KYBERNETIKA — VOLUME 27 (1991), NUMBER 1

FUNDAMENTAL THEOREM OF STATE FEEDBACK: THE CASE OF INFINITE POLES

PETR ZAGALAK, VLADIMÍR KUČERA

The limits of state feedback in altering the dynamics of a linear system at infinity are studied. A necessary and sufficient condition is established for a list of integers to make the infinite pole structure of a polynomial system obtained by state feedback from the given system. The condition consists of inequalities which involve the infinite zeros of the system, its controllability indices and, of course, the list of integers. A procedure is given for the calculation of a feedback gain which achieves the desired dynamics.

1. INTRODUCTION

The result concerning the limits of linear state feedback in altering the dynamics of linear systems is known as the fundamental theorem of state feedback. Rosenbrock ([8], Chap. 5, Thm. 4.2) was the first to prove such a result for *ordinary state-space systems*

$$\dot{x} = Fx + Gu \tag{1}$$

where F and G are respectively $n \times n$ and $n \times m$ matrices with entries in $\mathbb{R}$, the field of real numbers. These systems have only finite poles and their dynamics are fully described by the invariant polynomials of $sI_n - F$.

The result is as follows. Suppose that (1) is controllable with controllability indices $n_1, n_2, \ldots, n_m$. Then there exists a state feedback $u = Kx + v$ such that the dynamics of $\dot{x} = (F + GK)x + Gv$ are given by $c_1(s), c_2(s), \ldots, c_m(s)$, a list of polynomials over $\mathbb{R}$ having the total degree n and the property that $c_i(s)$ divides $c_{i-1}(s)$, if and only if the set of inequalities

$$\sum_{i=1}^{j} \deg c_i(s) \geqq \sum_{i=1}^{j} n_i \tag{2}$$

hold for $j = 1, 2, \ldots, m$. This shows that the (finite) poles of (1) can be shifted by state feedback to any (finite) positions but the structure of repeated poles is limited by (2).

This result was extended by Kučera and Zagalak ([4], Thm. 2) to *generalized state-space systems*

$$E\dot{x} = Fx + Gu \tag{3}$$

where E is an $n \times n$ matrix of rank r having entries in $\mathbb{R}$. Only regular systems are considered for which $sE - F$ is invertible. We call the system (3) *proper* and *polynomial* if the rational matrix $(sE - F)^{-1}$ is proper and polynomial, respectively. The system (3) can have both finite and infinite poles; the proper systems have no infinite poles while the polynomial systems have no finite poles. Thus the extension of the fundamental theorem (2) consists in allowing the given system to have infinite poles.

The result is as follows. Suppose that (3) is controllable with controllability indices $n_1, n_2, \ldots, n_m$. Then there exists a state feedback $u = Kx + v$ such that the system $E\dot{x} = (F + GK)x + Gv$ is regular, proper, and its dynamics are given by $c_1(s), c_2(s), \ldots, c_m(s)$, a list of polynomials over $\mathbb{R}$ having the total degree r and the property that $c_i(s)$ divides $c_{i-1}(s)$, if and only if the set of inequalities

$$\sum_{i=1}^{j} \deg c_i(s) \geqq \sum_{i=1}^{j} n_i \tag{4}$$

hold for $j = 1, 2, \ldots, m$. This shows that the poles of (3), no matter whether finite or infinite, can be shifted by state feedback to any finite positions while the structure of repeated poles is limited by (4).

The result just described is one extreme. The other extreme is where state feedback is applied to the generalized state-space system (3) so as to shift its finite poles to infinity and restructure the infinite pole.

The infinite pole calls for a special treatment. This part of the system dynamics cannot be described by the invariant polynomials of $sE - F$. Of course, it is a routine matter to apply to $(sE - F)^{-1}$ a bilinear transformation

$$w = \frac{as + b}{cs + d}, \quad c \neq 0$$

which sends the point $s = \infty$ to $w = a/c$ and study the infinite pole in s as a finite one in w. The new system, however, will have a zero at $w = a/c$ whenever the original system has a zero at $s = \infty$. And while the existence of finite zeros of (3) is ruled out by the assumption of controllability, there may be an infinite zero. The fact that poles are to be shifted to a position in which the system has a zero is not allowed for in the existing fundamental theorem. So (4) does not apply in this situation.

We shall extend the fundamental theorem so as to accommodate the effect of zeros. It turns out that the presence of zeros further limits the structure of the pole to be shifted under the zero, in addition to the constraint imposed by the set of inequalities (4).

Related to our results are various pole placement theorems. Wonham [10] showed that the (finite) poles of system (1) can be shifted by state feedback to any (finite) positions if and only if the system is controllable. Cobb [1] proved that the infinite poles of system (3) can be shifted by state feedback to any finite position if and only if the system is impulse controllable. And recently Fahmy and O'Reilly [2] studied the other extreme where state feedback is used to move all finite poles of system (3) to infinity.

These pole placement theorems can be considered special cases of the fundamental theorem of state feedback. They address just the aspect of positioning but say nothing about the achievable structure of repeated poles. Further insight is provided by Loiseau [7].

2. BASIC CONCEPTS

2.1. Matrix Fraction Description

Any $p \times q$ polynomial matrix $P(s)$ can be written either as

$$P(s) = \operatorname{diag}\left[s^{d_{r1}}, \ldots, s^{d_{rp}}\right] P_{hr} + \text{terms of lower degree in } s$$

or as

$$P(s) = P_{hc}\left[\operatorname{diag} s^{d_{c1}}, \ldots, s^{d_{cq}}\right] + \text{terms of lower degree in } s$$

where d_{ri} is the degree of row i of $P(s)$ and P_{hr} is its highest row-degree coefficient matrix while d_{ci} is the degree of column i of $P(s)$ and P_{hc} is its highest column-degree coefficient matrix. We say that $P(s)$ is *row proper* if P_{hr} has rank p and *column proper* if P_{hc} has rank q. If the rows of $P(s)$ are arranged so that $d_{ri} \geqq d_{rj}$ for $i < j$ then $P(s)$ is *row degree ordered* while if the columns of $P(s)$ are arranged so that $d_{ci} \geqq d_{cj}$ for $i < j$ then $P(s)$ is *column degree ordered.* Finally, $P(s)$ is said to be *irreducible* if its Smith form is I_p when $p = q$, $[I_p\ 0]$ when $p < q$, and $\begin{bmatrix} I_q \\ 0 \end{bmatrix}$ when $p > q$. Now consider a generalized state-space system over $\mathbb{R}$ governed by the equation (3),

$$E\dot{x} = Fx + Gu\,,$$

where E is an $n \times n$ matrix of rank r, F is $n \times n$ and G is $n \times m$. We say that (3) is *regular* if $sE - F$ is invertible and define its transfer function by

$$T(s) = (sE - F)^{-1}\, G\,.$$

Polynomial matrices $A(s)$, $B(s)$ such that

(a) $\quad (sE - F)^{-1}\, G = B(s)\, A^{-1}(s)\,;$ $\qquad$ (5)

(b) $\quad \begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$ is irreducible, column proper, and column degree ordered;

are said to form a right *standard matrix fraction description* (MFD) of the regular system (3).

2.2. Controllability

The problem of pole assignment by state feedback in linear systems is closely related to the notion of controllability [6, 9]. We say that a regular system (3) is *controllable* if $[sE - F \;\; G]$ is irreducible and has full rank for all complex s including $s = \infty$.

For use in our development, however, we need a more detailed structure of system (3) with respect to controllability. Let $A(s)$, $B(s)$ be a right standard MFD of a regular system (3) and for $i = 1, 2, \ldots, m$ let n_i denote the degree of column i of $\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$. Then the integers $n_1, n_2, \ldots, n_m$ are called the *controllability indices* of (3). It is shown in Kučera and Zagalak ([4], Thm. 1) that a regular system (3) is controllable if and only if

$$\sum_{i=1}^{m} n_i = r \,. \tag{6}$$

2.3. Poles and Zeros

The *poles* of a regular system (3) are the zeros of $sE - F$. The structure of the finite poles is given by the invariant polynomials of $sE - F$, which appear in the Smith form of $sE - F$ over the ring of polynomials in s. The structure of the infinite pole is given by an ordered list of integers $p_1 \geqq p_2 \ldots \geqq p_k$ which appear as the negative powers of s in the Smith-McMillan form of $sE - F$ over the ring of proper rational functions in s.

The invariant *zeros* of a regular system (3) are the zeros of

$$\begin{bmatrix} sE - F & G \\ I_n & 0 \end{bmatrix}. \tag{7}$$

The structure of the finite zeros is given by the invariant polynomials of (7), which appear in the Smith form of (7) over the ring of polynomials in s. The structure of the infinite zero is given by an ordered list of integers $z_1 \geqq z_2 \geqq \ldots \geqq z_q$ which appear as the *negative* powers of s in the Smith-McMillan form of (7) over the ring of proper rational functions in s.

Let (3) be a regular system and $T(s)$ its transfer function. Then if (3) is controllable, its poles and zeros can be obtained as poles and zeros of the rational matrix $T(s)$. In particular, a controllable system (3) has no finite zeros. According to Hautus and Heymann [3] this property is characteristic for the transfer functions which map the input of the system to its *state*.

3. PROBLEM STATEMENT

Consider a regular generalized state-space system governed by the equation (3),

$$E\dot{x} = Fx + Gu \,,$$

where E is an $n \times n$ matrix of rank r, F is an $n \times n$ matrix and G is an $n \times m$ matrix, all with entries in $\mathbb{R}$. To avoid trivia, we shall assume that $E \neq 0$ and G has rank m.

Let $p_1 \geqq p_2 \geqq \ldots \geqq p_k$ be an ordered list of positive integers whose sum is r. The problem considered in the paper can be stated as follows. Does there exist a state feedback

$$u = Kx + v \tag{8}$$

where K is an $m \times n$ matrix with entries in $\mathbb{R}$, such that the system

$$E\dot{x} = (F + GK)\,x + Gv \tag{9}$$

is regular, polynomial, and its infinite pole structure is $p_1, p_2, \ldots, p_k$? If so, give conditions for existence and a procedure to calculate K.

The motivation for the problem is the investigation of the limits of state feedback (8) in shifting all poles of (3) to infinity and structuring them at will. This will in fact characterize the dynamics of all polynomial systems that can be obtained by state feedback from a given regular generalized state-space system.

4. PRELIMINARY LEMMAS

The following lemmas will be needed to prove the fundamental theorem. At the same time, they seem to be of independent interest.

Lemma 1. Let $A(s)$, $B(s)$ be a standard right MFD of a controllable system (3). Then, for any $m \times n$ constant matrix K such that either of the matrices $A(s) - KB(s)$ or $sE - F - GK$ is nonsingular, the other matrix is also nonsingular and both have the same structure of their infinite zeros.

Proof. The proof follows from a result on finite zeros by Kučera and Zagalak ([4], Lemma 1) on putting $s = 1/w$ thereby sending the point $s = \infty$ to $w = 0$. $\square$

Lemma 2. Let $C(s)$ be a column proper, polynomial $m \times m$ matrix with column degrees d_{ci}, $i = 1, 2, \ldots, m$. Suppose that $d_{c\alpha} > d_{c\beta}$ for some α and β. Then there exist unimodular matrices $U_1(s)$ and $U_2(s)$ such that $\bar{C}(s) = U_1(s)\,C(s)\,U_2(s)$ is column proper with column degrees

$$\bar{d}_{ci} = d_{ci}\,, \quad i \neq \alpha, \beta$$

$$\bar{d}_{c\alpha} = d_{c\alpha} - 1$$

$$\bar{d}_{c\beta} = d_{c\beta} + 1\,.$$

Proof. See Rosenbrock ([8], Chap. 5, Lemma 1) or Kučera and Zagalak ([4], Lemma 2). $\square$

Lemma 3. Let $A(s)$, $B(s)$ and $C(s)$ be $m \times m$, $n \times m$ and $m \times m$ matrices whose entries are polynomials with coefficients in $\mathbb{R}$. Then the equation

$$X A(s) + Y B(s) = C(s)$$

has a constant solution pair X, Y with entries in $\mathbb{R}$ such that X is nonsingular if and only if the rows of the matrices

$$\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}, \quad \begin{bmatrix} B(s) \\ C(s) \end{bmatrix}$$

span the same $\mathbb{R}$-linear space.

Proof. See Kučera and Zagalak ([5], Thm. 2). □

5. FUNDAMENTAL THEOREM

The main result is the extension of the fundamental theorem of state feedback to the case of infinite poles.

Theorem 1. Let (3) be a regular controllable system, $n_1 \geqq n_2 \geqq \ldots \geqq n_m$ the list of its controllability indices, and $z_1 \geqq z_2 \geqq \ldots \geqq z_q$ the structure of its infinite zero. Let $p_1 \geqq p_2 \geqq \ldots \geqq p_k$ be a list of positive integers such that

$$\sum_{i=1}^{k} p_i = r \,. \tag{10}$$

Then there exists a constant matrix K such that the system (9) is regular, polynomial, and its infinite pole structure is $p_1, p_2, \ldots, p_k$ if and only if

$$k \leqq m - q \tag{11}$$

and

$$\sum_{i=1}^{j} p_i \geqq \sum_{i=1}^{j} n_i \,, \quad j = 1, 2, \ldots, m \tag{12}$$

where $p_i = 0$ for $i > k$.

Proof. We shall first prove the necessity of (11) and (12). To this end, we suppose that a state feedback (8) exists such that the system (9) is regular, polynomial, and its infinite pole structure is $p_1, p_2, \ldots, p_k$.

Inasmuch as

$$\begin{bmatrix} sE - F - GK & G \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} sE - F & G \\ I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -K & I_m \end{bmatrix} \tag{13}$$

the zeros of (3) are unaffected by state feedback (8). In particular, the infinite zero structure of (9) is $z_1, z_2, \ldots, z_q$.

We further deduce from (13) that the controllability of (3) is not affected by state feedback (8), either. Hence the transfer function of (9), when put to the Smith-McMillan form over the ring of proper rational functions, features the diagonal entries

$$s^{p_1}, \ldots, s^{p_k}, 1, \ldots, 1, s^{-z_q}, \ldots, s^{-z_1} \,.$$

Since there are at most m elements in the list, it follows that $k + q \leqq m$. This proves (11).

Now let $A(s)$, $B(s)$ be a right standard MFD for the system (3) and consider the $m \times m$ polynomial matrix $A(s) - K\,B(s)$. By Lemma 1, $A(s) - K\,B(s)$ has no finite zeros and the structure of its infinite zero is given by $p_1, p_2, \ldots, p_k$, the infinite pole structure of (9). The column degrees of $A(s) - K\,B(s)$ are those of $\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$ and hence equal to $n_1, n_2, \ldots, n_m$, the controllability indices of (3).

Consider the matrix

$$[A(w^{-1}) - K\,B(w^{-1})]\ \mathrm{diag}\,[w^{n_1}, \ldots, w^{n_m}] \tag{14}$$

which is a polynomial matrix in $w = 1/s$. Obviously, it has column degrees n_i, $i = 1, 2, \ldots, m$ and invariant polynomials w^{p_i}, $i = 1, 2, \ldots, m$ where $p_i = 0$ for $i > k$.

The product $w^{p_{j+1}} \ldots w^{p_m}$ is the greatest common divisor of all minors of order $m - j$ in (14). It follows that

$$\sum_{i=j+1}^{m} p_i \leqq \sum_{i=j+1}^{m} n_i\,, \quad j = 0, 1, \ldots, m-1\,. \tag{15}$$

By (6) and (10),

$$\sum_{i=1}^{m} p_i = r = \sum_{i=1}^{m} n_i. \tag{16}$$

Therefore equality holds in (15) when $j = 0$ and the inequalities can be reordered to give (12).

Sufficiency will be proved by construction. Let $p_1 \geqq p_2 \geqq \ldots \geqq p_k$ where $k \leqq m - q$ be positive integers that satisfy (11) and (12). We are going to construct an $m \times n$ constant matrix K such that the system (9) is regular, polynomial, and its infinite pole structure is given by $p_1, p_2, \ldots, p_k$.

First let $A(s)$, $B(s)$ be a right standard MFD of (3) and define the polynomial matrices $A'(w)$, $B'(w)$ by

$$\begin{bmatrix} A'(w) \\ B'(w) \end{bmatrix} = \begin{bmatrix} A(w^{-1}) \\ B(w^{-1}) \end{bmatrix} \mathrm{diag}\,[w^{n_1}, \ldots, w^{n_m}]\,. \tag{17}$$

In view of (5),

$$(w^{-1}E - F)^{-1}\,G = B'(w)\,A'^{-1}(w)\,.$$

Obviously the matrix (17) is irreducible, column proper and column degree ordered with column degrees $n_1, n_2, \ldots, n_m$. Next we form the $m \times m$ matrix

$$\hat{C}(w) = \mathrm{diag}\,[w^{p_1}, \ldots, w^{p_m}]$$

where $p_i = 0$ for $i > k$. If $p_i = n_i$ for all $i = 1, 2, \ldots, m$ we put $\bar{C}(w) = \hat{C}(w)$. If there is a column α such that $p_\alpha > n_\alpha$ there must be a column β such that $p_\beta < n_\beta$, for (16) holds. Moreover, the inequalities (12) imply that $p_\alpha > p_\beta$. Then Lemma 2 can be applied, several times if necessary, to bring $\hat{C}(w)$ to a matrix $\bar{C}(w)$ which is column proper with column degrees $n_1, n_2, \ldots, n_m$ and whose invariant polynomials remain to be $w^{p_1}, w^{p_2}, \ldots, w^{p_m}$. If $\bar{C}(w)$ is such that the matrix $\begin{bmatrix} B'(w) \\ \bar{C}(w) \end{bmatrix}$ is irreducible, we put $C'(w) = \bar{C}(w)$. If not, there is a zero at $w = 0$ common to $B'(w)$ and $\bar{C}(w)$. It follows, possibly after constant column operations, that the constant matrix $\begin{bmatrix} B'(0) \\ \bar{C}(0) \end{bmatrix}$ has a zero column. The unimodular transformations implied by Lemma 2 do not change the order of columns, so the last $m - k$ columns of $\bar{C}(0)$ are $\mathbb{R}$-linearly independent. Hence the zero columns are contained in the first k columns of $\begin{bmatrix} B'(0) \\ \bar{C}(0) \end{bmatrix}$ and their total number, say l, cannot exceed $q \leqq m - k$. We denote $\gamma_1, \gamma_2, \ldots, \gamma_l$ the positions of these zero columns and add column $m - i + 1$ of $\bar{C}(w)$ to column γ_i, $i = 1, 2, \ldots, l$ in order to bring $\bar{C}(w)$ to a matrix $C'(w)$ which complements $B'(w)$ to an irreducible matrix and whose column degrees and invariant polynomials remain unchanged.

Further we consider the equation

$$X A'(w) + Y B'(w) = C'(w) . \tag{18}$$

Since the matrices

$$\begin{bmatrix} A'(w) \\ B'(w) \end{bmatrix}, \quad \begin{bmatrix} B'(w) \\ C'(w) \end{bmatrix}$$

are both irreducible and column proper with column degrees equal to $n_1, n_2, \ldots, n_m$, the rows of these matrices span the same $\mathbb{R}$-linear spaces. By Lemma 3, the equation (18) has a constant solution pair X, Y such that X is nonsingular.

Using (17) with $s = 1/w$ in equation (18) we obtain the equation

$$X A(s) + Y B(s) = C(s)$$

where

$$C(s) = C'(s^{-1}) \operatorname{diag}\left[s^{n_1}, \ldots, s^{n_m}\right] .$$

Then

$$K = -X^{-1} Y$$

is a state feedback gain which makes the system (9) regular and polynomial with the infinite pole structure $p_1, p_2, \ldots, p_k$. This is an immediate consequence of Lemma 1.

□

6. CONSTRUCTION

The sufficiency part of the proof of Theorem 1 provides a construction of K which achieves the desired dynamics at infinity. The major steps of the procedure are summarized below.

Given: E, F, G and $p_1, p_2, \ldots, p_k$

Find: K

Step 1: Calculate polynomial matrices $A'(w)$, $B'(w)$ such that

$$(E - wF)^{-1}\, wG = B'(w)\, A'^{-1}(w)$$

and $\begin{bmatrix} A'(w) \\ B'(w) \end{bmatrix}$ is irreducible, column proper and column degree ordered.

Step 2: Read out $n_1, n_2, \ldots, n_m$, the column degrees of $\begin{bmatrix} A'(w) \\ B'(w) \end{bmatrix}$ and q, the defect of $B'(0)$.

Step 3: Check for the existence of K using (11) and (12).

Step 4: Construct a nonsingular polynomial matrix $C'(w)$ having invariant polynomials $w^{p_1}, w^{p_2}, \ldots, w^{p_k}$ and column degrees $n_1, n_2, \ldots, n_m$ and making $\begin{bmatrix} B'(w) \\ C'(w) \end{bmatrix}$ irreducible and column proper.

Step 5: Solve the equation

$$XA'(w) + YB'(w) = C'(w)$$

for a constant solution pair X, Y such that X is nonsingular.

Step 6: Put $K = -X^{-1}Y$.

It is to be noted that the feedback gain K furnished by this procedure is by no means the only one to achieve the desired dynamics.

7. EXAMPLES

Two examples are included which show how to convert an integrator into a differentiator and how to split a chain of differentiators.

Example 1. Given a proper system (3) by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{19}$$

can one make it into a polynomial system (with $p_1 = 1$) by state feedback (8)?

Step 1 gives

$$A'(w) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B'(w) = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$n_1 = 1, \quad n_2 = 0 \quad \text{and} \quad q = 1$$

in *Step* 2. *Step* 3 verifies (11) and (12).

An obvious candidate for $C'(w)$ is

$$\begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}$$

which meets all the requirements listed in *Step* 4 but the irreducibility. Adding the second column to the first gives

$$C'(w) = \begin{bmatrix} w & 0 \\ 1 & 1 \end{bmatrix}.$$

Step 5 furnishes

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

so that *Step* 6 yields a desired gain

$$K = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

Note the importance of the scalor which accompanies the pure integrator in (19).

Example 2. Given a polynomial system (3) by

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

can we split the two differentiators by state feedback (8)?

Step 1:

$$A'(w) = \begin{bmatrix} w & 0 & 0 \\ -1 & w & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B'(w) = \begin{bmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: $n_1 = 1$, $n_2 = 1$, $n_3 = 0$ and $q = 1$

Step 3: $p_1 = 1$, $p_2 = 1$

Step 4:

$$C'(w) = \begin{bmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 5:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 6:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The input is differentiated at most once in the resulting system.

8. CONCLUSIONS

The dynamics of all *polynomial* regular systems obtainable by applying state feedback to a given regular controllable system have been characterized in Theorem 1. It is called a fundamental theorem of state feedback.

This result is one extreme. The other extreme is where one looks for the dynamics of all *proper* regular systems which can be obtained by applying state feedback to a given regular controllable system. This type of fundamental theorem was proved by Kučera and Zagalak [4].

A major step in proving the present result has been the accommodation of zeros. Since the given system may have an infinite zero one faces the problem of assigning a pole under a zero. The presence of an infinite zero restricts the infinite pole structure by limiting the number of its cyclic chains whereas the controllability indices limit the sizes of these chains.

The case of infinite poles complements the results on finite poles thereby providing a more complete picture as to what can be accomplished by state feedback with regard to altering the dynamics of a regular controllable system.

(Received December 22, 1989).

REFERENCES

[1] J. D. Cobb: Feedback and pole placement in descriptor variable systems. IEEE Trans. Automat. Control *AC-29* (1981), 1076—1082.

[2] M. M. Fahmy and O'Reilly: Matrix pencil of closed-loop descriptor systems: infinite-eigenvalue assignment. Internat. J. Control *49* (1989), 1421—1431.

[3] M. L. J. Hautus and M. Heymann: Linear feedback — an algebraic approach. SIAM J. Control. Optim. *16* (1970), 83—105.

[4] V. Kučera and P. Zagalak: Fundamental theorem of state feedback for singular systems. Automatica *24* (1988), 653—658.

[5] V. Kučera and P. Zagalak: Constant solutions of polynomial equations. Preprints IFAC Workshop on System Structure and Control, Prague 1989, pp. 33—35.

[6] F. L. Lewis: A survey of linear singular systems. J. Circuits, Systems, Signal Proc., Special Issue on Singular Systems *5* (1986), 3—36.

[7] J. J. Loiseau: Pole placement and connected problems. Preprints IFAC Workshop on System Structure and Control, Prague 1989, pp. 193—196.

[8] H. H. Rosenbrock: State-space and Multivariable Theory. Wiley, New York 1970.

[9] G. C. Verghese, B. C. Lévy and T. Kailath: A generalized state-space for singular systems. IEEE Trans. Automat. Control *AC-26* (1981), 811—831.

[10] W. M. Wonham: On pole assignment in multi-input controllable linear systems. IEEE Trans. Automat. Control *AC-12* (1967), 660—665.

Ing. Petr Zagalak, CSc., Ing. Vladimír Kučera, DrSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.