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OPTIMIZATION OF UNIMODAL MONOTONE PSEUDOBOOLEAN FUNCTIONS

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In this paper unimodal strictly monotone pseudoboolean functions and unimodal monotone pseudoboolean functions having constancy sets are investigated. Employing the obtained properties of pseudoboolean functions, regular algorithms for optimization are constructed. Estimates of efficiency of the suggested algorithms are also obtained.

1. INTRODUCTION

The classical problem of pseudoboolean optimization can be formulated as follows (cf. [1]):

$$\varkappa(X) \to \min ,$$

where

$$\varkappa\colon \mathscr{B}_{2^n} \to \mathbb{R}^1 , \quad \mathscr{B}_{2^n} = \{X \mid x_j \in \mathscr{B}_2 , \quad j = 1, \ldots, n\} , \quad \mathscr{B}_2 = \{0, 1\} ,$$

or, after "embedding" the problem in $\mathbb{R}^n$:

$$\varkappa(X) \to \min_{X \in \mathscr{D}} ,$$

where

$$\mathscr{D} = \{X \in \mathbb{R}^n \mid x_j = 0 \vee 1\} , \quad \varkappa(X) \in \mathbb{R}^1 .$$

First shall present necessary definitions.

Definition 1.1. We shall call points X^1, $X^2 \in \mathscr{D}$ k-neighbouring if they differ only in the values of k coordinates $(k = 1, \ldots, n)$. 1-neighbouring points will be called simply neighbouring.

Definition 1.2. The set $\mathcal{O}_k(X)$ $(k = 1, \ldots, n)$ of points that are k-neighbouring to the point $X \in \mathscr{D}$ will be called the kth level of the point X $(\mathcal{O}_0(X) = X)$. The point $X \in \mathscr{D}$ is introduced as k-neighbouring to the set $\mathscr{B} \subset \mathscr{D}$ if $\mathscr{B} \cap \mathcal{O}_k(X) \neq \emptyset \wedge \forall l = 0, \ldots, k - 1$: $\mathscr{B} \cap \mathcal{O}_l(X) = \emptyset$. The set $\mathcal{O}_k(\mathscr{B}) \subset \mathscr{D}$ of all points of $\mathscr{D}$ which are k-neighbouring to the set $\mathscr{B}$ will be called the kth level of set $\mathscr{B}$, $\mathcal{O}_0(\mathscr{B}) = \mathscr{B}$.

Remark 1.1. It is obvious that for any $k = 1, \ldots, n$ card $\mathcal{O}_k(X) = \mathbf{C}_n^k$. Here (and in the sequel) $\mathbf{C}_n^k$ is the number of combinations from n on k.

The function $\varkappa: \mathcal{D} \to \mathbb{R}^1$ will be called a pseudoboolean function.

Definition 1.3. A point $X^* \in \mathcal{D}$ for which $\varkappa(X^*) < \varkappa(X)\ \forall X \in \mathcal{O}_1(X^*)$ will be called a local minimum of the pseudoboolean function $\varkappa$.

Definition 1.4. A pseudoboolean function which has only one local minimum on $\mathcal{D}$ will be called unimodal.

Lemma 1.1. If $X^k \in \mathcal{O}_k(X) \subset \mathcal{D}$, $k = 1, \ldots, n$, then $\mathcal{O}_1(X^k)$ consists of k points of the set $\mathcal{O}_{k-1}(X)$ and $(n - k)$ points of set $\mathcal{O}_{k+1}(X)$.

Proof. From Definition 1.1 it follows that the points X^k and X have k different components:

$$x_{j_i}^k = 1 - x_{j_i}, \quad i = 1, \ldots, k. \tag{1.1}$$

Let $Y \in \mathcal{O}_1(X^k)$, i.e. (by Definition 1.1) Y is a neighbouring point for X^k and it differs from X^k in the lth component: i.e. $y_1 = x_1^k, \ldots, y_{l-1} = x_{l-1}^k$, $y_l = 1 - x_l^k$, $y_{l+1} = x_{l+1}^k, \ldots, y_n = x_n^k$. If $l \neq j_i$, $i = 1, \ldots, k$, then $x_l^k = x_l$ and $y_l = 1 - x_l^k = 1 - x_l$, i.e. Y and X have $(k + 1)$ different components and by Definition 1.1 $Y \in \mathcal{O}_{k-1}(X)$. From (1.1) and $X \subset \mathbb{R}^n$ follows that there exists $(n - k)$ such points. If $l = j_i$, $i = 1, \ldots, k$, then $x_l^k = 1 - x_l$ and $y_l = 1 - x_l^k = 1 - 1 + x_l = x_l$ and the points Y and X have $(k - 1)$ different components, i.e. by Definition 1.1 $Y \in \mathcal{O}_{k-1}(X)$. From (1.1) it follows that there will be k such points. □

Corollary 1.1. For any $k = 1, \ldots, n$ there are no neighbouring points among the points $X_j^k \in \mathcal{O}_k(X) \subset \mathcal{D}$, $j = 1, \ldots. \mathbf{C}_n^k$.

Corollary 1.2. For any point $Y \in \mathcal{O}_1(X^k)$ and different from X^k in the jth ($j = 1, \ldots, n$) coordinate it holds

$$y_j = \begin{cases} x_j & \text{if} \quad Y \in \mathcal{O}_1(X^k) \cap \mathcal{O}_{k-1}(X), \\ 1 - x_j & \text{if} \quad Y \in \mathcal{O}_1(X^k) \cap \mathcal{O}_{k+1}(X) \end{cases} \tag{1.2}$$

implying that $X^k \in \mathcal{O}_k(X) \subset \mathcal{D}$.

2. OPTIMIZATION OF STRICTLY MONOTONE PSEUDOBOOLEAN FUNCTIONS

Definition 2.1. A unimodal pseudoboolean function $\varkappa$ will be called strictly monotone on $\mathcal{D}$ if

$$\varkappa(X^{k-1}) < \varkappa(X^k)\ \forall X^{k-1} \in \mathcal{O}_{k-1}(X^*) \wedge \forall X^k \in \mathcal{O}_k(X^*), \quad k = 1, \ldots, n. \tag{2.1}$$

Taking into account Definition 2.1 and employing Lemma 1.1 we can construct an algorithm for optimization of unimodal functions strictly monotone on $\mathcal{D}$.

The algorithm requires calculation of the optimized function in $(n + 1)$-th points of $\mathscr{D}$ for exact location of the minimum point X^* regardless the initial point X^0.

Algorithm 1.

1. The point $X^0 \in \mathscr{D}$ is chosen arbitrarily.
2. By sequential replacing values of the components of the point X^0 with the opposite ones, we find all points $X_j^1 \in \mathcal{O}_1(X^0), j = 1, \ldots, n$.
3. The values $\varkappa(X^0)$ and $\varkappa(X_j^1)$ for any $j = 1, \ldots, n$ are calculated.
4. The coordinates of the point X^* are found by the following rule $(j = 1, \ldots, n)$:

$$x_j^* = \begin{cases} x_j^0 & \text{if} \quad \varkappa(X_j^1) > \varkappa(X^0)\,, \\ 1 - x_j^0 & \text{if} \quad \varkappa(X_j^1) < \varkappa(X^0)\,. \end{cases} \tag{2.2}$$

Actually if $\varkappa(X_j^1) > \varkappa(X^0)$ then according to Lemma 1.1 and Definition 2.1 the point X_j^1 lies in $\mathcal{O}_{k+1}(X^*)$ where k is equal to the number of the points $X_j^1 (j = 1, \ldots, n)$ for which $\varkappa(X_j^1) < \varkappa(X^0)$ and by (1.2) $x_j^* = x_j^0$; if $\varkappa(X_j^1) < \varkappa(X^0)$ then, $X_j^1 \in \mathcal{O}_{k-1}(X^*)$ and by (1.2) $x_j^* = 1 - x_j^0$.

3. OPTIMIZATION OF MONOTONE PSEUDOBOOLEAN FUNCTIONS HAVING CONSTANCY SETS

Definition 3.1. The set of points $\mathscr{W}(X^0, X^l) = \{X^0, X^1, \ldots, X^i, \ldots, X^l\} \subset \mathscr{D}$ will be called the curve between the points X^0 and X^l if for all $i = 1, \ldots, l$, the point X^i is neighbouring for the point X^{i-1}.

Definition 3.2. The set $\mathscr{A} \subset \mathscr{D}$ is called the connected set if for any $X^0, X^l \in \mathscr{D}$ there exists a curve $\mathscr{W}(X^0, X^l) \subset \mathscr{A}$.

Definition 3.3. The connected set of points $\Pi_C \subset \mathscr{D}$, card $\Pi_C \geqq 2$, such that $\varkappa(X) = C\,(C = \text{const})$ for any $X \in \Pi_C$ is called the constancy set of the function $\varkappa$ on $\mathscr{D}$.

Remark 3.1. It is obvious that the number of the levels of a constancy set $\Pi_C \subset \mathscr{D}$ of the function $\varkappa$ onto $\mathscr{D}$ is equal to N where (and in the sequel)

$$N = \max_{Y \in \mathscr{D} \setminus \Pi_C} \min_{X \in \Pi_C} \sum_{j=1}^{n} |x_j - y_j|\,.$$

Definition 3.4. A unimodal function $\varkappa$ will be called monotone on $\mathscr{D}$ if

$$\varkappa(X^{k-1}) \leqq \varkappa(X^k)\, \forall X^{k-1} \in \mathcal{O}_{k-1}(X^*) \wedge \forall X^k \in \mathcal{O}_k(X^*)\,, \quad k = 1, \ldots, n\,, \tag{3.1}$$

or equivalently

$$\max_{X^{k-1} \in \mathcal{O}_{k-1}(X^*)} \varkappa(X^{k-1}) \leqq \min_{X^k \in \mathcal{O}_k(X^*)} \varkappa(X^k) \quad \forall k = 1, \ldots, n\,. \tag{3.2}$$

Remark 3.2. From Definition 3.4 follows that a function monotone on $\mathscr{D}$ may have constancy sets.

Definition 3.5. The constancy set Π_{C*} of a pseudoboolean function $\varkappa$ such that for any $X^1 \in \mathcal{O}_1(\Pi_{C*})$: $\varkappa(X^1) > C^*$ will be called the extended local minimum of the function $\varkappa$.

Remark 3.3. By analogy it is not difficult to define pseudoboolean functions which are unimodal, unimodal strictly monotone and unimodal monotone with respect to the extended minimum.

Definition 3.6. We shall call the first points of the set Π_C the points of the set $\{X_j^I\} = \mathcal{O}_I(X^*) \cap \Pi_C$ where Π_C is a constancy set of a unimodal pseudoboolean function $\varkappa$ if $\mathcal{O}_I(X^*) \cap \Pi_C \neq \emptyset \wedge \forall k = 1, \ldots, I - 1$: $\mathcal{O}_k(X^*) \cap \Pi_C = \emptyset$.

Definition 3.7. We shall call the last points of the constancy set Π_C the points of the set $\{\overline{X}_j^L\} = \mathcal{O}_L(X^*) \cap \Pi_C$ where Π_C is a constancy set of a unimodaj pseudoboolean function $\varkappa$ if $\mathcal{O}_L(X^*) \cap \Pi_C \neq \emptyset \wedge \forall k = L + 1, \ldots, n$: $\mathcal{O}_k(X^*) \cap \Pi_C = \emptyset$.

Remark 3.4. It is obvious that $0 \leqq I \leqq L \leqq n$. If $I = 0$ and $L = n$ then the function $\varkappa$ is constant on $\mathscr{D}$.

Lemma 3.1. If Π_C is a constancy set of a unimodal monotone on $\mathscr{D}$ function $\varkappa$ then for any $X_j^t \in \mathcal{O}_t(X^*)$ $(I < t < L, j = 1, \ldots, \mathbf{C}_n^t)$ $X_j^t \in \Pi_C$.

Proof. From (3.2) we have

$$\max_{X_j^I \in \mathcal{O}_I(X^*)} \varkappa(X_j^I) \leqq \min_{X_j^t \in \mathcal{O}_t(X^*)} \varkappa(X_j^t) \leqq \max_{X_j^t \in \mathcal{O}_t(X^*)} \varkappa(X_j^t) \leqq \min_{X_j^L \in \mathcal{O}_L(X^*)} \varkappa(X_j^L) . \tag{3.3}$$

Since $\mathcal{O}_I(X^*) \cap \Pi_C \neq \emptyset$ and $\mathcal{O}_L(X^*) \cap \Pi_C \neq \emptyset$ (I and $L$are the level issues of the first and last points of the set Π_C)

$$\max_{X_j^I \in \mathcal{O}_I(X^*)} \varkappa(X_j^I) = \min_{X_j^L \in \mathcal{O}_L(X^*)} \varkappa(X_j^L) = C . \tag{3.4}$$

According to (3.4) from (3.3) we have

$$\min_{X_j^t \in \mathcal{O}_t(X^*)} \varkappa(X_j^t) = \max_{X_j^t \in \mathcal{O}_t(X^*)} \varkappa(X_j^t) = C ,$$

i.e. $\{X_j^t,\ I < t < L,\ \ j = 1, \ldots, \mathbf{C}_n^t\} \subset \Pi_C$. □

Corollary 3.1. For any $\Pi_C \subset \mathscr{D}$ of a unimodal function $\varkappa$

$$\Pi_C = \{\tilde{X}_j^I\} \cup \left(\bigcup_{t=I+1}^{L-1} \mathcal{O}_t(X^*) \right) \cup \{\overline{X}_j^L\} .$$

Remark 3.5. If Π_{C_1} and Π_{C_2} are constancy sets of a unimodal function monotone on $\mathscr{D}$ then it is obvious that $C_1 < C_2$ if $L_1 \leqq I_2$ where L_1 is the level issue of the last points of Π_{C_1} and I_2 is the level issue of the first points of Π_{C_2}.

Definition 3.8. A constancy set of a pseudoboolean function $\varkappa$ – Π_{C_1} will be called isolated if for any $q = 2, \ldots, Q$:

$$\Pi_{C_1} \cap \mathcal{O}_1(\Pi_{C_q}) = \emptyset . \tag{3.5}$$

Here Q is the number of constancy sets of the function $\varkappa$.

Definition 3.9. A constancy set of a pseudoboolean function $\varkappa - \boldsymbol{\Pi}_{C_1}$ will be called weakly adjacent if for some q $(q = 2, \ldots, Q)$ the condition (3.5) is broken but

$$I_1 \neq L_q \wedge L_1 \neq I_q , \tag{3.6}$$

where I_1, I_q are the issues of the first and L_1, L_q are the issues of the last levels of the sets $\boldsymbol{\Pi}_{C_1}$ and $\boldsymbol{\Pi}_{Cq}$ respectively. If for set $\boldsymbol{\Pi}_{C_1}$ both condition (3.5) and condition (3.6) are broken then the constancy set $\boldsymbol{\Pi}_{C_1}$ will be called strongly adjacent.

Lemma 3.2. Let $\boldsymbol{\Pi}_{C_1}$ be an isolated or weakly adjacent constancy set of a unimodal function $\varkappa$ monotone on $\mathscr{D}$, $\boldsymbol{X}^0 \in \mathcal{O}_1(\boldsymbol{\Pi}_{C_1}) \cap \mathcal{O}_k(\boldsymbol{X}^*)$. Then

$$\begin{aligned} \varkappa(\boldsymbol{X}^0) &> \varkappa(\boldsymbol{X}^1_{j_i}), \quad i = 1, \ldots, k, \\ \varkappa(\boldsymbol{X}^0) &\leqq \varkappa(\boldsymbol{X}^1_{j_i}), \quad i = k + 1, \ldots, n, \end{aligned} \tag{3.7}$$

if $k \geqq L_1$ and

$$\begin{aligned} \varkappa(\boldsymbol{X}^0) &\geqq \varkappa(\boldsymbol{X}^1_{j_i}), \quad i = 1, \ldots, k, \\ \varkappa(\boldsymbol{X}^0) &< \varkappa(\boldsymbol{X}^1_{j_i}), \quad i = k + 1, \ldots, n, \end{aligned} \tag{3.8}$$

if $k \leqq I_1$ where I_1 and L_1 are the level issues of the first and the last points of the set $\boldsymbol{\Pi}_{C_1}$, $\{\boldsymbol{X}^1_{j_i}, i = 1, \ldots, k\} = \mathcal{O}_1(\boldsymbol{X}^0) \cap \mathcal{O}_{k-1}(\boldsymbol{X}^*)$, $\{\boldsymbol{X}^1_{j_i}, i = k + 1, \ldots, n\} = \mathcal{O}_1(\boldsymbol{X}^0) \cap \cap \mathcal{O}_{k+1}(\boldsymbol{X}^*)$.

Proof. Put $k \geqq L_1$. From Lemma 1.1 and the condition $\boldsymbol{X}^0 \in \mathcal{O}_k(\boldsymbol{X}^*)$ it follows that the set $\mathcal{O}_1(\boldsymbol{X}^0)$ consists of k points of the $(k-1)$-th level of $\boldsymbol{X}^*$: $\boldsymbol{X}^1_{j_i} \in \mathcal{O}_1(\boldsymbol{X}^0) \cap \cap \mathcal{O}_{k-1}(\boldsymbol{X}^*)$, $i = 1, \ldots, k$, and $(n - k)$ points of the $(k + 1)$-th level of $\boldsymbol{X}^*$: $\boldsymbol{X}^1_{j_i} \in \in \mathcal{O}_1(\boldsymbol{X}^0) \cap \mathcal{O}_{k+1}(\boldsymbol{X}^*)$, $i = k + 1, \ldots, n$.

Then from monotonicity of $\varkappa$ and the condition $\boldsymbol{X}^0 \notin \boldsymbol{\Pi}_{C_1}(\boldsymbol{X}^0 \in \mathcal{O}_1(\boldsymbol{\Pi}_{C_1}))$, where $\boldsymbol{\Pi}_{C_1}$ is not a strongly adjacent set, according to the conditions of Lemma 3.2 we have

$$\varkappa(\boldsymbol{X}^0) > \varkappa(\boldsymbol{X}^1_{j_i}) \quad \forall i = 1, \ldots, k. \tag{3.9}$$

If $\boldsymbol{\Pi}_{C_1}$ is an isolated constancy set then by Definition 3.7 $\boldsymbol{X}^0 \in \bigcup_{q=2}^{Q} \boldsymbol{\Pi}_{C_q}$ and from monotonicity of $\varkappa$ it follows

$$\varkappa(\boldsymbol{X}^0) > \varkappa(\boldsymbol{X}^1_{j_i}) \quad \forall i = k + 1, \ldots, n. \tag{3.10}$$

If the constancy set $\boldsymbol{\Pi}_{C_1}$ is weakly adjacent with certain constancy set $\boldsymbol{\Pi}_{C_2}(C_1 < C_2)$ then it is possible that $\boldsymbol{X}^0 \in \boldsymbol{\Pi}_{C_2}$. In this case if $\boldsymbol{\Pi}_{C_2}$ contains more than one level of $\boldsymbol{X}^*$ for a part or all $i = k + 1, \ldots, n$:

$$\varkappa(\boldsymbol{X}^0) = \varkappa(\boldsymbol{X}^1_{j_i}) \tag{3.11}$$

(if $\boldsymbol{\Pi}_{C_2}$ contains one level only we have (3.10)). Gathering (3.9)–(3.11) we come to (3.7).

The relation (3.6) for the case $k \leqq I_1$ can be proved similarly. □

Remark 3.6. The in equalities (3.9) hold for a part of $i = k - 1, \ldots, n$ in case $\boldsymbol{\Pi}_{C_2} = \{\bar{\boldsymbol{X}}^{I_2}_j\} \cup \{\bar{\boldsymbol{X}}^{L_2}_j\} \wedge \{\bar{\boldsymbol{X}}^{I_2}_j\} \neq \mathcal{O}_{I_2}(\boldsymbol{X}^*)$, $\{\bar{\boldsymbol{X}}^{L_2}_j\} \neq \mathcal{O}_{L_2}(\boldsymbol{X}^*)$. Definition 3.2 allows existence of similar constancy sets for monotone functions.

Remark 3.7. If we assume, under the conditions of Lemma 3.2, that Π_{C_1} is strongly adjacent constancy set then instead of (3.7) and (3.8) we get

$$\begin{aligned} \varkappa(X^0) &\geqq \varkappa(X^1_{j_i})\,, \quad i = 1, \ldots, k\,, \\ \varkappa(X^0) &\leqq \varkappa(X^1_{j_i})\,, \quad i = k+1, \ldots, n\,. \end{aligned} \tag{3.12}$$

The obtained results enable us to formulate the following algorithm for minimization of unimodal functions which are monotone on $\mathscr{D}$ and have constancy sets.

Algorithm 2.

1. The point $X^0 \in \mathscr{D}$ is chosen arbitrarily.
2. By sequential replacing the component values of the point X^0 with opposite ones we find all points $X^1_j \in \mathcal{O}_1(X^0)$, $j = 1, \ldots, n$.
3. $\varkappa(X^0)$ and $\varkappa(X^1_j)$, $j = 1, \ldots, n$, are calculated. If

$$\varkappa(X^0) \neq \varkappa(X^1_j) \text{ for any } j = 1, \ldots, n, \tag{3.13}$$

then we define the coordinates of the extremal point X^* by rule (2.1) if

$$\mathscr{J} = \{j \in \{1, \ldots, n\} \mid \varkappa(X^0) = \varkappa(X^1_j)\} \neq \emptyset \wedge \mathscr{J} \neq \{1, \ldots, n\} \tag{3.14}$$

then go to 4, otherwise we suppose that $t = 1$ and go to 5.

4. If $\{1, \ldots, n\} = \mathscr{F} \cup \mathscr{F}'$ where $\mathscr{F}' = \{j \in \{1, \ldots, n\} \mid \varkappa(X^0) < \varkappa(X^1_j)\}$ then

$$x^*_j = \begin{cases} x^0_j & \text{for } j \in \mathscr{F}'\,, \\ 1 - x^0_j & \text{for } j \in \mathscr{F}\,, \end{cases}$$

if $\{1, \ldots, n\} = \mathscr{F} \cup \mathscr{F}''$ where $\mathscr{F}'' = \{j \in \{1, \ldots, n\} \mid \varkappa(X^0) > \varkappa(X^1_j)\}$ then

$$x^*_j = \begin{cases} x^0_j & \text{for } j \in \mathscr{F}\,, \\ 1 - x^0_j & \text{for } j \in \mathscr{F}''\,, \end{cases}$$

if $\{1, \ldots, n\} = \mathscr{F} \cup \mathscr{F}' \cup \mathscr{F}'' \wedge \mathscr{F}' \neq \emptyset, \mathscr{F}'' \neq \emptyset$ then by the rule

$$x^{1*}_j = \begin{cases} x^0_j & \text{if } \varkappa(X^1_j) > \varkappa(X^0)\,, \\ 1 - x^0_j & \text{if } \varkappa(X^1_j) \leqq \varkappa(X^0)\,, \end{cases} \tag{3.15}$$

$j = 1, \ldots, n$ the point X^{1*} is defined and by the rule

$$x^{2*}_j = \begin{cases} x^0_j & \text{if } \varkappa(X^1_j) \geqq \varkappa(X^0)\,, \\ 1 - x^0_j & \text{if } \varkappa(X^1_j) < \varkappa(X^0)\,, \end{cases} \tag{3.16}$$

$j = 1, \ldots, n$, the point X^{2*} is defined. We calculate $\varkappa(X^{1*})$ and $\varkappa(X^{2*})$. $\varkappa(X^*) = \min\{\varkappa(X^{1*}), \varkappa(X^{2*})\}$.

5. For any $j = 1, \ldots, \mathbf{C}^t_n$ all points $X^{1j}_i \in \mathcal{O}_1(X^t_j)$, $i = 1, \ldots, n$, are defined. We suppose $t = t + 1$ and select the set of points $X^t_j \in \mathcal{O}_t(X^0)$, $j = 1, \ldots, \mathbf{C}^t_n$ (in which values of the function have not been calculated yet). We calculate $\varkappa(X^t_j)$, $j = 1, \ldots, \mathbf{C}^t_n$ and go to 6.
6. If $\varkappa(X^t_j) = \varkappa(X^0)$ for any $j = 1, \ldots, \mathbf{C}^t_n$ then go to 5, otherwise go to 7.

7. Let $X^t_{j_1}$ be the first point in sequence of points of the set $\mathcal{O}_t(X^0)$: $X^t_1, \ldots, X^t_{C_n^t}$ for which $\varkappa(X^t_{j_1}) \neq \varkappa(X^0)$. In this case we suppose $X^0 = X^t_{j_1}$ and define all points $X^1_j \in \mathcal{O}_1(X^0)$ and values of the function in them, i.e. $\varkappa(X^1_j)$, $j = 1, \ldots, n$. Next we verify the conditions (3.13) and (3.14). If (3.13) is correct then we define the coordinates of the point by rule (2.1) if condition (3.14) is correct the coordinates of the point X^* are defined by rule (3.15) (if $\varkappa(X^0) < C$) or by rule (3.16) (if $\varkappa(X^0) > C$).

Remark 3.8. If condition (3.10) holds for an arbitrarily chosen point X^0 then Algorithm 2 coincides with Algorithm 1. In this sense Algorithm 2 is an extension of Algorithm 1.

Remark 3.9. It is obvious that under optimizing of unimodal monotone pseudo-boolean functions having strongly adjacent constancy sets the considered algorithm can produce an error in location of X^* (in case when (3.12) holds). A simple modification of the algorithm enables us to reduce the error – in Step 4, if $\{1, \ldots, n\} = \mathcal{F} \cup \mathcal{F}' \cup \mathcal{F}'' \wedge \mathcal{F}' \neq \emptyset$, $\mathcal{F}'' \neq \emptyset$, it is necessary to suppose that $X^0 = X^1_j$, where j is an index from the set $\mathcal{F}' \cup \mathcal{F}''$, and then relations (3.15) and (3.16) will be correct for the point X^0. Similar situation also arises in the case when (3.12) holds in going from a constancy set (cf. Step 5, 7). Strict theoretical proof of the statement "then relations (3.15) and (3.16) will be correct for the point X^0" requires rather bulky calculations. Therefore, since the remark is of no fundamental importance, we shall regard the statement to be obvious.

4. EFFECTIVENESS OF OPTIMIZATION

When real-life optimization problems are solved numerically the principal cost of search of extremum is connected with computations of values of the minimized functional in different points of $\mathcal{D}$ (see Antamoshkin [2, 3]). Therefore as a rule (see e.g. Himmelblau [4]) effectiveness of the optimization algorithm is estimated by the number of computations of the minimized function which are required for locating an extremum of the function for any initial point.

As it was previously pointed out, in optimization of unimodal functions strictly monotone on $\mathcal{D}$ Algorithm 2 coincides with Algorithm 1 and hence requires $(n + 1)$ computations of the function for any initial point. The same estimate is also correct for Algorithm 2 used for optimization of unimodal functions monotone on $\mathcal{D}$ if for an initial point X^0 condition (3.13) is correct.

If for the point X^0 condition (3.14) is correct Algorithm 2 requires $(n + 2)$ computations of the function (or $2n$ computations the modification given in Remark 3.9).

It remains to estimate effectiveness of the algorithm when $\mathcal{O}_1(X^0) \subset \Pi_C$. Two cases are possible: $\mathcal{O}_n(X^*) \subset \Pi_C$ and $\mathcal{O}_n(X^*) \not\subset \Pi_C$. We snall consider them separately.

Theorem 4.1. Locating of the minimum point X^* (some point of the extended minimum, i.e. $X \in \Pi_{C^*}$) of a unimodal function $\varkappa$ monotone on $\mathscr{D}$ for which the condition

$$\varkappa(X^n) \neq \varkappa(X_j^{n-1})\ \forall X_j^{n-1} \in \mathcal{O}_{n-1}(X^*), \quad X^n \in \mathcal{O}_n(X^*), \tag{4.1}$$

holds, from the initial point $X^0 \in \mathcal{O}_k(X^*) \subset \Pi_C$ such that $\mathcal{O}_1(X^0) \subset \Pi_C$ by Algorithm 2 requires T_1 computations of $\varkappa$.

$$T_1 = \sum_{i=0}^{M} \mathbf{C}_n^i + S + 1, \tag{4.2}$$

$$M = \min\{L - k, k - I\} \tag{4.3}$$

(I and $L$are the level issues of the first and the last points of the set Π_C),

$$S = \begin{cases} I - 1 & \text{if} \quad M = k - I, \\ n - L & \text{if} \quad M = L - k. \end{cases} \tag{4.4}$$

Proof. If $\{\bar{X}_j^I\} = \mathcal{O}_I(X^*)$ and $\{\bar{X}_j^L\} = \mathcal{O}_L(X^*)$ then by Corollary 3.1

$$\Pi_C = \bigcup_{t=I}^{L} \mathcal{O}_t(X^*)$$

besides from Condition (4.1) it follows $L \leqq n - 1$; $I \geqq 1 - \Pi_C \neq \Pi_{C^*}$. According to Algorithm 2 the values of $\varkappa$ are calculated in the points

$$X^0 \subset \Pi_C\ (\varkappa(X^0) = C), \quad X_j^1 \in \mathcal{O}_1(X^0) \subset \Pi_C\ (\varkappa(X_j^1) = C, \quad j = 1, \ldots, n),$$
$$\ldots, X_j^M \subset \mathcal{O}_M(X^0) \subset \Pi_C\ (\varkappa(X_j^M) = C, \quad j = 1, \ldots, \mathbf{C}_n^M),$$

where M is found by (4.3). Thus we shall carry out $T_1^1 = \sum_{i=0}^{M} \mathbf{C}_n^i$ calculations. Then the value in some point $X_j^{M+1} \in \mathcal{O}_{M+1}(X^0) \not\subset \Pi_C$ is calculated, i.e. $\varkappa(X_j^{M+1}) \neq C$. $T_1^2 = 1$ more calculations have been done. Now, according to the algorithm, we must do $T_1^3 = S$ calculations of $\varkappa$ for locating X^*, where S is found by (4.4). Summing up T_1^1, T_1^2 and T_1^3 we have (4.2). □

Corollary 4.1.

$$\max_k T_1 = \sum_{i=0}^{\alpha} \mathbf{C}_n^i + S + 3$$

where

$$\alpha = \begin{cases} (L - I)/2 & \text{if} \quad (L - I) \quad \text{is even,} \\ \text{the integer part of the number} \\ (L - I)/2 & \text{if} \quad (L - I) \quad \text{is odd.} \end{cases}$$

$$\bar{T}_1 = \max_{I,L} \max_k T_1 = \sum_{i=0}^{\beta} \mathbf{C}_n^i + 2, \tag{4.5}$$

where

$$\beta = \begin{cases} (n - 2)/2 & \text{if} \quad (n - 2) \quad \text{is even,} \\ \text{the integer part of the number} \quad (n - 2)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Remark 4.1. In Theorem 4.1 the case when $\{\bar{X}_j^I\} = \mathcal{O}_I(X^*)$ and $\{\bar{X}_j^L\} = \mathcal{O}_L(X^*)$ was considered. If $\{\bar{X}_j^I\} \subset \mathcal{O}_I(X^*)$ and $\{\bar{X}_j^L\} \subset \mathcal{O}_L(X^*)$ then estimate (4.2) may be reduced.

Theorem 4.2. Locating of the minimum point X^* (some points of the extended minimum, i.e. $X \in \boldsymbol{\Pi}_{C^*}$) of a unimodal function $\varkappa$ monotone on $\mathcal{D}$, which satisfies the condition

$$\mathcal{O}_{n-1}(X^*) \cup \mathcal{O}_n(X^*) \subset \boldsymbol{\Pi}_C , \tag{4.6}$$

from the initial point $X^0 \in \mathcal{O}_k(X^*) \subset \boldsymbol{\Pi}_C$ such that $\mathcal{O}_1(X^0) \subset \boldsymbol{\Pi}_C$, by Algorithm 2 requires T_2 computations of $\varkappa$.

$$T_2 = \sum_{i=0}^{k-I} \mathbf{C}_n^i + I \tag{4.7}$$

where I is the level issue of the first points of the set $\boldsymbol{\Pi}_C$.

Proof. Supposing $\{\bar{X}_j^I\} = \mathcal{O}_I(X^*)$ and taking into account that for a constancy set which is defined by Condition (4.6) $\{\bar{X}_j^L\} = \emptyset$, according to Corollary 3.1 we have $\boldsymbol{\Pi}_C = \bigcup_{t=I}^{n} \mathcal{O}_t(X^*)$ where $I \geqq 1$, otherwise $\varkappa$ is constant on $\mathcal{D}$. According to Algorithm 2 the values of $\varkappa$ are calculated in the points $X^0 \subset \boldsymbol{\Pi}_C$ $(\varkappa(X^0) = C)$, $X_j^1 \in \mathcal{O}_1(X^0) \subset \boldsymbol{\Pi}_C$ $(\varkappa(X_j^1) = C\,,\ \ j = 1, \ldots, n)$,

$$\ldots, X_j^{k-I} \in \mathcal{O}_{k-I}(X^0) \subset \boldsymbol{\Pi}_C \ (\varkappa(X_j^{k-I}) = C\,,\ \ j = 1, \ldots, \mathbf{C}_n^{k-I}).$$

Totally, there have been done $T_2^1 = \sum_{i=0}^{k-I} \mathbf{C}_n^i$ calculations. Then the value of $\varkappa$ in some point $X_j^{k-I+1} \in \mathcal{O}_{k-I+1}(X^0) \not\subset \boldsymbol{\Pi}_C$, i.e. $\varkappa(X_j^{k-I+1}) \neq C$, are calculated. There have been performed $T_2^2 = 1$ calculations more. Now, according to the algorithm for locating of X^*, we must perform $T_2^3 = I - 1$ calculations of $\varkappa$ $(X_j^{k-I+1} \in \mathcal{O}_{k-I+1}(X^0) \cap \cap\, \mathcal{O}_{I-1}(X^*))$. Summing up T_2^1, T_2^2 and T_2^3 we have (4.7). □

Corollary 4.2. If $X^0 \in \boldsymbol{\Pi}_{C^*}$ then Algorithm 2 for justifying this fact requires no more than $T_3 = \sum_{i=0}^{L} \mathbf{C}_n^i + n - L$ calculations of $\varkappa$, where L is a level issue of the last points of the set $\boldsymbol{\Pi}_{C^*}$.

Corollary 4.3.

$$\max_k T_2 = \sum_{i=0}^{n-I} \mathbf{C}_n^i + I\,,$$
$$\max_I \max_k T_2 = \sum_{i=0}^{n-1} \mathbf{C}_n^i + 1 = 2^n. \tag{4.8}$$

Remark 4.2. The case when $\{\bar{X}_j^I\} = \mathcal{O}_I(X^*)$ have been considered. If $\{\bar{X}_j^I\} \subset \subset \mathcal{O}_I(X^*)$ then estimate (4.7) may be reduced.

As it follows from estimate (4.8), Algorithm 2 degenerates into the total sorting under optimization of a unimodal function $\varkappa$ having a constancy set of the form

$$\Pi_C = \bigcup_{t=1}^{n} \mathcal{O}_t(X^*) \tag{4.9}$$

from the point $X^0 \in \mathcal{O}_n(X^*)$. But the event $X^0 \in \mathcal{O}_n(X^*)$ is low probable. The next theorem gives a more objective estimate of effectiveness of the algorithm.

Theorem 4.3. Locating the minimum point of a unimodal pseudoboolean function having a constancy set of the form (4.9) by Algorithm 2 requires on the average T_4 calculations of $\varkappa$

$$T_4 = \frac{1}{2^n}\left[(n+1)^2 + \sum_{k=2}^{n} \mathbf{C}_n^k \left(\sum_{i=0}^{k-1} \mathbf{C}_n^i + 1\right)\right] \tag{4.10}$$

Proof. According to the algorithm the point X^0 is chosen arbitrarily, hence it may be assumed that $\forall X \in \mathcal{D}\ \mathsf{P}\{X^0 = X\} = 1/2^n$. Whence taking into account that card $\mathcal{O}_k(X^*) = \mathbf{C}_n^k$, $k = 1, \ldots, n$ we have

$$\mathsf{P}\{X^0 \in \mathcal{O}_k(X^*)\} = \mathbf{C}_n^k/2^n. \tag{4.11}$$

For a constancy set of form (4.9) estimate (4.7) will assume the form

$$T_3' = \sum_{i=0}^{k-1} \mathbf{C}_n^i + 1\,. \tag{4.12}$$

From (4.11), (4.12) and the fact that for $X^0 \in X^* \cup \mathcal{O}_1(X^*)$ Algorithm 2 requires $(n+1)$ calculations of $\varkappa$, we have for the mathematical expectation of the number of calculations of $\varkappa$ required for locating X^*

$$T_4 = \frac{\mathbf{C}_n^1 + 1}{2^n}(n+1) + \sum_{k=2}^{n} \frac{\mathbf{C}_n^k}{2^n}\left(\sum_{i=0}^{k-1} \mathbf{C}_n^i + 1\right) =$$

$$= \frac{1}{2^n}\left[(n+1)^2 + \sum_{k=2}^{n} \mathbf{C}_n^k \left(\sum_{i=0}^{k-1} \mathbf{C}_n^i + 1\right)\right]. \qquad \square$$

Remark 4.2. Performing the summation in (4.10), for the estimator of T_4 we have

$$T_4 = 2^{n-1} + 1 - \frac{\mathbf{C}_{2n}^n}{2^{n+1}} + \frac{n^2}{2^n} \tag{4.13}$$

from which it is possible to obtain the asymptotic estimator

$$T_4 \approx 2^{n-1}\left(1 - \frac{1}{\sqrt{(\pi n)}}\right) + 1\,.$$

Remark 4.3. Estimator (4.7) (also estimators (4.10), (4.13) respectively) is an accessible estimator of the algorithm "from the top".

5. CONCLUSION

It follows from estimators (4.5) and (4.13) that optimizing any unimodal monotone pseudoboolean function by Algorithm 2 on the average requires no more than $2n$ computations of the function for the exact locating of the extremum from any initial point (after the modification considered in Remark 3.8 the algorithm will require $(2n + 1)$ computations), i.e. in comparison with the total sorting in which 2^n computations of the function are necessary, Algorithm 2 on the average requires a number of computations of the minimized function in two (or more) times less.

The review of the existing methods of pseudoboolean optimization is given by the present authors in [5]. Comparison of effectiveness of these methods (by the estimators given in [2, 3]) and effectiveness of the proposed algorithms shows the advantage of the approach suggested in this paper.

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