SOME ASYMPTOTIC RESULTS FOR ROBUST PROCEDURES FOR TESTING THE CONSTANCY OF REGRESSION MODELS OVER TIME

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The aim of the present paper is to derive the asymptotic distribution of test statistics connected with a robust version of CUSUM (cumulative sums) procedure used for testing of the constancy of the regression relationship over time. The obtained results are, in fact, certain extensions of the Darling-Erdös theorem (cf. [4]).

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be independent random variables, X_i distributed according to the distribution function (d.f.) $F(x - \boldsymbol{c}_i'\boldsymbol{\theta}_i)$, where $\boldsymbol{c}_i = (c_{i1}, \ldots c_{ip})'$, $i = 1, \ldots, n$, are known regression vectors, $\boldsymbol{\theta}_i$, $i = 1, \ldots, n$, are unknown parameters, F fulfils certain regularity conditions (and unknown otherwise). For testing problem:

$$\mathrm{H}_0: \boldsymbol{\theta} = \ldots = \boldsymbol{\theta}_n = \boldsymbol{\theta}_0 \quad \text{(known or unknown)}$$

against

$$\mathrm{H}_1: \text{there exists} \quad 1 \leqq m < n \quad \text{such that}$$

$$\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_m \neq \boldsymbol{\theta}_{m+1} = \ldots = \boldsymbol{\theta}_n$$

(called testing of the constancy of the regression relationship over time) there were developed many different test procedures (for further information see survey papers, e.g. [2], [3], [8], [9], [12]. The recursive procedures (CUSUM – cumulative sums and MOSUM – moving sums) were developed and deeply studied for F normal in [5]. They are based on the recursive residuals

$$X_i - \boldsymbol{c}_i'\boldsymbol{\theta}_{i-1}, \quad i = p + 1, \ldots, n, \tag{1.1}$$

where $\boldsymbol{\theta}_{i-1}$ is the least squares estimator of $\boldsymbol{\theta}_0$ based on $X_1, \ldots, X_{i-1}$. The important feature of these residuals (for F normal) is that they are independent and normally distributed.

For F fulfilling only some regularity conditions a robust version of recursive procedures related to M-estimators were developed. They are called recursive M-procedures (CUSUM M-test and MOSUM M-test) and are based on the *M-recursive residuals*

$$W_i = \psi(X_i - c_i'\theta_{i-1}(\psi)), \quad p + 1 \leqq i \leqq n, \tag{1.2}$$

where ψ is a score function from $\mathbb{R}^1$ into $\mathbb{R}^1$ (usually monotone), $\theta_{i-1}(\psi)$ is an M-estimator of θ_0 (or an estimator related to it) generated by the function ψ based on $X_1, \ldots, X_{i-1}$. For $\psi(x) = x, x \in \mathbb{R}^1$, one obtains classical recursive procedures based on the recursive residuals (1.1).

In the present paper we shall concentrate on the CUSUM M-tests which are based on the statistics

$$\{|\sum_{j=p+1}^{k} W_j|\, \sigma_k^{-1},\ k_0 \leqq k \leqq n\}, \tag{1.3}$$

where σ_k^2 is a consistent estimator of $\int\psi^2(x)\,\mathrm{d}F(x)$. Typically, critical regions of the CUSUM M-tests are of the form:

$$\bigcup_{k=k_0}^{n} \{|\sum_{j=p+1}^{k} W_j|\, \sigma_k^{-1} > w(\alpha, k, n)\}, \tag{1.4}$$

where $w(\alpha, k, n)$, $k = k_0, \ldots, n$, are chosen in such a way that the asymptotic level is α (or $\leqq\alpha$). This test can be described as follows: after the kth ($k_0 \leqq k < n$) observation one compute $|\sum_{j=p+1}^{k} W_j|\,\sigma_k^{-1}$, if $|\sum_{j=p+1}^{k} W_j|\,\sigma_k^{-1} > w(\alpha, k, n)$, one rejects the null hypothesis and stops with observations, in the opposite case one proceeds with observations, for $k = n$, if $|\sum_{j=p+1}^{n} W_j|\,\sigma_k^{-1} > w(\alpha, n, n)$, one rejects the null hypothesis H_0, in the opposite case one accepts H_0.

The critical values $w(\alpha, k, n)$ are not uniquely determined (different arguments lead to different values $w(\alpha, k, n)$). This is discussed in the papers [1], [7]. One should remark that the first CUSUM M-test was introduced and studied by Sen in [11]. He proposed to take $w(\alpha, k, n) = n^{1/2}\, w_\alpha$, where w_α is determined by

$$\mathsf{P}\left(\sup_{t\in(0,1)} |W(t)| < w_\alpha\right) = 1 - \alpha \tag{1.5}$$

with $\{W(t), t \in (0, 1)\}$ being the standardized Wiener process.

Here we shall focus on the case when

$$w(\alpha, k, n) = k^{1/2}\, w(\alpha, n). \tag{1.6}$$

More exactly, we shall study the asymptotic behavior of

$$Z_n = \max_{p<k\leqq n} \{k^{-1/2}|\sum_{j=p+1}^{k} W_j|\}\, (2 \log\log n)^{1/2}\, \sigma^{-1} - \\ - 2\log\log n - (\log\log\log n)/2 + (\log 4\pi)/2, \tag{1.7}$$

where W_j is defined by (1.2) and $\sigma^2 = \int\psi^2(x)\,\mathrm{d}F(x)$, under the null hypothesis and some alternatives. Theorem 2.1 below says that under the null hypothesis and contiguous alternatives

$$\mathsf{P}(Z_n \leqq y) \to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{1.8}$$

which implies that in (1.6)

$$w(\alpha, n) = a(\log n - \log\log(1-\alpha)^{-1/2})\,, \tag{1.9}$$

$$a(t, y) = (y + 2\log t + (\log\log t)/2 - (\log 4\pi)/2)(2\log t)^{-1/2}\,. \tag{1.10}$$

One should remark that the result (1.8) was first proved by Darling and Erdös in [4] for the case W_j/σ, $j = p + 1, \ldots, n$, being i.i.d. random variables with zero mean, unit variance and the finite third moment and later on extended to more general situations, however, they do not cover our ones.

The rest of the paper is organized as follows. Section 2 contains the main assertions on asymptotic behavior of Z_n (Theorem 2.1) under the null hypothesis and the alternative:

$$\begin{aligned} &\mathrm{H}_{1n}(\boldsymbol{\delta}_n)\text{: there exists } \ 1 \leqq m < n \ \text{ such that} \\ &\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_m = \boldsymbol{\theta}_0 \neq \boldsymbol{\theta}_{m+1} - \ldots = \boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \boldsymbol{\delta}_n \\ &m/n \to \lambda \in (0, 1) \quad \text{as} \quad n \to \infty\,. \end{aligned} \tag{1.11}$$

2. MAIN RESULTS

In the present section we shall adapt the following assumptions.

Assumption A. ψ is nondecreasing, $0 < \int|\psi(x)|^3\,\mathrm{d}F(x) < +\infty$, there exist positive constants D_1, D_2 such that

$$\int(\psi(x-a) - \psi(x-b))^2\,\mathrm{d}F(x) \leqq D_2|a-b|^s$$

for $|a| \leqq D_1$, $|b| \leqq D_1$ and some $2 \geqq s \geqq 1$.

Assumption B. The function $\lambda(a) = -\int\psi(x-a)\,\mathrm{d}F(x)$, $a \in \mathbb{R}^1$, fulfils: $\lambda(0) = 0$, there exists the first derivative λ' such that $\lambda'(0) > 0$ and

$$|\lambda'(a) - \lambda'(b)| \leqq D_3|a-b|^r$$

for $|a| \leqq D_4$, $|b| \leqq D_4$ and some $D_3 > 0$, $D_4 > 0$ and $r > 0$.

Assumption C. The regression vectors $\boldsymbol{c}_i = (c_{i1}, \ldots, c_{ip})'$, $i = 1, \ldots, n$, fulfil:

$$n^{-1}\sum_{i=1}^{[nt]} \boldsymbol{c}_i\boldsymbol{c}_i' \to t\boldsymbol{C} \quad \text{as} \quad n \to \infty \quad \text{for} \quad t \in \langle 0, 1\rangle$$

$$\limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n} c_{ij}^2 < +\infty\,, \quad \limsup_{n\to\infty} \max_{1\leqq i\leqq n}\{c_{ij}^2 n^{-1}\log^3 n\} < +\infty\,,$$

$j = 1, \ldots, p$, where $\boldsymbol{C}$ is a positive definite matrix and $[a]$ denotes the integer part of a.

Under mild conditions on F typical ψ-functions fulfil Assumptions A and B, e.g., the Huber ψ-function, $\psi(x) = x$, $x \in \mathbb{R}^1$, ψ being a step function with a finite number of jumps.

The main assertion of the present section is the following:

Theorem 2.1. Let Assumptions A, B, C be satisfied and let $\boldsymbol{\theta}_k(\psi)$, be an estimator of $\boldsymbol{\theta}_0$ based on $X_1, \ldots, X_k$, $p + 1 \leqq k \leqq n$, such that

$$\max_{p<k\leqq k_n} \{\|\boldsymbol{\theta}_k(\psi) - \boldsymbol{\theta}_0\|\} = O_p(1) \quad \text{as} \quad n \to \infty\,, \tag{2.1}$$

$$\max_{k_n\leqq k\leqq n} \{\|\lambda'(0)\, \boldsymbol{C}_k^{1/2}(\boldsymbol{\theta}_k(\psi) - \boldsymbol{\theta}_0) - \boldsymbol{C}_k^{-1/2} \sum_{i=p+1}^{k} \boldsymbol{c}_i \psi(X_i - \boldsymbol{c}_i'\boldsymbol{\theta}_0)\|\, k^v\} = o_p(1)$$
$$\text{as} \quad n \to \infty\,, \tag{2.2}$$

where $\|\cdot\|$ denotes the Euclidean norm for some $v > 0$ and some sequence $\{k_n\}$ such that $k_n^2 = o(\log\log n)$ and $k_n \to \infty$ as $n \to \infty$. Then under both the null hypothesis

$$\mathsf{P}(Z_n \leqq y) \to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{2.3}$$

where Z_n is defined by (1.8), holds true.

Moreover, under the alternative $\mathrm{H}_{n1}(\boldsymbol{\delta}_n)$ (defined by (1.11)) with

$$\|\boldsymbol{\delta}_n\|^{r+1} = o((n \log\log n)^{-1/2}) \quad \text{as} \quad n \to \infty \tag{2.4}$$

the following holds:

$$\mathsf{P}(\max_{p<k\leqq n} \{k^{-1/2}| \sum_{i=p+1}^{k} \psi(X_i - \boldsymbol{c}_i'\boldsymbol{\theta}_{i-1}(\psi)) - \mathrm{I}\{k > m\}\,.$$
$$.\sum_{i=m+1}^{k} \boldsymbol{c}_i'\boldsymbol{C}_{i-1}^{-1}\boldsymbol{C}_m\boldsymbol{\delta}_n\lambda'(0)|\}\,(2\log\log n)^{1/2}\,\sigma^{-1} -$$
$$- 2\log\log n - (\log\log\log n)/2 + (\log 4\pi)/2 \leqq y) \to$$
$$\to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{2.5}$$

where $\mathrm{I}\{A\}$ denotes the indicator of the set A.

Remark 2.1. Reasonable candidates for estimators $\boldsymbol{\theta}_k(\psi)$, $p + 1 \leqq k \leqq n$, are the usual M-estimators, the recursive M-estimators and the stochastic approximation type estimators all generated by the function ψ. From the computational point of view the usual M-estimators are less appealing (because they are defined implicitly and after each observation one must apply an iterative procedure to get the estimator), while the recursive M-estimators are easy to compute, however, one needs a stronger version of Assumption C. The stochastic approximation type estimator possesses property (2.2) (according to Theorem 2.2 and Remark 2.3 in [7]) and relatively easy to compute. Then regarding assumption (2.1) one can use the following estimators: the estimators $\boldsymbol{\theta}_k(\psi)$, $p < k \leqq k_n$, can be chosen arbitrary (close to $\boldsymbol{\theta}_0$), $\{k_n\}$ has the property: $k_n^2 = o(\log\log n)$, $k_n \to \infty$ as $n \to \infty$, as $\boldsymbol{\theta}_{k_n}(\psi)$ can be taken the usual M-estimator (generated by the function ψ) and for $k_n \leqq k < n$

one can use:

$$\boldsymbol{\theta}_{k+1}(\psi) = \boldsymbol{\theta}_k(\psi) + \gamma_k^{-1} \sum_{i=1}^{k+1} \boldsymbol{C}_{k+1}^{-1} \boldsymbol{c}_i' \psi(X_i - \boldsymbol{c}_i'(d)\, \boldsymbol{\theta}_k(\psi)), \quad k_n \leqq k \leqq n\,, \tag{2.6}$$

where

$$\boldsymbol{C}_{k+1} = \sum_{i=1}^{k+1} \boldsymbol{c}_i \boldsymbol{c}_i' \tag{2.7}$$

$$\begin{aligned} \boldsymbol{c}_i(d) &= \boldsymbol{c}_i & \|\boldsymbol{c}_i\| &\leqq d \\ &= \boldsymbol{c}_i d \|\boldsymbol{c}_i\|^{-1} & \|\boldsymbol{c}_i\| &> d \end{aligned} \tag{2.8}$$

$$\begin{aligned} \gamma_k &= \gamma_k^* & a_k \leqq \gamma_k^* &\leqq a_k^{-1} \\ &= a_k & \gamma_k^* &< a_k \\ &= a_k^{-1} & \gamma_k^* &> a_k^{-1} \end{aligned} \tag{2.9}$$

$$\begin{aligned} \gamma_k^* = (2tk^{1/2})^{-1} \sum_{i=1}^{k} (\psi(X_i - \boldsymbol{c}_i'(d)\, \boldsymbol{\theta}_k(\psi) + tk^{-1/2}) - \\ - \psi(X_i - \boldsymbol{c}_i'(d)\, \boldsymbol{\theta}_k(\psi) - tk^{-1/2})) \end{aligned} \tag{2.10}$$

with $t > 0$ fixed, $a_k \searrow 0$, $d > 0$ large.

Remark 2.2. The second part of Theorem 2.1 says that the power of the test described in Section 1 with $w(\alpha, k, n)$ defined by (1.9) for contiguous alternatives converge to α as $n \to \infty$, which is quite unpleasant.

However, if

$$\liminf_{n \to \infty} \|\boldsymbol{\delta}_n\|\, n^{1/2} \log \log n > 0 \tag{2.11}$$

$$\frac{1}{m-k} \sum_{i=m+1}^{k} \boldsymbol{c}_i \to \boldsymbol{c}^* \neq 0 \quad \text{as} \quad m - k \to \infty \tag{2.12}$$

the asymptotic distribution of Z_n under H_0 and $\mathrm{H}_{n1}(\boldsymbol{\delta}_n)$ are different.

Moreover, it can be easily seen that

$$\max_{m \leqq k \leqq n} \left\{ k^{-1/2} \left| \sum_{i=m+1}^{k} \boldsymbol{c}_i' \boldsymbol{C}_{i-1}^{-1} \boldsymbol{C}_m \boldsymbol{\delta}_n - m \boldsymbol{c}^{*\prime} \boldsymbol{\delta}_n \log \frac{k}{m} \right. \right\} = o(1) \text{ as } n \to \infty\,, \tag{2.13}$$

so that

$$\begin{aligned} \omega_n(k) &= k^{-1/2} m \boldsymbol{c}^{*\prime} \boldsymbol{\delta}_n \left(\log \frac{k}{m} \right) \sigma^{-1} & k &> m \\ &= 0 & k &\leqq m \end{aligned} \tag{2.14}$$

represents certain drift corresponding to the alternative $H_{n1}(\boldsymbol{\delta}_n)$.

Finally, we should remark that since (3.2), (3.5), (3.7) and (3.10) the maximum of $k^{-1/2} \left| \sum_{i=p+1}^{k} W_i \right| \sigma_k^{-1}$ over $p < k \leqq n$ can be attained with probability close to 1 and n large only for $\log n < k < n^{\lambda_0}$, for $\lambda_0 \in (0, 1)$.

Theorem 2.1 follows from Theorem 2.2 and Theorem 2.3 below which are of their own importance. Theorem 2.2 extends the Darling-Erdös result [4] to a linear combination of i.i.d. random variables. Theorem 2.3 concerns certain uniform asymptotic linearity results related to M-estimators.

Theorem 2.2. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with zero mean, unit variance and the finite third absolute moment. Let Assumption C be satisfied. Then

$$\mathsf{P}\big(\max_{1 \leqq k \leqq n} \{k^{-1/2}|\sum_{i=1}^{k} (Y_i - c_i' C_{i-1}^{-1} \sum_{j=1}^{i-1} c_j Y_j)|\} \leqq a(\log n, y)\big)$$
$$\to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{2.15}$$

where $a(\log n, y)$ is defined by (1.10).

Theorem 2.3. Let Assumptions A, B, C be satisfied. Let $\boldsymbol{\theta}_i^*$, $p \leqq i \leqq n$, be either nonrandom or Borel functions of $X_1, \ldots, X_i$ satisfying:

$$\max_{p < i \leqq n} \{i^{1/2} \|\boldsymbol{\theta}_{i-1}^* - \boldsymbol{\theta}_0\| \log^{-1} n\} \leqq D_5 \quad \text{for some} \quad D_5 > 0\,. \tag{2.16}$$

Then under the null hypothesis:

1) for each $k^* > 0$ there exists $\varkappa^* > 0$ such that for all $\varkappa > \varkappa^*$ and $n > k^*$

$$\mathsf{P}\big(\max_{k^* \leqq k \leqq n} \{k^{-1/2}|\sum_{i=1}^{k} (\psi(X_i - c_i'\boldsymbol{\theta}_{i-1}^*) - \psi(X_i - c_i'\boldsymbol{\theta}_0) + \lambda'(0)\, c_i'(\boldsymbol{\theta}_{i-1}^* - \boldsymbol{\theta}_0))|\}$$
$$\geqq \varkappa\big) \leqq \varkappa^{-2} k^{*-1/2} (\log k^*)\, D_6 \tag{2.17}$$

for some $D_6 > 0$.

2) there exists $D_7 > 0$ and $\varkappa_0 > 0$ such that for all $\varkappa > \varkappa_0$ and $n > p$

$$\mathsf{P}\big(\max_{p < k \leqq n} \{k^{-1/2}|\sum_{i=1}^{k} (\psi(X_i - c_i'\boldsymbol{\theta}_{i-1}^*) - \psi(X_i - c_i'\boldsymbol{\theta}_0) + \lambda'(0)\, c_i'(\boldsymbol{\theta}_{i-1}^* - \boldsymbol{\theta}_0))|\}$$
$$\geqq \varkappa\big) \leqq \varkappa^{-2} D_7. \tag{2.18}$$

3. PROOFS OF THEOREMS

Proof of Theorem 2.1. Let us start with the null hypothesis. Define $\{\boldsymbol{\theta}_k^*\}_k$ as follows:

$$\begin{aligned} \boldsymbol{\theta}_k^* &= \boldsymbol{\theta}_k && p < k \leqq k_n \\ &= \boldsymbol{\theta}_k(\psi) && \text{if} \quad \|\boldsymbol{C}_k^{1/2}(\boldsymbol{\theta}_k(\psi) - \boldsymbol{\theta}_0)\| \leqq D_5\,, \quad k_n < k \leqq n\,, \\ &= \boldsymbol{\theta}_k^0 && \text{of} \quad \|\boldsymbol{C}_k^{1/2}(\boldsymbol{\theta}_k(\psi) - \boldsymbol{\theta}_0)\| > D_5\,, \quad k_n < k \leqq n\,, \end{aligned} \tag{3.1}$$

where $\boldsymbol{\theta}_k^0$ is an arbitrary point from $\{\boldsymbol{\theta};\ \|\boldsymbol{C}_k^{1/2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)\| \leqq D_5\}$ (D_5 is a constant from Theorem 2.3). Then due to the assumptions (2.1) and (2.2) one observes that

$$\mathsf{P}\big(\max_{k_n \leqq k \leqq n} \{k^{-1/2}|\sum_{i=p+1}^{k} \psi(X_i - c_i'\boldsymbol{\theta}_{i-1}(\psi))| \geqq (\log\log n)^{-1}\big) -$$

$$- \mathsf{P}\big(\max_{k_n \leqq k \leqq n} \{k^{-1/2} | \sum_{i=p+1}^{k} \psi(X_i - c_i'\theta_{i-1}^*)| \geqq (\log\log n)^{-1}\big) -$$

$$\to 0 \quad \text{as} \quad n \to \infty\,, \tag{3.2}$$

which means that it is suffices to prove (2.3) with $\theta_{i-1}(\psi)$ replaced by θ_{i-1}^*.

Applying Theorem 2.2 with $Y_i = \psi(X_i - c_i'\theta_0)$ and n replaced by $n_0 = \log n$ one receives

$$\mathsf{P}\big(\max_{p<k \leqq n_0} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta_0) - c_i'C_{i-1}^{-1} \sum_{j=p+1}^{i-1} c_j \psi(X_j - c_j'\theta_0))|\} \leqq$$

$$\leqq a(\log n_0, y)\big) \to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n_0 \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{3.3}$$

where $a(m, y)$ is defined by (1.10). Clearly,

$$\frac{a(\log n_0, y)}{a(\log n, y)} \to 0 \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,, \tag{3.4}$$

which in combination with Theorem 2.3.2) and (3.1) implies that for any $\varepsilon > 0$ there exists n_ε such that for all $n \geqq n_\varepsilon$

$$\mathsf{P}\big(\max_{p<k \leqq n_0} \{k^{-1/2} | \sum_{i=p+1}^{k} \psi(X_i - c_i'\theta_{i-1}(\psi)| > a(\log n, y)\big)\} < \varepsilon\,. \tag{3.5}$$

Next, applying Theorem 2.3.1) with $k^* = n_0 = \log n$, $\varkappa = \log^{-\alpha} n$, $0 < \alpha < r/4$, one observes

$$(\log n)^{\alpha/2} \max_{n_0 \leqq k \leqq n} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0) +$$

$$+ \lambda'(0)\, c_i'(\theta_{i-1}^* - \theta_0))|\} = o_p(1) \quad \text{as} \quad n \to \infty \tag{3.6}$$

which together with (3.1) and the assumption (2.2) ensures that

$$(\log n)^{\beta} \max_{n_0 \leqq k \leqq n} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0) +$$

$$+ c_i'C_{i-1}^{-1} \sum_{j=p+1}^{i-1} c_j' \psi(X_j - c_i'\theta_0))|\} = o_p(1) \quad \text{as} \quad n \to \infty \tag{3.7}$$

for some $\beta > 0$. This relation together with Theorem 2.2 with $Y_i = \psi(X_i - c_i'\theta_0)$, $n_0 \leqq i \leqq n$, implies

$$\mathsf{P}\big(\max_{n_0 \leqq k \leqq n} \{k^{-1/2} | \sum_{i=1}^{k} \psi(X_i - c_i'\theta_{i-1}^*)|\} \leqq a(\log n, y)\big)$$

$$\to \exp\{-2\exp\{-y\}\} \quad \text{as} \quad n \to \infty\,, \quad y \in \mathbb{R}^1\,. \tag{3.8}$$

The assertion of Theorem 2.1 under H_0 can be easily concluded from (3.2), (3.5) and (3.8).

As for the alternative (1.11), one realises that (2.2), (3.5), (3.8) holds true even in this case. Consequently, the asymptotic distribution of Z_n is the same as that of

$$\max_{n_0 \leq k \leq n} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta_0) - c_i' C_{i-1} \sum_{j=p+1}^{i-1} c_j \psi(X_j - c_j'\theta_0))|\} \, .$$

$$. (2 \log \log n)^{1/2} - 2 \log \log n - (\log \log \log n)/2 + (\log 4\pi)/2 \, . \tag{3.9}$$

Next, one should notice that by the Chow inequality for all $0 < \lambda_1 < \lambda_2 \leqq 1$

$$\max_{[n\lambda_1] \leq k \leq [n\lambda_2]} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta_0) - \mathsf{E}\psi(X_i - c_i'\theta_0) -$$

$$- c_i' C_{i-1}^{-1} \sum_{j=p+1}^{i-1} c_j (\psi(X_j - c_j'\theta_0) - \mathsf{E}\psi(X_i - c_i'\theta_0)))|\} = O_p(1) \tag{3.10}$$

as $n \to \infty$, and, moreover,

$$\max_{[n\lambda_1] \leq k \leq [n\lambda_2]} \{k^{-1/2} | \sum_{i=p+1}^{k} \mathsf{E}(\psi(X_i - c_i'\theta_0) - c_i' C_{i-1} \sum_{j=p+1}^{i-1} c_j \psi(X_j - c_j'\theta_0))|\} =$$

$$= \max_{[n\lambda_1] \leq k \leq [n\lambda_2]} \{k^{-1/2} | \sum_{i=m+1}^{k} (\lambda(-c_i'\delta_n n^{-1/2}) - c_i' C_{i-1}^{-1} \sum_{j=m+1}^{i-1} c_j \lambda(-c_j'\delta_n n^{-1/2}))|\}$$

$$= \max_{[n\lambda_1] \leq k \leq [n\lambda_2]} \{k^{-1/2} | \sum_{i=m+1}^{k} (c_i' - c_i' C_{i-1}^{-1} \sum_{j=m+1}^{i-1} c_j c_j') \, \delta_n \, n^{-1/2} \lambda'(0)| \, (1 + o(1))\}$$

$$= O(1) \quad \text{as} \quad n \to \infty \, . \tag{3.11}$$

The last two relations imply that both under H_0 and $\mathrm{H}_{n1}(\delta_n)$:

$$\max_{[n\lambda_1] \leq k \leq [n\lambda_2]} \{k^{-1/2} | \sum_{i=p+1}^{k} (\psi(X_i - c_i'\theta) - c_i' C_{i-1}^{-1} \sum_{j=p+1}^{i-1} c_j \psi(X_j - c_j'\theta_0))|\}$$

$$= O_p(1) \, , \quad \text{as} \quad n \to \infty \, , \tag{3.12}$$

which easily implies that the asymptotic distribution of Z_n is the same under both H_0 and $H_{n1}(\delta_n)$. □

Proof of Theorem 2.2. The proof can follow the same line as that of the Darling-Erdös theorem which is based on a suitable application of the Berry-Esseen theorem and the law of iterated logarithm.

Going carefully through the proof of Darling-Erdös theorem [4] we find that, in our case, it suffices to show:

(i) $\operatorname{var}(S_k - S_\nu) = k - \nu + \sum_{i=\nu+1}^{k} c_i' C_{i-1}^{-1} c_i \, , \quad p < \nu < k \leqq n \, ,$

where

$$S_k = \sum_{i=1}^{k} (Y_i - c_i' C_{i-1}^{-1} \sum_{j=1}^{i-1} c_j Y_j) \, , \quad k \geqq 1 \, ;$$

(ii) $\sup_x |\mathsf{P}(S_k < x \sqrt{(\operatorname{var} S_k)}) - \Phi(x)| \leqq Q_1 k^{-1/2} \, , \quad k \geqq 1 \, , \quad x \in \mathbb{R}^1$

for some constant $Q_1 > 0$ (not depending on k), Φ is the distribution function of $N(0, 1)$ (the Berry-Esseen bound);

(iii) $\limsup_{k\to\infty} \frac{|S_k|}{(k \log\log k)^{1/2}} < +\infty$ a.s.

(the law of iterated logarithm).

The property (i) follows by direct computations.

Now, we turn to (iii), by Theorem 10.2.2 in Petrov [10], if for arbitrary $\varepsilon > 0$

$$\limsup_{k\to\infty} \left(\sum_{i=1}^{k} c_{ij}^2\right)^{-1} \sum_{k=k_0}^{k} \int \mathrm{I}\{c_{ij}^2 x^2 \geqq \varepsilon^2 \sum_{s=1}^{k} c_{sj}^2 / \log\log\left(\sum_{s=1}^{k} c_{sj}^2\right)\} \, . \, c_{ij}^2 x^2 \, \mathrm{d}F(x) < +\infty , \quad j = 1, \ldots, p , \tag{3.13}$$

and

$$\sum_{k=k_0}^{\infty} \left(\sum_{s=1}^{k} c_{sj}^2 \log\log\left(\sum_{q=l}^{k} c_{qj}^2\right)\right)^{-1} \int \mathrm{I}\{c_{kj}^2 x^2 \geqq \varepsilon^2 \sum_{s=1}^{k} c_{sj}^2 / \log\log\left(\sum_{q=1}^{k} c_{qj}^2\right)\} \, . \, c_{kj}^2 x^2 \, \mathrm{d}F(x) < +\infty , \quad j = 1, \ldots, p , \tag{3.14}$$

for some k_0, where $\mathrm{I}\{A\}$ denotes the indicator of a set A, F is the distribution function of Y_i is fulfilled, then

$$\limsup_{k\to\infty} \left|\sum_{i=1}^{k} c_{ij} Y_i\right| \left(\sum_{i=1}^{k} c_{ij}^2 \log\log\left(\sum_{i=1}^{k} c_{ij}^2\right)\right)^{-1/2} = 1 \quad \text{a.s.} \tag{3.15}$$

$j = 1, \ldots, p$. Validity of (3.13) and (3.14) is easily implied by the following three inequalities:

$$\int \mathrm{I}\{c_{ij}^2 x^2 \geqq \varepsilon^2 \sum_{s=1}^{k} c_{sj}^2 / \log\log\left(\sum_{q=1}^{k} c_{qj}^2\right)\} \, c_{ij}^2 x^2 \, \mathrm{d}F(x) \leqq$$
$$\leqq \varepsilon^{-1} |c_{ij}|^3 \, \mathsf{E}|Y_1|^3 \left(\log\log\left(\sum_{q=1}^{k} c_{qj}^2\right)\right)^{1/2} \left(\sum_{s=1}^{k} c_{sj}^2\right)^{-1/2} , \tag{3.16}$$

$$k^{-1} \sum_{i=k_0}^{k} |c_{ij}|^3 (\log\log i)^{1/2} \, i^{-1/2} \leqq k^{-1} \left(\sum_{i=k_0}^{k} c_{ij}^4\right)^{3/4} \left(\sum_{i=k_0}^{k} (i^{-1} \log\log i)^2\right)^{1/4} \leqq$$
$$\leqq Q_2 k^{-1/4} , \tag{3.17}$$

$$\sum_{i=k_0}^{\infty} i^{-3/2} |c_{ij}|^3 (\log\log i)^{-1/2} \leqq \tfrac{2}{3} (\log\log k_0)^{-1/2} \sum_{i=k_0}^{\infty} |c_{ij}|^3 \sum_{s=i}^{\infty} i^{-5/2} \leqq$$
$$\leqq Q_3 (\log\log k_0)^{-1/2} \sum_{s=k_0}^{\infty} s^{-3/2} < +\infty \tag{3.18}$$

for some $Q_2 > 0$, $Q_3 > 0$, $k_0 > 0$. Hence (3.15) is true. By Assumption C, the relation (3.15) can be rewritten as follows:

$$\limsup_{k\to\infty} \left|\sum_{i=1}^{k} c_{ij} Y_i\right| (c_{ij}^2 k \log\log k)^{-1/2} = 1 \, . \quad \text{a.s.} \quad j = 1, \ldots, p , \tag{3.19}$$

where $0 < c_j^2 = \lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} c_{ij}^2$, which immediately implies

$$\limsup_{k\to\infty} \left|\sum_{i=1}^{k} \mathbf{c}_i' \mathbf{C}_{i-1}^{-1} \sum_{j=1}^{i-1} \mathbf{c}_j Y_j\right| (k \log\log k)^{-1/2} \leqq$$

$$\leqq Q_4 \limsup_{k\to\infty} \sum_{i=1}^{k} \|c_i\| \cdot \|C_{i-1}^{-1}\| \,(i \log\log i)^{1/2}\,(k \log\log k)^{-1/2} \leqq$$

$$\leqq Q_5 \limsup_{k\to\infty} \sum_{i=1}^{k} \|c_i\|\, i^{-1/2} k^{-1/2} < +\infty\,. \tag{3.20}$$

Hence (iii) is proved.

It remains to show (ii). Since S_k is the sum of independent random variables one has the Berry-Esseen bound

$$\sup_x |\mathrm{P}(S_k < (\mathrm{var}\, S_k)^{1/2}) - \Phi(x)| \leqq$$

$$\leqq Q_6 \sum_{i=1}^{k} \mathrm{E}|Y_i|^3 \cdot |1 - c_i' \sum_{j=i+1}^{k} C_{j-1}^{-1} c_j|^3 (\mathrm{var}\, S_k)^{-3/2}, \; k \geqq 1,\; x \in \mathbb{R}^1 \tag{3.21}$$

for some $Q_6 > 0$. Thus to obtain (iii) it remains (due to (ii)) to show that

$$\sum_{i=1}^{k} |1 - c_i' \sum_{j=i+1}^{k} C_{j-1}^{-1} c_j|^3 \leqq Q_7 k \tag{3.22}$$

for some $Q_7 > 0$. Clearly, the left hand side of (3.22) is smaller or equaled to

$$k + 3 \sum_{i=1}^{k} \sum_{j=i+1}^{k} |c_i' C_{j-1}^{-1} c_j| + 3 \sum_{i=1}^{k} \|c_i\|^2 \Big(\sum_{j=i+1}^{k} \|C_{j-1}^{-1} c_j\|\Big)^2 +$$

$$+ \sum_{i=1}^{k} \|c_i\|^3 \Big(\sum_{j=i+1}^{k} \|C_{j-1}^{-1} c_j\|\Big)^3 . \tag{3.23}$$

Applying some elementary inequalities and Assumption C one receives

$$\sum_{i=1}^{k} \|c_i\|^3 \Big(\sum_{j=i+1}^{k} \|C_{j-1}^{-1} c_j\|\Big)^3 = 6 \sum_{i=1}^{k} \|c_i\|^3 \sum\sum\sum_{i<j_1<j_2<j_3\leqq k} \|C_{j_1}^{-1} c_{j_1}\| \cdot$$

$$\|C_{j_2}^{-1} c_{j_2}\| \, \|C_{j_3}^{-1} c_{j_3}\| + 3 \sum_{i=1}^{k} \|c_i\|^3 \sum\sum_{i<j_1<j_2\leqq k} (\|C_{j_1}^{-1} c_{j_1}\|^2 \|C_{j_2}^{-1} c_{j_2}\| +$$

$$+ \|C_{j_1}^{-1} c_{j_1}\| \, \|C_{j_2}^{-1} c_{j_2}\|^2) + \sum_{i=1}^{k} \|c_i\|^3 \sum_{j=i+1}^{k} \|C_j^{-1} c_j\|^3 \leqq$$

$$\leqq Q_8 \Big(\sum\sum\sum\sum_{1\leqq i<j_1<j_2<j_3\leqq k} \|c_i\|^3 \prod_{\nu=1}^{3} j_\nu^{-1} \|c_{j_\nu}\| +$$

$$+ \sum\sum\sum_{1\leqq i<j_1<j_2\leqq k} \|c_i\|^3 \prod_{\nu=1}^{2} j_\nu^{-1} \|c_{j_\nu}\| \,(j_1^{-1} \|c_{j_1}\| + j_2^{-1} \|c_{j_2}\|) +$$

$$+ \sum\sum_{1\leqq i<j\leqq k} \|c_i\|^3 j^{-3} \|c_j\|^3 \Big) \leqq Q_9 k\,. \tag{3.24}$$

The other terms in (3.23) can be treated in the same way. They are also bounded from above by k multiplied by a constant (not depending on k). Hence the inequality (3.22) is true. The assertion (ii) is proved. □

Proof of Theorem 2.3. To be brief we prove only (2.17) for (2.18) can be proved in the same way.

Since

$$\{\sum_{i=1+p}^{k}(\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0) + \lambda(c_i'(\theta_{i-1}^* - \theta_0)),\ p+1 \leqq k \leqq n\}$$

is a martingale one has by the Chow inequality:

$$\mathsf{P}\big(\max_{k^*\leqq k\leqq n}\{k^{-1/2}|\sum_{i=p+1}^{k}(\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0) + \lambda(c_i'(\theta_{i-1}^* - \theta_0)))|\} \geqq$$

$$\geqq \varkappa) \leqq \varkappa^{-2}\{\sum_{k=k^*}^{n-1}(k^{-1} - (k+1)^{-1})\sum_{i=p+1}^{k}\mathsf{E}(\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0)))^2 +$$

$$+ n^{-1}\sum_{i=p+1}^{n}\mathsf{E}(\psi(X_i - c_i'\theta_{i-1}^*) - \psi(X_i - c_i'\theta_0))^2\} \leqq$$

$$\leqq Q_{10}\varkappa^{-2}\sum_{k=k^*}^{n-1}k^{-2}\sum_{i=p+1}^{k}\|c_i\|^r i^{-r/2}\log^r i + n^{-1}\sum_{i=p+1}^{n}\|c_i\|\, i^{-r/2}\log^r i \leqq$$

$$\leqq Q_{11}\varkappa^{-2}k^{*-r/2}\log^r k^*. \tag{3.25}$$

Next, by Assumption B

$$|\lambda(c_i'(\theta_{i-1}^* - \theta_0)) - c_i'(\theta_{i-1}^* - \theta_0)\lambda'(0)| \leqq \int_0^{c_i'(\theta^*_{i-1}-\theta_0)}|\lambda'(y) - \lambda'(0)|\,\mathrm{d}y$$

$$\leqq D_3|c_i'(\theta_{i-1}^* - \theta_0)|^{r+1}(r+1)^{-1}$$

$$\leqq D_3 Q_{12}\|c_i\|^{r+1}((i-1)^{-1/2}\log(i-1))^{r+1} \tag{3.26}$$

which implies

$$\max_{k^*\leqq k\leqq n}\{k^{-1/2}|\sum_{i=p+1}^{k}(\lambda(c_i'(\theta_{i-1}^* - \theta_0)) - \lambda'(0)\,c_i'((\theta_{i-1}^* - \theta_0))|\} \leqq$$

$$\leqq Q_{13}\max_{k^*\leqq k\leqq n}\{k^{-1/2}\sum_{i=p+1}^{k}\|c_i\|^{r+1}((i-1)^{-1/2}\log(i-1))^{r+1}\} \leqq$$

$$\leqq Q_{14}k^{*-r/2}(\log k)^{r+1}. \tag{3.27}$$

Taking into account the assumptions, the needed assertion follows from (3.27) and (3.25). □

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