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ON THE PENALTY APPROXIMATION OF QUADRATIC PROGRAMMING PROBLEM

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An upper bound for the difference of the exact solution of the problem of minimization of quadratic functional on a subspace and its penalty approximation has been given. The paper is supplied with a numerical example.

1. INTRODUCTION

It is an easy consequence of the general theory of the penalty method [1] that the solution $\bar{x}$ of the problem

$$\min \{f(x): x \in \mathscr{V}\}\,, \tag{1}$$

where $f(x) = \frac{1}{2}x^{\mathrm{T}}Ax - b^{\mathrm{T}}x$ is a convex quadratic functional and $\mathscr{V}$ is a subspace of the real Euclidean space $\mathbb{R}^n$, may be approximated by the solution $\bar{x}_\varepsilon$ of the problem

$$\min \{f(x) + \tfrac{1}{2}\varepsilon^{-1}x^{\mathrm{T}}Px:\ x \in \mathbb{R}^n\} \tag{2}$$

where $\varepsilon > 0$ and P is any $n \times n$ matrix with the range $\mathscr{V}$.

Even though there is a number of exact methods for transformation of the problem (1) to the unconstrained one [3], hardly any of them is as simple and cheap from the point of view of numerical realisation as the penalty approximation (2). In a special case when P is diagonal its application consists simply in overwriting corresponding diagonal entries of A by a large number and is widely used [2]. In a more general case, when values of constrained variables are not fully determined by constraints, the application is complicated by the fact that adding a too large matrix to A may destroy the corresponding part of A. This has motivated us to examine in detail the difference $\bar{x} - \bar{x}_\varepsilon$ and to carry out some numerical experiments in order to assess the applicability of the penalty approach.

2. AN ESTIMATE

Let $f(x) = \frac{1}{2}x^{\mathrm{T}}Ax - b^{\mathrm{T}}x$ where $A \in \mathbb{R}^{n \times n}$ is a real positive semidefinite matrix and $b \in \mathbb{R}^n$. Denote by $\mathscr{V}$ and $\mathscr{U}$ a subspace of $\mathbb{R}^n$ and its orthogonal complement.

Let $V \in \mathbb{R}^{n \times m}$ and $U \in \mathbb{R}^{n \times (n-m)}$ be two matrices whose columns are formed by the orthonormal bases of $\mathscr{V}$ and $\mathscr{U}$, respectively. Denote by P a conjugate projector on $\mathscr{V}$. Let us remind that a matrix P is a conjugate projector on a subspace $\mathscr{V}$ iff $P^2 = P$ and $P^{\mathrm{T}}A(I - P) = 0$. In this case $I - P$ is a conjugate projector on the null space of P. Notice that AP is symmetric and positive semidefinite as $P^{\mathrm{T}}A = P^{\mathrm{T}}AP$.

Theorem. Let f, U, V. $\mathscr{V}$ be defined as above, let $\bar{x}$ be the solution of the problem $\min_{\mathscr{V}} f(x)$, and let $\bar{x}_\varepsilon$ denote for each $\varepsilon > 0$ the solution of the problem $\min\,(f(x) + \frac{1}{2}\varepsilon^{-1}x^{\mathrm{T}}UU^{\mathrm{T}}x)$.

If $V^{\mathrm{T}}AV$ is invertible, then

$$|\bar{x} - \bar{x}_\varepsilon| \leqq \varepsilon(|P - VV^{\mathrm{T}}| + 1)^2\,|b|\,, \tag{3}$$

where $P = V(V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}A$ is a conjugate projector on $\mathscr{V}$.

Proof. Under the assumption of the theorem, obviously $\bar{x} = V(V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}b$ and $A + \varepsilon^{-1}UU^{\mathrm{T}}$ is positive definite.

Now notice that $Q = (V, U)$ is an $n \times n$ orthogonal matrix, so that

$$(A + \varepsilon^{-1}UU^{\mathrm{T}})^{-1} = Q(Q^{\mathrm{T}}(A + \varepsilon^{-1}UU^{\mathrm{T}})\,Q)^{-1}\,Q^{\mathrm{T}}\,.$$

As

$$Q^{\mathrm{T}}(A + \varepsilon^{-1}UU^{\mathrm{T}})\,Q = \begin{pmatrix} V^{\mathrm{T}}AV, & V^{\mathrm{T}}AU \\ U^{\mathrm{T}}AV, & U^{\mathrm{T}}AU + \varepsilon^{-1}I \end{pmatrix},$$

we can apply the formula for the inverse of 2×2 block matrix to get

$$\bar{x}_\varepsilon = (V, U)\begin{pmatrix} (V^{\mathrm{T}}AV)^{-1} + CZC^{\mathrm{T}}, & -CZ \\ -ZC^{\mathrm{T}} & Z \end{pmatrix}\begin{pmatrix} V^{\mathrm{T}} \\ U^{\mathrm{T}} \end{pmatrix} b =$$

$$= \bar{x} + VCZ^{\mathrm{T}}V^{\mathrm{T}}b - VCZU^{\mathrm{T}}b - UZC^{\mathrm{T}}V^{\mathrm{T}}b + UZU^{\mathrm{T}}b\,, \tag{4}$$

where

$$C = (V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}AU$$

and

$$Z = (U^{\mathrm{T}}AU + \varepsilon^{-1}I - U^{\mathrm{T}}AV(V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}AU)^{-1}\,.$$

Let us now estimate the norm of Z. First notice that

$$Z = \varepsilon(I - \varepsilon U^{\mathrm{T}}A(V(V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}A - I)\,U)^{-1}\,. \tag{5}$$

Since $P = V(V^{\mathrm{T}}AV)^{-1}\,V^{\mathrm{T}}A$ is a conjugate projector on $\mathscr{V}$, we can rewrite Z into the form

$$Z = \varepsilon(I + \varepsilon U^{\mathrm{T}}A(I - P)\,U)^{-1}\,,$$

where the inverted matrix is obviously positive definite with all the eigenvalues greater or equal to 1. It follows that

$$|Z| \leqq \varepsilon\,.$$

Further, notice that

$$|VC| = |V(V^TAV)^{-1}\, V^TAU| = |PU| = |P(I - VV^T)| =$$
$$= |P - PVV^T| = |P - VV^T|\,.$$

Taking into account that both $|U| \leqq 1$ and $|V| \leqq 1$, we get

$$|x - x_\varepsilon| = |VCZC^TV^Tb - VCZU^Tb - UZC^TV^Tb + UZU^Tb| \leqq$$
$$\leqq (|VC|^2 + 2|VC| + 1)\,|Z|\,|b| \leqq \varepsilon(|VC| + 1)^2\,|b| =$$
$$= \varepsilon(|P - VV^T| + 1)^2\,|b|\,.$$

Consider the following trivial example: $A = I$, $b = (1, 0\ldots)^T$, $\mathscr{V}$ is the set of all vectors with the first coordinate equal to 0. Then obviously $|b| = 1$,

$$|\bar{x} - \bar{x}_\varepsilon| = \varepsilon(1 + \varepsilon)^{-1}\,,$$

while (3) yields

$$|\bar{x} - \bar{x}_\varepsilon| \leqq \varepsilon\,.$$

This shows that the constant $(|P - VV^T| + 1)^2$ can not be improved without additional assumptions. □

The following corollary gives us an idea about the constant in (3) in terms of the spectrum of A.

Corollary. Let f, U, V, A, $\bar{x}$ and $\bar{x}_\varepsilon$ be those of the theorem, and let A be invertible. Then

$$|\bar{x} - \bar{x}_\varepsilon| \leqq \varepsilon(\varkappa(A) + 1)^2\,|b|\,, \tag{4}$$

where $\varkappa(A)$ is the spectral condition number of A.

Proof. Notice that

$$|P - VV^T| = |(P - VV^T)(UU^T + VV^T)| = |PUU^T| \leqq |P|$$

and

$$|P| = |V(V^TAV)^{-1}\, V^TA| \leqq |(V^TAV)^{-1}|\,|A| \leqq |A^{-1}|\,|A|\,.$$

Thus $|P - VV^T| \leqq \varkappa(A)$, which substituted into (3) yields (4). □

3. NUMERICAL EXAMPLE

As an illustration, consider the problem

$$\min_{x \in \mathscr{V}} \tfrac{1}{2}x^TAx - b^Tx\,,$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\mathscr{V} = \{x\colon x_1 + x_3 = 0\}\,.$$

Direct computations show that $\varkappa(A) = (1 + \sqrt{2})/(1 - \sqrt{2})$, so that we get the estimate

$$|x - x_\varepsilon| \leqq 80{\cdot}77\varepsilon .$$

The values of both sides are tabulated in Table 1. All the penalty computations were carried out in double precision and no serious effect of computer arithmetics has

Table 1.

ε	10^{-1}	10^{-4}	10^{-7}
$\|x - x_\varepsilon\|$	1·716E-1	1·837E-4	1·837E-7
$\varepsilon(1 + \varkappa(A))^2 \|b\|$	8·077E0	8·077E-3	8·077E-6

been observed with $\varepsilon^{-1} \leqq 10^{10}$. We conclude that the estimate (4) may give us some idea about the error of penalty computations, though in many cases it will be too pesimistic.

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