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ON THE CONVERGENCE RATE OF EMPIRICAL ESTIMATES IN CHANCE CONSTRAINED STOCHASTIC PROGRAMMING

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The chance constrained stochastic programming problems with unknown probability laws were discussed e. g. in [9], [11]. In these papers empirical distribution functions were used to get some estimates on the optimal value and the optimal solution. In particular, some sufficient conditions for the consistency of the estimates on the optimal value were introduced in [9] and in [11] convergence rate of estimates on the optimal solution in a rather special case was studied.

In this paper we present some results on the optimal value estimates under rather general conditions including also some type of stochastic dependent samples. The paper employs the methods used in [7], [8], [9].

1. INTRODUCTION

Let $(\Omega, \mathcal{S}, P)$ be a probability space,

$\xi = \xi(\omega) = [\xi_1(\omega), \ldots, \xi_l(\omega)]$ be an l-dimensional random vector defined on $(\Omega, \mathcal{S}, P)$,

$F(z)$ be the distribution function of the random vector $\xi(\omega)$,

$\xi^k = \xi^k(\omega) = [\xi_1^k(\omega), \ldots, \xi_l^k(\omega)]$ be a sequence of random vectors such that for every $k = 1, 2, \ldots$ the random vector $\xi^k(\omega)$ has the same distribution function as the random vector $\xi(\omega)$,

$f_i(x)$, $i = 1, 2, \ldots, l$ be a real valued continuous function defined on E_n (E_n, $n \geqq 1$ denotes an n-dimensional Euclidean space), and

$$f(x) = [f_1(x), \ldots, f_l(x)] .$$

Let, further, $g(x, z)$ be a real-valued, continuous function defined on $E_n \times E_l$.

If we define the function $U_k(z, \omega) = U_k(z)$, $F_N(z, \omega) = F_N(z)$, $Z = [z_1, \ldots, z_l] \in E_l$, $\omega \in \Omega$, $k = 1, 2, \ldots, N$, $N = 1, 2, \ldots$ by

$$U_k(z, \omega) = 1 \Leftrightarrow \xi_j^k(\omega) < z_j \quad \text{for all} \quad j = 1, 2, \ldots, l ,$$

$$= 0 \Leftrightarrow \xi_j^k(\omega) \geqq z_j \quad \text{for at least one } j \in \{1, 2, \ldots, l\} ,$$

$$F_N(z, \omega) = \frac{1}{N} \sum_{k=1}^{N} U_k(z, \omega) .$$

then we can define the sets $Z(x)$, $X(\alpha)$, $X_N(\alpha) = X_N(\alpha, \omega)$, $N = 1, 2, \ldots, x \in E_n^+$, in the following way

$$Z(x) = \{z \in E_l^+ : z = (z_1, \ldots, z_l) : f_i(x) \leqq z_i, \, i = 1, 2, \ldots, l\};$$
$$X(\alpha) = \{x \in E_n^+ : P[Z(x)] \geqq \alpha\}, \quad \text{for} \quad \alpha \in \langle 0, 1 \rangle,$$
$$X(\alpha) \equiv X(1) \quad \text{for} \quad \alpha > 1,$$
$$X(\alpha) \equiv X(0) \quad \text{for} \quad \alpha < 0,$$
$$X_N(\alpha) = X_N(\alpha, \omega) = \{x \in E_n^+ : P_N[Z(x)] \geqq \alpha\}, \quad \text{for} \quad \alpha \in \langle 0, 1 \rangle,$$

where $P[Z(x)] = \mathsf{P}\{\omega: \xi(\omega) \in Z(x)\}$, $P_N[\cdot] = P_N\{\cdot, \omega\}$ is the empirical probability measure corresponding to the distribution function F_N, $E_n^+ = \{x \in E_n: x = (x_1, \ldots, x_n): x_i \geqq 0, \, i = 1, 2, \ldots, n\}$.

Denoting by E and E_N, $N = 1, 2, \ldots$, the theoretical and the empirical mathematical expectation, respectively, it is easy to see that

$$\inf_{X_N(\alpha)} \mathsf{E}_N \, g(x, \xi(\omega)) \tag{1}$$

estimates the theoretical value

$$\inf_{X(\alpha)} \mathsf{E} g(x, \xi(\omega)) . \tag{2}$$

Remark. It can generally happen that some symbols in (1) and (2) are not meaningful. However, this situation cannot appear under the assumptions considered in this paper.

Some sufficient assumptions under which (1) is a consistent estimate of (2) are presented in [9]. The present paper goes deeper in this direction. Namely, we shall study the convergence rate. In particular, we shall try to find an upper bound on the following probabilities

$$\mathsf{P}\{\omega: |\inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega)) - \inf_{X(\alpha)} \mathsf{E} g(x, \xi(\omega))| > t\},$$
$$\mathsf{P}\{\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\} \tag{3}$$

$t \in E_1$, $t > 0$, $N = 1, 2, \ldots$.

These results generalize some of the author's previous results for stochastic programming problems in which the optimum is sought with respect to the deterministic constraints only (cf. [7], [8]). We restrict our consideration to the special (however from the practical point of view important enough) case.

Let $\alpha, \delta > 0$, $\alpha \in (0, 1)$ be arbitrary. We make the following assumptions:

i) $f_i(x)$, $i = 1, 2, \ldots, l$ are real valued, continuous function on E_n^+ such that

a) $f_i(0) = 0$, $i = 1, 2, \ldots, l$, $0 \in E_n$,

b) there exists a constant $\gamma_1 > 0$ such that

$$f_i(x) - f_i(x') \geqq \gamma_1 \sum_{j=1}^{n} (x_j - x'_j)$$

for every $x = (x_1, \ldots, x_n)$, $x' = (x'_1, \ldots, x'_n)$, $x \geqq x'$ componentwise, $i = 1, 2, \ldots, l$, $x, x' \in E_n^+$;

c) there exists a constant $\gamma_2 > 0$ such that

$$|f_i(x) - f_i(x')| \leqq \gamma_2 \|x - x'\|$$

for $i = 1, 2, \ldots, l$, $x, x' \in X(\alpha, 2\delta)$, $x < x'$ componentwise. $X(\alpha, \delta)$ is defined by the following relations

$$X(\alpha, \delta) = \{x = x_1 + x_2, x_1 \in X(\alpha), x_2 \in B(\delta)\} \tag{4}$$

where $B(\delta)$ denotes the δ-surrounding of $0 \in E_n$. $\|\cdot\|$denotes the Euclidean norm in E_n.

ii) $\xi(\omega)$ fulfils the conditions

a) the probability measure of the random vector $\xi(\omega)$ is absolutely continuous with respect to the Lebesgue measure in E_l. We denote by $h(y)$ the probability density corresponding to the distribution function $F(\cdot)$ of the random vector $\xi(\omega)$.

b) there exists a real number $c_j, j = 1, 2, \ldots, l$ such that $c_j \geqq 0$ and that

$$\mathsf{P}\{\omega: \xi(\omega) \in \prod_{j=1}^{l} \langle 0, c_j \rangle\} = 1\,,$$

c) there exists $\vartheta_2 > 0$, $\vartheta_2 \in E_1$ such that

$$h(z) \leqq \vartheta_2 \quad \text{for all} \quad z \in E_l^+\,,$$

d) there exists a constant $\vartheta_1 > 0$ such that

$$\vartheta_1 \leqq h(z) \quad \text{for every} \quad z \in \prod_{j=1}^{l} \langle 0, c_j \rangle\,.$$

Remark. It is easy to see that under these assumptions the sets $X(\alpha)$ for $\alpha \in (0, 1\rangle$ are compact.

2. SOME AUXILIARY DEFINITIONS

The Hausdorff distance between two subsets in E_n is defined in the following way:

Definition 1. If $X. X' \subset E_n$, $n \geqq 1$ are two non-empty sets then the Hausdorff distance of these sets $\Delta_n(X', X'')$ is defined by

$$\Delta_n(X', X'') = \max\,[\delta_n(X', X''), \delta_n(X'', X')]\,,$$

$$\delta_n(X', X'') = \sup_{x' \in X'} \inf_{x' \in X''} \|x' - x''\|\,,$$

(we usually omit the subscripts in the symbols Δ_n, δ_n).

Let, further, $\{\eta^k\}_{k=-\infty}^{\infty}$ be an l-dimensional strong stationary random sequence defined on $(\Omega, \mathcal{S}, P)$, $\mathcal{B}(-\infty, a)$ be the σ-algebra given by $\ldots, \eta^{a-1}, \eta^a$, $\mathcal{B}(b, +\infty)$ be the σ-algebra given by $\eta^b, \eta^{b+1}, \ldots$ (a, b are integers).

If $\mathbb{N}$ denotes the set of natural numbers, $\Phi(\cdot)$ a non-negative real-valued function defined on $\mathbb{N}$ then we can define the Φ-mixing random sequence by the following definition.

Definition 2. We say that the strong stationary random sequence $\{\eta^k\}_{k=-\infty}^{\infty}$ fulfils the condition of Φ-mixing if

$$|P(A_1 \cap A_2) - P(A_1)\, P(A_2)| \leqq \Phi(N)\, P(A_1)$$

for $A_1 \in B(-\infty, m)$, $A_2 \in B(m + N, +\infty)$, $-\infty < m < +\infty, N \geqq 1$.

Remark. Of course, it is assumed that $\lim_{N \to \infty} \Phi(N) = 0$.

If $D \subset E_n$ be a bounded set then there exist $d'_j, d''_j \in E_1, j = 1, 2, \ldots, n$ and natural numbers $m_j = m_j(D, d)$, $j = 1, 2, \ldots, n$ for $d > 0$, $d \in E_1$, $d < \inf_j (d''_j - d'_j)$ such that

$$d'_j(D) = d'_j = \inf \{x_j \colon x = (x_1, \ldots, x_n) \in D\},$$
$$d''_j(D) = d''_j = \sup \{x_j \colon x = (x_1, \ldots, x_n) \in D\}$$
$$D_j n/d \leqq m_j \leqq D_j n/d + 1, \quad D_j = d''_j - d'_j.$$

Further, we can define $x_{j_1}, \ldots, x_{j,m_j}$, $j = 1, 2, \ldots, n$ such that $d'_j = x_{j1}, x_{jr} = x_{j_{r-1}} + d/n$, $r = 1, 2, \ldots, m_j$, $x_{j,m_j-1} < d''_j$ $x_{jm_j} \geqq d''_j$, $j = 1, 2, \ldots, n$.

It is easy to see that we can define the system S as follows:

$$S = S(D, d) = \{x = [x_1, \ldots, x_n] \colon x_r \in [x_{rj_1}, \ldots, x_{rm_r}],\ r = 1, 2, \ldots, n\}.$$

It holds that

$$\inf_{x' \in S} \|x - x'\| \leqq d \quad \text{for all} \quad x \in D$$

$$\inf_{x' \in S} \|x - x'\| \leqq d \quad \text{for all} \quad x \in \prod_{j=1}^{n} \langle d'_j, d''_j \rangle$$

and

$$m = \prod_{j=1}^{n} m_j,$$

where we denote by $m = m(D, d)$ the number of elements of the system S.

3. MAIN RESULTS

In this section we present some upper bounds for the expression given by relations (3). First, we shall consider the case of independent random samples.

a) Independent Case

Theorem 1. Let $\alpha \in (0, 1)$, $\delta > 0$. If assumptions i), ii) are fulfilled (for these α, δ)

and if $\{\xi^k\}_{k=1}^{\infty}$ is a sequence of independent random vectors then, for $t > 0$, $d > 0$ such that $d < \delta$, $\sqrt{n/\gamma_1}\ \sqrt[l]{(2t/\vartheta_1)} < \delta$, $\vartheta_2\gamma_2 d \sum_{i \neq 1}^{l} \prod_{v \neq 1} c_v < t/6$, it holds

1. $\mathsf{P}\{\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\} \geqq 1 - 2m[X(\alpha, 2\delta), d] \exp[-Nt^2/18]$,
2. $\mathsf{P}\{\omega: \Delta[X(\alpha), X_N(\alpha)] \leqq \sqrt[l]{(2t/\vartheta_1)}\ \sqrt{n/\gamma_1}\} \geqq 1 - 2m[X(\alpha, 2\delta), d] \exp[-Nt^2/18]$.

Theorem 2. Let $\alpha \in (0, 1)$, $\delta > 0$, $d > 0$, $t > 0$, $t_0/2 = \sqrt{n/\gamma_1}\ \sqrt[l]{(2t/\vartheta_1)}$. If assumptions i), ii) are fulfilled (for this α, δ) and if

i') $\{\xi^k\}_{k=1}^{\infty}$ is a sequence of independent random vectors,

ii') $d < \min(\delta, t/6)$, $\sqrt{n/\gamma_1}\ \sqrt[l]{(2t/\vartheta_1)} < \delta$

$$\vartheta_2\gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/6\,,$$

iii') $g(x, z)$ is for every $z \in E_l^+$ a Lipschitz function of $x \in X(\alpha, 2\delta)$ with Lipschitz constant L not depending on z then

$$\mathsf{P}\{\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| \geqq t_0 L\} \leqq$$
$$\leqq 2m[X(\alpha, 2\delta), d] \exp[-Nt^2/18] + 2m[X(\alpha, 2\delta), d] \exp[-Nt_0^2L^2/(4.18M^2)]$$

where the constant M fulfils the inequality

$$|g(x, z)| < M \quad \text{for} \quad x \in X(\alpha, 2\delta)\,.$$

Corollary 1. If the assumptions of Theorem 2 are fulfilled and if there exists a constant I such that

$$0 < I \leqq \inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega))$$

then

$$\mathsf{P}\left\{\omega: \frac{|\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))|}{\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega))} \geqq t_0 L\right\} \leqq$$
$$\leqq 2m[X(\alpha, 2\delta), d] \exp[-N(tI)^2/18] +$$
$$+ 2m[X(\alpha, 2\delta), d] \exp[-N(t_0IL)^2/(4.18M)]\,.$$

Since the proof of the presented assertions is rather complicated and long we present it in detail in the Appendix. In this section we introduce further some similar results for dependent samples fulfilling also the conditions of Φ-mixing.

b) Dependent Case

Theorem 3. Let $\alpha \in (0, 1)$, $\delta > 0$. If assumptions i), ii) are fulfilled (for this α, δ) and if $\{\xi^k\}_{k=-\infty}^{\infty}$ is a random sequence fulfilling the conditions of Φ-mixing then for $t > 0$, $d > 0$ such that $d < \delta$,

$$\vartheta_2\gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/_3\,, \quad \sqrt[l]{(2t/\vartheta_1)}\ \sqrt{n/\gamma_1} < \delta\,,$$

1. $\mathsf{P}\{\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\} \geqq$

$$\geqq 1 - 2m[X(\alpha, 2\delta), d]\, 36/(t^2N^2)\, [N + \sum_{k=1}^{N} (N - k)\, \Phi(k)]\,,$$

2. $\mathsf{P}\{\omega: \Delta[X(\alpha), X_N(\alpha)] \leqq \sqrt[l]{(2t/\vartheta_1)}\, \sqrt{n}/\gamma_1\} \geqq$

$$\geqq 1 - 2m[X(\alpha, 2\delta), d]\, 36/(t^2N^2)\,[N + \sum_{k=1}^{N} (N - k)\, \Phi(k)]\,.$$

Theorem 4. Let $\alpha \in (0, 1)$, $\delta > 0$, d, $t \in E_1^+$, $t_0/2 = \sqrt{n}/\gamma_1 \sqrt[l]{(2t/\vartheta_1)}$. If assumptions i), ii) are fulfilled (for this α, δ) and if

i″) $\{\xi^k\}_{k=-\infty}^{\infty}$ is a random sequence fulfilling the conditions of Φ-mixing,

ii″) the assumptions ii′), iii′) of Theorem 2 are fulfilled then

$$\mathsf{P}\{\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L\} \leqq$$

$$\leqq 2m[X(\alpha, 2\delta), d]\, [N + \sum_{k=1}^{N} (N - k)\, \Phi(k)] \left[\frac{36.4}{t^2N^2} + \frac{4.36M^2}{L^2t_0^2N^2}\right].$$

Corollary 2. If the assumptions of Theorem 2 are fulfilled and if there exists a constant I such that

$$0 < I \leqq \inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega))$$

then

$$\mathsf{P}\left\{\omega: \frac{|\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))|}{\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega))} > t_0 L\right\} \leqq$$

$$\leqq m[X(\alpha, \delta), d]\, [N + \sum_{k=1}^{N} (N - k)\, \Phi(k)] \left[\frac{36}{t^2LN^2} + \frac{4.36M^2}{L^2t_0^2N^2I^2}\right].$$

4. APPENDIX

The aim of this section is to give the proof of the former assertions. First we prove some auxiliary results.

Lemma 1. Let $\alpha \in (0, 1)$. If assumptions ia), ib), iia), iid) are fulfilled then

$$\Delta\left[X(\alpha), X\left(\alpha - \vartheta_1\left(\frac{d}{\sqrt{n}}\right)^l\right)\right] < \frac{d}{\gamma_1} \quad \text{for } d > 0 \quad \text{such that} \quad \left(\alpha - \vartheta_1\left(\frac{d}{\sqrt{N}}\right)^l\right) > 0\,.$$

Proof. Let $d > 0$ be arbitrary given such that the assumptions of Lemma 1 are fulfilled. Since

$$\Delta\left[X(\alpha), X\left(\alpha - \vartheta_1\left(\frac{d}{\sqrt{n}}\right)^l\right)\right] =$$

$$= \max\left[\sup_{x \in X(\alpha)} \inf_{x' \in X(\alpha - \vartheta_1(d/\sqrt{n})^l)} \|x - x'\|\,, \sup_{x' \in X(\alpha - \vartheta_1(d/\sqrt{n})^l)} \inf_{x \in X(\alpha)} \|x - x'\|\right]$$

and since

$$X(\alpha) \subset X(\alpha - \vartheta_1(d/\sqrt{n})^l),$$

it is necessary to prove the inequality

$$\sup_{x \in X(\alpha - \vartheta_1(d/\sqrt{n})^l)} \inf_{x' \in X(\alpha)} \|x - x'\| < d/\gamma_1, \tag{8}$$

only

Let $x = (x_1, \ldots, x_n) \in X(\alpha - \vartheta_1(d/\sqrt{n})^l)$ be an arbitrary point. To prove (8) it is sufficient to find $x^* = x^*(x)$ such that $\|x - x^*\| < d/\gamma_1$, $x^* \in X(\alpha)$.

It is easy to see that if $x \in X(\alpha)$ then we can set $x = x^*$. So it remains to consider the case $x \notin X(\alpha)$. If we define in this case the point $x' = (x'_1, \ldots, x'_n)$, $x'_i = x_i - - d/(\gamma_1 \sqrt{n})$, $i = 1, 2, \ldots, n$ we get $\|x - x'\| = d/\gamma_1$. Two different cases can happen

a) there exists an $r \in \{1, 2, \ldots, n\}$ such that $x'_r \geqq 0$

b) $x'_j < 0$ for $j = 1, 2, \ldots, n$.

Let us, first, consider the case a). In this case we can define the point $x^* = (x_1^*, x_2^*, \ldots \ldots, x_n^*)$ by the following prescription $x_r^* = x'_r$, $x_j^* = x_j, j \neq r$. It follows from the assumption ib) that $f_i(x^*) < f_i(x)$, $|f_i(x^*) - f_i(x)| \geqq d/\sqrt{n}$, $i = 1, 2, \ldots, l$ and simultaneously $\|x^* - x\| = d/(\gamma_1 \sqrt{n}) \leqq d/\gamma_1$.

Further, it follows from the properties of the probability measure that for d fulfilling the assumptions we get

$$\begin{aligned}
&\mathsf{P}\{\omega: f_i(x^*) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} \geqq \mathsf{P}\{\omega: f_i(x) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} + \\
&+ \mathsf{P}\{\omega: f_i(x^*) \leqq \xi_i(\omega) \leqq f_i(x), i = 1, 2, \ldots, l\} \geqq \\
&\geqq \alpha - \vartheta_1(d/\sqrt{n})^l + \vartheta_1 \mu\{\prod_{i=1}^{l} < f_i(x^*), f_i(x) >\} \geqq \\
&\geqq \alpha - \vartheta_1(d/\sqrt{n})^l + \vartheta_1(d/\sqrt{n})^l = \alpha
\end{aligned}$$

($\mu(\cdot)$ denotes the Lebesgue measure in E_l)}.

It remains to consider the case b). However as we have $\|x\| \leqq d/\gamma_1$, in this case, the assertion of Lemma 1 follows from the assumption iia). □

Lemma 2. Let $\alpha \in (0, 1)$. If assumptions ia), ib), iia), iid) are fulfilled then for $\beta > 0$, $\alpha - \beta > 0$ the inequality

$$\Delta[X(\alpha), X(\alpha - \beta)] < \frac{\sqrt{n}}{\gamma_1} \sqrt[l]{\frac{\beta}{\vartheta_1}}$$

holds.

Proof. The assertion of Lemma 2 follows immediately from the assertion of Lemma 1. □

Lemma 3. Let $\alpha \in (0, 1)$. If assumptions ia), ib), iia), iid) are fulfilled then for $t > 0$, $\alpha - \vartheta_1(t\gamma_1/\sqrt{n})^l > 0$ it is

$$\Delta[X(\alpha), X(\alpha - \vartheta_1(t\gamma_1/\sqrt{n})^l)] < t.$$

Proof. The assertion of Lemma 3 follows immediately from the assertion of Lemma 1. □

Lemma 4. Let $\alpha, \delta > 0$, $\alpha \in (0, 1)$. If assumptions ia), ic), iia), iib) iic) are fulfilled then for $t > 0$, $t \in E_1$, $d > 0$ such that $d < \delta$,

$$\vartheta_2 \gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/3 \tag{9}$$

there exists a system $S(X(\alpha, 2\delta), d)$ such that

$$\mathsf{P}\{\omega: |P(Z(x)) - P_N(Z(x))| > t \text{ for at least one } x \in X(\alpha, \delta)\} \leqq$$
$$\leqq \sum_{x^v \in S(X(\alpha, 2\delta), d)} \mathsf{P}\{\omega: |P_N(Z(x^v)) - P(Z(x^v))| > t/3\} .$$

Proof. First we get from the definition of the system $S(X(\alpha, 2\delta), d)$ and from the assumptions that

$$x \in X(\alpha, 2\delta) \Rightarrow \inf_{x^j \in S(X(\alpha, \delta), d)} |f_1(x) - f_i(x^j)| \leqq d\gamma_2 \quad i = 1, 2, \ldots, l$$

and moreover there exist $x^j, x^r \in S(X(\alpha, \delta), d)$, $x^j < x^r$

componentwise such that $f_i(x^j) \leqq f_i(x) \leqq f_i(x^r)$, $i = 1, 2, \ldots, l$

and simultaneously $\|x^r - x^j\| \leqq d$.

However, it follows from this and from the assumptions that for $x \in X(\alpha, \delta)$ there exists $x^j = x^j(x)$, $x^r = x^r(x)$, x^j, $x^r \in S(X(\alpha, \delta), d)$, $x^j < x^r$ componentwise, $\|x^j - x^r\| < d$ such that

$$\mathsf{P}\{\omega: f_i(x^r) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} \leqq \mathsf{P}\{\omega: f_i(x) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} \leqq$$
$$\leqq \mathsf{P}\{\omega: f_i(x^j) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\}$$

and simultaneously

$$\mathsf{P}\{\omega: f_i(x^j) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} - \mathsf{P}\{\omega: f_i(x^r) \leqq \xi_i(\omega), i = 1, 2, \ldots, l\} \leqq$$
$$\leqq \vartheta_2 \sum_{i=1}^{l} \mu\{\prod_{v \neq i} \langle 0, c_i \rangle \times \langle f_v(x^j), f_v(x^r) \rangle\} \leqq \vartheta_2 \gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/3 .$$

So we have

$$\mathsf{P}\{\omega: |P_N(Z(x)) - P(Z(x))| > t \text{ for at least one } x \in X(\alpha, \delta)\} \leqq$$
$$\leqq \mathsf{P}\{\omega: |P_N(Z(x^v)) - P(Z(x^v))| > t/3 \text{ for at least one } x^v \in S(X(\alpha, 2\delta), d)\}$$
$$\leqq \sum_{x^v \in S(X(\alpha, 2\delta), d)} \mathsf{P}\{\omega: |P_N(Z(x^v)) - P(Z(x^v))| > t/3\} .$$

□

Lemma 5. If $\alpha \in (0, 1)$, $\delta > 0$. If the assumptions i), ii) are fulfilled (for this α, δ) then for $t > 0$, $d > 0$ such that $d < \delta$,

$$\frac{\sqrt{n}}{\gamma_1} \sqrt[l]{\frac{t}{\vartheta_1}} < \delta , \quad \vartheta_2 \gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/3 , \quad i = 1, 2, \ldots, l \tag{10}$$

there exists a system $S(X(\alpha, 2\delta), d)$ such that

$$\mathsf{P}\{\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\} \geqq$$
$$\geqq 1 - \sum_{x^\nu \in S(X(\alpha, 2\delta), d)} \mathsf{P}\{\omega: |P_N(Z(x^\nu)) - P(Z(x^\nu))| > t/3\} .$$

Proof. First, it follows from the assumptions and from Lemma 2 that

$$X(\alpha + t) \subset X(\alpha - t) \subset X(\alpha, \delta) .$$

Further, since

$$x \in X_N(\alpha, \omega) \Rightarrow x' \in X_N(\alpha, \omega) \quad \text{for all} \quad x' < x, x' \geqq 0 \quad \text{(componentwise)} ,$$

we get

$$\{\omega: \text{there exists } x \in X_N(\alpha, \omega) \text{ such that } x \notin X(\alpha - t)\} \subset$$
$$\{\omega: \text{there exists } x \in X_N(\alpha, \omega),\ x \in X(\alpha, \delta) \text{ such that } x \notin X(\alpha - t)\} \subset$$
$$\subset \{\omega: \text{there exists } x \in X(\alpha, \delta) \text{ such that } |P_N(Z(x)) - P(Z(x))| > t\} .$$

and moreover the relation

$$\Omega - [\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)] \subset$$
$$\subset \{\omega: \text{there exists } x \in X(\alpha, \delta) \text{ such that } |P_N(Z(x)) - P(Z(x))| > t\}$$

too.

However from this and from the assertion of Lemma 4 we obtain that

$$\mathsf{P}\{\omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\} \geqq$$
$$\geqq 1 - \sum_{x^\nu \in S(X(\alpha, 2\delta), d)} \mathsf{P}\{\omega: |P(Z(x^\nu)) - P_N(Z(x^\nu))| > t/3\} .$$

This completes the proof.

Remark. It is easy to see that if $X', X'' \subset E_n$ are two sets such that $X(\alpha + t) \subset \subset X' \subset X(\alpha - t), X(\alpha + t) \subset X'' \subset X(\alpha - t)$ then $\Delta[X', X''] \leqq \Delta[X(\alpha + t), X(\alpha - t)]$.

Lemma 6. If $\alpha \in (0, 1)$, $\delta > 0$. If further the assumptions i), ii) are fulfilled (for this α, δ) and if $t > 0, d > 0$ satisfy the inequalities $d < \delta, \sqrt{n}/\gamma_1 \sqrt[l]{(t/\vartheta_1)} < \delta$,

$$\vartheta_2 \gamma_2 d \sum_{i=1}^{l} \prod_{v \neq i} c_v < t/3 ,$$

then there exists a system $S(X(\alpha, 2\delta))$ such that

$$\mathsf{P}\{\omega: \Delta[X(\alpha), X_N(\alpha)] \leqq \sqrt[l]{(2t/\vartheta_1)}\ \sqrt{n}/\gamma_1\} \geqq$$
$$\geqq 1 - \sum_{x^\nu \in S(X(\alpha, 2\delta), d)} \mathsf{P}\{\omega: |P_N(Z(x^\nu)) - P(Z(x^\nu))| > t/3\} .$$

Proof. First, it follows from the assumptions and the assertion of Lemma 2 that $X(\alpha - t) \subset X(\alpha, 2\delta)$.

If we denote by $\Omega''(t)$ the set such that

$$\Omega'' = \{\omega \in \Omega: X(\alpha + t) \subset X_N(\alpha) \subset X(\alpha - t)\}$$

then we get

$$\Delta[X(\alpha), X_N(\alpha)] \leqq \Delta[X(\alpha + t), X(\alpha - t)] \quad \text{for} \quad \omega \in \Omega'' .$$

Employing again Lemma 2 we obtain that

$$\Omega'' \subset \{\omega: \Delta[X(\alpha), X_N(\alpha)] \leqq \sqrt{n}/\gamma_1 \sqrt[l]{(2t/\vartheta_1)}\}$$

and further that

$$\mathsf{P}\{\omega: \Delta[X(\alpha), X_N(\alpha)] \leqq \sqrt{n}/\gamma_1 \sqrt[l]{(2t/\vartheta_1)}\} \geqq \mathsf{P}(\Omega''\} \text{ too} .$$

The assertion of Lemma 6 follows immediately from the last inequality and from the assertion of Lemma 5. □

Lemma 7. Let $\alpha \in (0, 1)$, $\delta > 0$, $t_0 > 0$, $d < t_0/6$, $d > 0$, $t_0/2 < \delta$. If the assumptions i), ii) are fulfilled (for this α, δ) and if $g(x, z)$ is for every $z \in \prod_{i=1}^{l} < 0, c_i\rangle$ a Lipschitz function of $x \in X(\alpha, 2\delta)$ with Lipschitz constant L not depending on z then there exists a system $S(X(\alpha, 2\delta), d)$ such that

$$\begin{aligned}
&\mathsf{P}\{\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > Lt_0\} \leqq \\
&\leqq \mathsf{P}\{\omega: \Delta[X(\alpha), X_N(\alpha)| > t_0/2\} + \\
&+ \mathsf{P}\{\omega: |\mathsf{E}g(x^v, \xi(\omega)) - \mathsf{E}_N g(x^v, \xi(\omega))| > t_0 L/6 \quad \text{for at least one} \\
&x^v \in S(X(\alpha, 2\delta), d)\} .
\end{aligned}$$

Proof. Let $t_0 > 0$, $t_0 \in E_1$ be arbitrary fulfilling the assumptions.

We can define the sets Ω_1, Ω_2 in the following way

$$\Omega_1 = \{\omega \in \Omega: \Delta[X(\alpha), X_N(\alpha)] \leqq t_0/2\} ,$$

$$\Omega_2 = \Omega - \Omega_1 .$$

Since it is easy to see that

$$\begin{aligned}
&\mathsf{P}\{\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L\} \leqq \mathsf{P}\{\Omega_2\} + \\
&+ \mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L\}
\end{aligned} \tag{11}$$

we can deal with

$$\mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L\}$$

only.

Since the triangular inequality implies

$$|\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| \leqq$$
$$\leqq |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega))| +$$
$$+ |\inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))|$$

we get

$$\mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L\} \leqq$$
$$\leqq \mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega))| > t_0 L/2 +$$
$$+ \mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/2\}\,.$$

Further, it follows from the definition of the set Ω_1 and the assumption that

$$\mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega))| > t_0 L/2\} = 0\,.$$

So to prove the assertion of Lemma 7 it is sufficient to prove the inequality

$$\mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/2\} \leqq$$
$$\leqq \mathsf{P}\{\omega: |\mathsf{E}g(x, \xi(\omega)) - \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/2 \text{ for at least one }$$
$$x \in S(X(\alpha, 2\delta), d)\}\,.$$

It is easy to see that the functions $\mathsf{E}g(x, \xi)$, $\mathsf{E}_N g(x, \xi)$ are Lipschitz functions on $X(\alpha, 2\delta)$ with Lipschitz constant L. Using these facts along with the assumptions we get

$$\mathsf{P}\{\Omega_1 \cap [\omega: |\inf_{X_N(\alpha)} \mathsf{E}g(x, \xi(\omega)) - \inf_{X_N(\alpha)} \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/2\} \leqq$$
$$\leqq \mathsf{P}\{\omega: |\mathsf{E}g(x, \xi(\omega)) - \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/2 \quad \text{for at least one}$$
$$x \in X(\alpha, 2\delta)\} \leqq \mathsf{P}\{\omega: |\mathsf{E}(x, \xi(\omega)) - \mathsf{E}_N g(x, \xi(\omega))| > t_0 L/6$$
$$\text{for at least one} \quad x \in S(X(\alpha, 2\delta), d)\}\,.$$

□

To prove Theorem 1 and Theorem 2 we have to recall here one well-known inequality first introduced in [2]. We shall present it in the special form suitable for our problem. If $\varkappa(z)$ is a measurable function defined on E_l such that $|\varkappa(z)| \leqq M$ for $z \in \prod_{i=1}^{l} \langle 0, c_i \rangle$ and if F_N corresponds to an independent random sequence $\{\xi^k\}_{k=1}^{\infty}$, then

$$\mathsf{P}\{\omega: \mathsf{E}_N \varkappa(\xi(\omega)) - \mathsf{E}\varkappa(\xi(\omega)) > y\} \leqq \exp\{-Ny^2/(2M^2)\} \tag{13}$$

for every $y > 0$, $y \in E_1$.

P r o o f o f T h e o r e m 1. Assertion 1 of Theorem 1 follows immediately from Lemma 5 and inequality (13). Assertion 2 follows in a quite similar way from Lemma 6 and the inequality (13).

□

Proof of Theorem 2. It is easy to see that we can obtain the assertion of Theorem 2 from Theorem 1, Lemma 7 and the inequality (13). □

The assertion of Corollary 1 follows immediately from the assertion of Theorem 2.

It remains to deal with the case, when the members of the random sequence $\{\xi^k\}_{k=1}^{\infty}$ are dependent. In [1] some inequalities are proved for random sequences fulfilling the conditions of Φ-mixing. We introduce one of them again in a special (for us suitable) form.

If $\varkappa(z)$ is a measurable function defined on E_l such that $|\varkappa(z)| \leqq M$ for $z \in \prod_{i=1}^{l} \langle 0, c_i \rangle$ and if the random sequence $\{\xi^k\}_{k=-\infty}^{\infty}$ fulfils the condition of Φ-mixing then

$$|\mathsf{E}[\varkappa(\xi^k(\omega)) - \mathsf{E}\varkappa(\xi(\omega))(\varkappa(\xi^r(\omega)) - \mathsf{E}\varkappa(\xi(\omega))]| \leqq 2M^2\Phi(|r - k|) \tag{14}$$

for every $r, k \in \{\ldots -1, 0, 1, 2, \ldots\}$.

It is easy to see that utilizing the last inequality we can prove the assertions of Theorem 3 and Theorem 4 similarly to the proof of the assertion of Theorem 1 and Theorem 2 employing inequality (14) instead of the inequality (13).

Proof of Theorem 3. Assertion 1 of Theorem 3 follows immediately from Lemma 5, the Chebyshev inequality and relation (14). Assertion (2) can be proved in a quite similar way utilizing Lemma 6 instead of Lemma 5. □

Proof of Theorem 4. Utilizing Lemma 7, Theorem 3, the Chebyshev inequality and the inequality (14) we get the assertion of Theorem 4. □

It is easy to see again that the assertion of Corollary 2 follows immediately from the assertion of Theorem 4.

Remarks. 1. Number $m(X(\alpha, 2\delta), d)$ was used in the assertions of this paper. The upper bound of this number is presented in [8]. 2. The case when the support of the random vector $\xi(\omega)$ is in the form $\prod_{i=1}^{l} \langle 0, c_i \rangle$ is considered in the paper. However it is easy to see that the assertions of this paper are valid for the support in the form $\prod_{i=1}^{l} \langle c_i', c_i'' \rangle$, $c_i', c_i'' \in E_1^+$, $c_i' < c_i''$, too.

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