becomes intuitive that n_{ie} provides an information about the degree of dependency at infinity between the ith output and the other ones.

In transfer matrix terms, the ith essential order is the polynomial degree of the i-th column of the interactor of $T(s)$. Recall that the interactor of a full row rank matrix is the inverse of its Hermite form [6].

We will not detail here the general decoupling condition for static state feedback. Roughly speaking it says that the structure of $\mathscr{R}^*$, the maximal controllability subspace contained in Ker C, must be rich enough to increase the infinite structure until reaching the list of essential orders. The complete proof for a class of systems (shifted systems) can be found in [4]. The obtained condition is then some kind of Rosenbrock's Theorem (see [20]).

For dynamic state feedback, the simpler condition is that the excess of the number of inputs over the number of outputs, $(m - p)$, is greater than or equal to the column rank deficiency at infinity of the interactor matrix, [5]. In both cases the simplest infinite structure achievable for the decoupled system is the list of the essential orders.

More complicated is the decoupling with stability. It is well known [6] that, even when decoupling is possible using regular feedback, stability problems can arise, due to some fixed poles. The minimal set of fixed poles coincides with the interconnection invariant zeros of the open loop system. These zeros are the invariant zeros of (C, A, B) which are not invariant zeros of the row subsystems (c_i, A, B), (c_i denotes here the ith row of C). These zeros have been characterized in a polynomial context by Koussiouris [10] and only recently in a geometric context [11]. The result is that decoupling can be achieved with stability using regular feedback if an only if there is no unstable interconnection invariant zero, [10].

The complete solution of this problem with static state feedback usually needs non regular laws (G only monic) and is not known for the moment: of course the structure of $\mathscr{R}^*$ plays a key role in the solution of this problem. In a transfer matrix approach, only the interconnection transmission zeros can be characterized through $T(s)$. The solution of decoupling with stability using dynamic feedback can be found in [5], and for static state feedback in [12], but in this last case only when shifted systems are considered.

5. CONCLUDING REMARKS

We have shown, in this communication, that the knowledge of some structural properties of the system is crucial when trying to control it. *Internal structures* like invariant zeros, zeros at infinity. Morse's list $\mathscr{I}_2, \ldots$, play a key role in the solution of control problems with *static state feedback*, while *external structures* like transmission zeros, zeros at infinity, kernel indices, ..., are related to *dynamic solutions*. Now, this way of tackling systems within, say, a "structural approach", is not only particularly efficient, but also not limited to classical linear systems:

– extension of this structural insight has been given for *generalized* (or implicit, descriptor, singular, ...) *linear systems* (see [16]),
– for proper but *infinite dimensional* linear systems, a first attempt has been developed in [17] where zeros at infinity have been introduced (for a particular class of systems in Hilbert spaces) and Disturbance Decoupling Problem with Disturbance Measurement has been solved in a structural way,
– extensions to strictly proper *non linear* (affine) systems are also available, see [18].

Our opinion is that some common structural skeleton must exist for systems, and structural solutions to control problems must be given and written under the same form, independently of their linearity, properness or finite dimensionality.

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Dr. Christian Commault, Laboratoire d'Automatique de Grenoble, E.N.S.I.E.G. (U.R.A. 228 CNRS), B. P. 46 Saint Martin d'Hères. France.

Prof. Dr. Jean F. Lafay, Dr. Michel Malabre, Laboratoire d'Automatique de Nantes, E.N.S.M. (U.R.A. 823 CNRS), 1 rue de la Noë, 44072 Nantes-cedex. France.