ON STATIONARITY OF A MULTIPLE DOUBLY STOCHASTIC MODEL*

JIŘÍ ANDĚL

A multiple linear process with random coefficients is investigated in the paper. Conditions for existence of such process are derived and its covariance function as well as the matrix of spectral densities are calculated. The results are applied to multiple AR(1) process with random coefficients, where the matrices of coefficients can be described by a stationary process. In this case conditions for existence and stationarity of the AR(1) process are given.

1. INTRODUCTION

Let ε_t be a white noise such that $\mathsf{E}\varepsilon_t = 0$, $\mathsf{E}\varepsilon_t^2 = \sigma^2$ and $\mathsf{E}\varepsilon_s\varepsilon_t = 0$ for $s \neq t$. The classical autoregressive process X_t is defined by the relation

$$X_t = \sum_{k=1}^{n} \beta_k X_{t-k} + \varepsilon_t , \tag{1.1}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)'$ are the autoregressive parameters. If

$$z^n - \beta_1 z^{n-1} - \ldots - \beta_n \neq 0 \quad \text{for} \quad |z| \geqq 1 , \tag{1.2}$$

then there exists a unique process X_t satisfying

$$X_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k} , \quad \sum_{k=0}^{\infty} c_k^2 < \infty \tag{1.3}$$

such that (1.1) holds. This process is called autoregressive and it is stationary. It is known that (1.2) is a necessary and sufficient condition under which there exists a process (1.3) such that (1.1) holds.

A process X_t can be defined by a more general relation

$$X_t = \sum_{k=1}^{n} b_k(t) X_{t-k} + \varepsilon_t$$

* Presented at the "Kolloquium über Mathematische Statistik im Rahmen der Wissenschaftlichen Kolloquien der Universität Hamburg und der Karls-Universität Prag", Hamburg, June 1989.

where

$$\mathbf{b}(t) = [b_1(t), \dots, b_n(t)]'$$

are random vectors such that

$$\mathsf{E}\mathbf{b}(t) = \boldsymbol{\beta}, \quad \operatorname{var} \mathbf{b}(t) = \boldsymbol{\Delta} .$$

The case when $\mathbf{b}(t)$ are independent is considerably simpler. Anděl [1] derived a condition under which X_t is stationary. Nicholls and Quinn [5] proposed a method for estimating parameters and in [6] generalized Anděl's result to multiple autoregressive models. They summarized their work in [7]. Anděl [2] analyzed a model with nonvanishing mean.

If $\mathbf{b}(t)$ are not independent, the conditions for stationarity are rather complicated. Koubková [4] investigated an AR(1) process with a random coefficient $b_1(t)$ such that $b_1(t)$ is a MA(1) process. She proved that X_t is stationary only if some complicated relations among the moments are satisfied. Tjøstheim [9] derived some conditions under which an AR(1) process with random and dependent coefficients is strictly stationary. His conditions do not guarantee the existence of any moments of X_t. Pourahmadi [8] presents conditions for stationarity and derives explicit results for the following cases: (i) log $b_1^2(t)$ is a stationary Gaussian process; (ii) log $b_1^2(t)$ is an AR (1) process; (iii) log $b_1^2(t)$ is a MA(q) process.

In our paper we generalize some of Pourahmadi's results to the multiple AR(1) process.

2. MULTIPLE LINEAR PROCESS WITH RANDOM COEFFICIENTS

Let ε_n be a p-dimensional white noise with $\mathsf{E}\varepsilon_n = \mathbf{0}$, $\operatorname{var} \varepsilon_n = \mathbf{V}$. Let $\mathbf{B}_n$ be a sequence of $p \times p$ random matrices, the elements of which have finite second moments. Assume that $\{\mathbf{B}_n\}$ and $\{\varepsilon_n\}$ are independent. Define

$$\mathbf{X}_t = \sum_{n=-\infty}^{\infty} \mathbf{B}_n \varepsilon_{t-n} , \tag{2.1}$$

if the series converges in the quadratic mean. The process $\{\mathbf{X}_t\}$ can be considered as a generalization of the linear process. Denote

$$\mathbf{W}_n = \mathsf{E}\mathbf{B}_n'\mathbf{B}_n$$

and

$$\mathbf{B}_n = \begin{pmatrix} \mathbf{b}_{1,n} \\ \dots \\ \mathbf{b}_{p,n} \end{pmatrix},$$

where $\mathbf{b}_{i,n}$ are row random vectors.

Lemma 2.1. The series (2.1) converges in the quadratic mean if and only if

$$\operatorname{Tr}\mathbf{V} \sum_{n=-\infty}^{\infty} \mathbf{W}_n < \infty . \tag{2.2}$$

Proof. For a fixed t and a given $N \geqq 0$ define

$$\mathbf{S}_N = \sum_{j=0}^{N} \mathbf{B}_j \varepsilon_{t-j} .$$

For $m \geqq 1$ we have

$$\mathsf{E}(\mathbf{S}_{N+m} - \mathbf{S}_N)' (\mathbf{S}_{N+m} - \mathbf{S}_N) = \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} \operatorname{Tr} \mathsf{E} \varepsilon_{t-k} \varepsilon'_{t-j} \mathbf{B}'_j \mathbf{B}_k =$$

$$= \sum_{k=N+1}^{N+m} \operatorname{Tr} \mathbf{V} \mathbf{W}_k = \operatorname{Tr} \mathbf{V} \sum_{k=N+1}^{N+m} \mathbf{W}_k .$$

Thus $\mathbf{S}_N$ is a Cauchy sequence in the quadratic mean if and only if

$$\operatorname{Tr} \mathbf{V} \sum_{n=0}^{\infty} \mathbf{W}_n < \infty .$$

The convergence of

$$\mathbf{s}_N = \sum_{j=-N}^{-1} \mathbf{B}_j \varepsilon_{t-j}$$

can be treated analogously. □

Lemma 2.2. Let the condition (2.2) be fulfilled. Then $\mathbf{X}_t$ is a stationary process with vanishing mean. Let $\gamma_{ij}^{(s)}$ be the (i, j)th element of the covariance function

$$\gamma^{(s)} = \mathsf{E}\, \mathbf{X}_{t+s} \mathbf{X}'_t .$$

Then

$$\gamma_{ij}^{(s)} = \operatorname{Tr} \mathbf{V} \sum_{k=-\infty}^{\infty} \mathsf{E} \mathbf{b}'_{j,k} \mathbf{b}_{i,k+s} .$$

Proof. The relation $\mathsf{E}\mathbf{X}_t = \mathbf{0}$ follows from (2.1) and from Lemma 2.1. Further we have

$$\mathsf{E}\mathbf{X}_{t+s}\mathbf{X}'_t = \mathsf{E} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{B}_j \varepsilon_{t+s-j} \varepsilon'_{t-k} \mathbf{B}'_k =$$

$$= \mathsf{E} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{B}_{j+s} \varepsilon_{t-j} \varepsilon'_{t-k} \mathbf{B}'_k = \mathsf{E} \sum_{k=-\infty}^{\infty} \mathbf{B}_{k+s} \mathbf{V} \mathbf{B}'_k .$$

The (i, j)th element of the last expression is

$$\mathsf{E} \sum_{k=-\infty}^{\infty} \mathbf{b}_{i,k+s} \mathbf{V} \mathbf{b}'_{j,k} = \operatorname{Tr} \mathbf{V} \sum_{k=-\infty}^{\infty} \mathsf{E} \mathbf{b}'_{j,k} \mathbf{b}_{i,k+s} .$$

It can be checked also directly that (2.2) implies $\gamma_{ii}^{(0)} < \infty$ for all i. From here it follows that $\gamma_{ij}^{(s)}$ exist and are finite. □

Lemma 2.3. Let the condition (2.2) be fulfilled and let

$$\sum_{s=-\infty}^{\infty} |\gamma_{ij}^{(s)}| < \infty , \quad i, j = 1, \ldots, p . \tag{2.3}$$

Then there exists the matrix $\boldsymbol{f}(\lambda) = (f_{uv}(\lambda))$ of spectral densities of the process $\boldsymbol{X}_t$ and

$$f_{uv}(\lambda) = (2\pi)^{-1} \operatorname{Tr} \boldsymbol{V} \mathsf{E}\Big(\sum_{n=-\infty}^{\infty} \boldsymbol{b}'_{v,n} \mathrm{e}^{in\lambda}\Big)\Big(\sum_{n=-\infty}^{\infty} \boldsymbol{b}_{u,n} \mathrm{e}^{-in\lambda}\Big) .$$

Proof. It is known (see Brillinger [3], Theorem 2.5.1) that a stationary process $\boldsymbol{X}_t$ satisfying (2.3) possesses the matrix of spectral densities $\boldsymbol{f}(\lambda)$ given by the formula

$$\boldsymbol{f}(\lambda) = (2\pi)^{-1} \sum_{s=-\infty}^{\infty} \gamma^{(s)} \mathrm{e}^{-is\lambda} .$$

Inserting for $\gamma^{(s)}$ we get after some computations the assertion of Lemma 2.3. □

For applications it is necessary to generalize the model (2.1) in such a way that the matrices $\boldsymbol{B}_n$ are allowed to depend also on t.

Theorem 2.4. Let $\boldsymbol{B}_{n,t}$ be $p \times p$ random matrices, the elements of which have finite second moments. Let $\{\boldsymbol{B}_{n,t}\}$ and $\{\varepsilon_t\}$ be independent. Define

$$\boldsymbol{X}_t = \sum_{n=-\infty}^{\infty} \boldsymbol{B}_{n,t} \varepsilon_{t-n} , \tag{2.4}$$

$$\boldsymbol{W}_{n,t} = \mathsf{E}\boldsymbol{B}'_{n,t}\boldsymbol{B}_{n,t} , \quad \boldsymbol{B}_{n,t} = \begin{pmatrix} \boldsymbol{b}^n_{1,t} \\ \cdots \\ \boldsymbol{b}^n_{p,t} \end{pmatrix} ,$$

where $\boldsymbol{b}^n_{i,t}$ are row random vectors. Then the series (2.4) converges in the quadratic mean if and only if

$$\operatorname{Tr} \boldsymbol{V} \sum_{n=-\infty}^{\infty} \boldsymbol{W}_{n,t} < \infty . \tag{2.5}$$

If (2.5) holds, then $\mathsf{E}\boldsymbol{X}_t = \boldsymbol{0}$ and the element $\gamma^{(s)}_{ij}$ of the matrix

$$\gamma^{(s)} = \mathsf{E}\boldsymbol{X}_{t+s}\boldsymbol{X}'_t$$

is given by

$$\gamma^{(s)}_{ij} = \operatorname{Tr} \boldsymbol{V} \sum_{n=-\infty}^{\infty} \mathsf{E}\boldsymbol{b}^{t\prime}_{jn}\boldsymbol{b}^{t+s}_{i,n+s} . \tag{2.6}$$

Proof. The first assertion follows from Lemma 2.1. Formula (2.6) can be proved in the same way as Lemma 2.2. Of course, $\gamma^{(s)}_{ij}$ as well as $\gamma^{(s)}$ generally may depend on t. □

Notice, however, that the assumptions of Lemma 2.4 do not guarantee the stationarity of the process $\boldsymbol{X}_t$. It is clear that $\boldsymbol{X}_t$ given by (2.4) is stationary if and only if (2.5) holds for all t and $\gamma^{(s)}_{ij}$ in formula (2.6) does not depend on t. This must be verified in special models separately.

3. MULTIPLE AR(1) PROCESS WITH RANDOM COEFFICIENTS

Consider a random process $\mathbf{X}_t$ generated by

$$\mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \varepsilon_t , \tag{3.1}$$

where $\mathbf{A}_t$ are $p \times p$ random matrices and ε_t is a p-dimensional white noise. Formula (3.1) can be rewritten into the form

$$\mathbf{X}_t = \sum_{n=0}^{\infty} \mathbf{B}_{n,t} \varepsilon_{t-n} \tag{3.2}$$

where

$$\mathbf{B}_{0,t} = \mathbf{I} , \quad \mathbf{B}_{n,t} = \prod_{i=0}^{n-1} \mathbf{A}_{t-i} \quad \text{for} \quad n \geqq 1 .$$

The sequence $\mathbf{A}_t$ is called strictly stationary, if for every integer $N \geqq 1$ and for arbitrary integers $t_1, \ldots . t_N$ the joint distribution of $(\mathbf{A}_{t_1+h}, \ldots \mathbf{A}_{t_N+h})$ does not depend on h.

Theorem 3.1. The relation (3.1) has a solution of the type (3.2) if and only if

$$\operatorname{Tr} \mathbf{V} \sum_{n=0}^{\infty} \mathsf{E}\Big(\prod_{i=0}^{n-1} \mathbf{A}_{t-i}\Big)' \Big(\prod_{j=0}^{n-1} \mathbf{A}_{t-j}\Big) < \infty$$

for all integers t.

Proof. Theorem 3.1 follows from Theorem 2.4. □

Theorem 3.2. If $\mathbf{A}_t$ is strictly stationary, then (3.1) has a stationary solution of the type (3.2) if and only if

$$\operatorname{Tr} \mathbf{V} \sum_{n=1}^{\infty} \mathsf{E} \mathbf{A}_1' \mathbf{A}_2' \ldots \mathbf{A}_n' \mathbf{A}_n \ldots \mathbf{A}_2 \mathbf{A}_1 < \infty . \tag{3.4}$$

Proof. We have from (3.3) that

$$\operatorname{Tr} \mathbf{V} \sum_{n=1}^{\infty} \mathsf{E} \mathbf{A}_{t-n+1}' \mathbf{A}_{t-n+2}' \ldots \mathbf{A}_t' \mathbf{A}_t \ldots \mathbf{A}_{t-n+2} \mathbf{A}_{t-n+1} < \infty \tag{3.5}$$

must hold for all integers t. Since $\mathbf{A}_t$ is strictly stationary, the left-hand side of (3.5) does not change when we subtract $t - n$ from each index. The proof, that (2.6) does not depend on t, is similar. □

Generally, it is extremely difficult to verify if the condition (3.4) is fulfilled. Only some special cases allow to write explicit solution.

Theorem 3.3. Let $\|\mathbf{A}\|$ be a norm of a matrix $\mathbf{A}$ satisfying $\|\mathbf{AB}\| \leqq \|\mathbf{A}\| . \|\mathbf{B}\|$. If $\|\mathbf{A}_t\| \leqq d < 1$ for all t, then (3.3) and (3.4) hold.

Proof is clear. □

(Received December 8, 1989.)

REFERENCES

[1] J. Anděl: Autoregressive series with random parameters. Math. Operationsforsch. Statist., Ser. Statistics *7* (1976), 735—741.

[2] J. Anděl: On autoregressive models with random parameters. In: Proc. 3rd Prague Symp. on Asymptotic Statistics (P. Mandl, M. Hušková, eds.), Elsevier, Amsterdam 1984, pp. 17—30.

[3] D. R. Brillinger: Time Series: Data Analysis and Theory. Holt, Rinehart and Winston, Inc., New York 1975.

[4] A. Koubková: First-order autoregressive processes with time-dependent random parameters. Kybernetika *18* (1982), 408—414.

[5] D. E. Nicholls and B. G. Quinn: The estimation of random coefficient autoregressive models. J. Time Ser. Anal. *1* (1980), 37—46.

[6] D. E. Nicholls and B. G. Quinn: Multiple autoregressive models with random coefficients. J. Multivariate Anal. *11* (1981), 185—198.

[7] D. E. Nicholls and B. G. Quinn: Random Coefficient Autoregressive Models: An Introduction. (Lecture Notes in Statistics 11.) Springer-Verlag, Berlin—Heidelberg—New York 1982.

[8] M. Pourahmadi: On stationarity of the solution of a doubly stochastic model. J. Time Ser. Anal. *7* (1986), 123—131.

[9] D. Tjøstheim: Some doubly stochastic time series models. J. Time Ser. Anal. *7* (1986), 51—72.

Prof. RNDr. Jiří Anděl, DrSc., Univerzita Karlova, matematicko-fyzikální fakulta (Charles University — Faculty of Mathematics and Physics), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.