ON EXPONENTIALLY DISCOUNTED ADAPTIVE CONTROL*

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A family of least squares estimates that are obtained from an exponentially discounted quadratic functional is investigated when the unknown parameters in a linear stochastic system are periodic functions. Some asymptotic properties of the family of estimates and a linear feedback control are given when the discount rate tends to zero.

1. INTRODUCTION

In many applications of adaptive control it may be clear if the unknown parameters are constants or are time varying functions. It may be especially difficult to determine if some of the parameters are constants or slowly varying functions of time. To determine parameter variations it is necessary to "forget" the past of the state. The approach of exponential discounting of past information has been studied for some practical applications ([1], [2]) and has often been used in other control problems. If it is unclear whether the unknown parameters are constants or time varying, then an estimator must compromise between the conflicting goals of accuracy in the case of constant parameters and response to changes in the parameters in the case of time varying parameters. In this paper a least squares estimator with exponential discounting is formed. Properties of this estimator are investigated for the case where the parameters are periodic functions and the rate of the exponential discounting approaches zero. This asymptotic behaviour of the estimator provides a quantitative way to compromise between the possibilities of constant or periodic parameters via discounting. The model that is considered here is a linear stochastic differential equation where the unknown parameters appear affinely in the drift. The asymptotic distribution of the estimator is obtained for an identification problem (where the feedback control is fixed) as the discount rate approaches zero. For an

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adaptive control problem where the feedback gain is a function of the parameters an asymptotic bound for the difference of the feedback gain depending on the discounted least squares estimate and a nonrandom function is given when the discount rate approaches zero. In addition the differences between a periodic quadratic (cost) functional and its "average" is estimated in terms of the discount rate.

2. PRELIMINARIES

The stochastic system is modelled by the following linear stochastic differential equation

$$\mathrm{d}X(t) = (f(a) + gk)\,X(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{1}$$

where $X(t) \in \mathbb{R}^n$, $g \in L(\mathbb{R}^m, \mathbb{R}^n)$, $k \in L(\mathbb{R}^n, \mathbb{R}^m)$, $(W(t), t \in \mathbb{R})$ is an n-dimensional Brownian motion with infinitesimal variance h and $f(a) \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$f(a) = f_0 + \sum_{i=1}^{q} a^i f_i \tag{2}$$

and $a = (a^1, \ldots, a^q)$ and k are fixed. Some conditions on the unknown parameter vector a will be specified subsequently.

From the second method of Lyapunov it is well known that $f(a) + gk$ is stable if and only if there is a $z > 0$ such that

$$z(f(a) + gk) + (f(a) + gk)'\,z + I \leqq 0\,. \tag{3}$$

If $f(a) + gk$ is stable then the stationary distribution is $N(0, v)$ where v satisfies

$$(f(a) + gk)\,v + v(f(a) + gk)' + h = 0\,. \tag{4}$$

If $r \in L(\mathbb{R}^n, \mathbb{R}^n)$ is symmetric then

$$\mathsf{E}_s\, X'(t)\, r\, X(t) = \operatorname{trace}(vr)$$

where E_s is expectation with respect to the stationary distribution. A dual way of obtaining $\mathsf{E}_s\, X'(t)\, r\, X(t)$ is to solve

$$w(f(a) + gk) + (f(a) + gk)'\,w + r = 0 \tag{5}$$

so that

$$\mathsf{E}_s\, X'(t)\, r\, X(t) = \operatorname{trace}(wh)\,.$$

Consider that a is a parameter that is unknown to the controller with true value α. Assume that for each a there is a desirable feedback gain $k(a)$ so that the system

$$\mathrm{d}X(t) = (f(a) + gk(a))\,X(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{6}$$

has some desirable properties such as pole placement or optimal stationary control. Let $\mathscr{K} \subset L(\mathbb{R}^n, \mathbb{R}^m)$ be the family of admissible feedback gains. Let $\mathscr{A} \subset \mathbb{R}^q$ be the set of possible values of a. To have systems with a stability property the following global Lyapunov condition is imposed.

Assumption 1. Assume that $\mathscr{A}$ and $\mathscr{K}$ are closed and bounded subsets of $\mathbb{R}^q$ and $L(\mathbb{R}^n, \mathbb{R}^m)$ respectively and that there is a $z > 0$ such that the inequality (3) is satisfied for all $a \in \mathscr{A}$ and $k \in \mathscr{K}$.

It is often convenient to express explicitly the dependence of v on a and k as $v(a, k)$.

If α is the true value of the unknown parameter vector a then the controller computes an estimate $\alpha^*(t)$ of α from the past trajectory of the system and forms the feedback gain $k(\alpha^*(t))$. The feedback control

$$U(t) = k(\alpha^*(t))\,X(t)$$

has the self-tuning property if $k(\cdot)$ is continuous and $\alpha^*(t) \to \alpha$ a.s. as $t \to \infty$.

If the observation started at time S then a least squares estimate at time $T > S$ is determined by minimizing the formal expression

$$\textstyle\int_S^T (\dot{X}(t) - f(a)\,X(t) - g\,U(t))'\,l\,(\dot{X}(t) - f(a)\,X(t) - g\,U(t))\,\mathrm{d}t \tag{7}$$

where $l \in L(\mathbb{R}^n, \mathbb{R}^n)$ is positive semidefinite. The undefined term $\int_S^T \dot{X}'(t)\,l\,\dot{X}(t)\,\mathrm{d}t$ is treated as a constant with respect to a and the other terms with $\dot{X}(t)$ occur as $\dot{X}(t)\,\mathrm{d}t$ which is rewritten as $\mathrm{d}X(t)$.

A natural necessary condition on (1) to obtain consistent estimates is the following.

Assumption 2. The family of matrices $(\sqrt{(l)}\,f_i\,\sqrt{(h)},\ i = 1, 2, \ldots, q)$ are linearly independent where $\sqrt{l}$ and $\sqrt{h}$ are the symmetric square roots of l and h respectively.

Some conditions for the strong consistency of this least squares estimator, that is, $\alpha^*(T) \to \alpha$ a.s. as $T \to \infty$, can be found in [3], [6].

Since it is often desirable to "forget" the past observations in the family of estimates of the unknown parameters an exponential discount factor is introduced in (7) and furthermore we let $S \to -\infty$ so that the formal expression to be minimized for the least squares estimate at time T given the infinite past history of the process is

$$\textstyle\int_{-\infty}^T \mathrm{e}^{\lambda t}(\dot{X}(t) - f(a)\,X(t) - g\,U(t))'\,l\,(\dot{X}(t) - f(a)\,X(t) - g\,U(t))\,\mathrm{d}t\,. \tag{8}$$

For this minimization it suffices to equate the partial derivatives of a^i for $i = 1, \ldots, q$ to zero. One obtains the system of equations

$$\sum_{j=1}^{q} \textstyle\int_{-\infty}^T \mathrm{e}^{\lambda t}\, X' f_i' l f_j X\,\mathrm{d}t\,\alpha^{*j}(T) =$$

$$= \textstyle\int_{-\infty}^T \mathrm{e}^{\lambda t}\, X' f_i' l(\mathrm{d}X(t) - f_0\,X(t)\,\mathrm{d}t - g\,U(t)\,\mathrm{d}t) \tag{9}$$

for $i = 1, 2, \ldots, q$. The dependence of X on t has been suppressed in the integrands for notational simplicity. This will often be done in this paper.

Since there is a trade-off between the accuracy of the estimator for constant parameters and the ability of the estimator to determine parameter changes we investigate the behavior of the estimator and the adaptive controls as the discount rate $\lambda \downarrow 0$. Let us define the true value of the parameter as a function that evaluated at time t is

$$\alpha(t) = a(\lambda t) \tag{10}$$

for $t \in (-\infty, \infty)$. The following conditions are imposed on $a(\cdot)$.

Assumption 3. The function $a(\cdot)$ is a periodic, piecewise continuously differentiable function mapping $\mathbb{R}$ into $\mathscr{A}$. The period of $a(\cdot)$ is $\tau > 0$.

The process $(X(t),\ t \in \mathbb{R})$ that satisfies the equation

$$\mathrm{d}X(t) = \big(f(\alpha(t)) + gk(\alpha^*(t))\big)\, X(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{11}$$

depends on λ by (9), (10).

Using (1), the equation for the estimator (9) can be rewritten as

$$\int_{-\infty}^{T} \mathrm{e}^{\lambda t}\, Q(t)\,(\alpha^*(T) - \alpha(t))\,\mathrm{d}t = \int_{-\infty}^{T} e^{\lambda t}\, L(t)\,\mathrm{d}W(t) \tag{12}$$

where $Q(t) = (X'(t) f_i' l f_j X(t))$ for $i, j \in \{1, \ldots, q\}$ and $L(t) = (X'(t) f_1' l, \ldots, X'(t) f_q' l'$. Let $(Y(T),\ T \in \mathbb{R})$ be the process defined by the equation

$$Y(T) = \int_{-\infty}^{T} \mathrm{e}^{\lambda(t-T)}\, Q(t)\,\mathrm{d}t\,. \tag{13}$$

Using (1), (12) it is easily seen that $(Y(t),\ \alpha^*(t),\ t \in \mathbb{R})$ satisfy the stochastic differential equations

$$\mathrm{d}Y(t) = Q(t)\,\mathrm{d}t - \lambda\, Y(t)\,\mathrm{d}t \tag{14}$$

$$\mathrm{d}\alpha^*(t) = Y^{-1}(t)\, Q(t)\,(\alpha(t) - \alpha^*(t))\,\mathrm{d}t + Y^{-1}(t)\, L(t)\,\mathrm{d}W(t)\,. \tag{15}$$

It is straightforward to verify the existence and uniqueness of the solution of the stochastic differential equations using the assumptions on $k(.)$ and the positivity of $(Y(t),\ t \in \mathbb{R})$. From this construction of solutions it is immediate that $(X(t),\ \alpha^*(t),\ Y(t);\ t \in \mathbb{R})$ is a Markov process with periodic transition probabilities. It will be assumed that $(X(t),\ \alpha^*(t),\ Y(t);\ t \in \mathbb{R})$ is in a periodic state, that is, its family of finite dimensional distributions are invariant with respect to the shift of magnitude τ/λ.

Clearly the results about the family of least squares estimates $(\alpha^*(t),\ t \in \mathbb{R})$ are more complete if there is no interaction between estimation and control, that is,

$$k(a) = k_0 \tag{16}$$

than if there is interaction between estimation and control. To apply the results to adaptive control it is assumed that the feedback gain is close to k_0, that is,

$$k(a) = k_0 + \varepsilon\, j(a) \tag{17}$$

where ε is a small parameter. Some conditions are imposed on $j(\cdot)$.

Assumption 4. The function $j\colon \mathbb{R}^q \to L(\mathbb{R}^n, \mathbb{R}^m)$ is bounded and Lipschitz continuous. For $\varepsilon > 0$ and $\lambda > 0$ sufficiently small there is a periodic state of $(X(t),\ \alpha^*(t),\ t \in \mathbb{R})$ such that $\mathsf{E}[\exp(p|X_0|^2)]$ is bounded for some $p > 0$.

Some verifiable conditions to ensure the validity of the statements in Assumption 4 can be obtained by combining some results on stationary distributions of Markov processes (e.g. [4], [7]) with some perturbation techniques.

3. STATEMENT OF THE MAIN RESULTS

Proposition 1. Let (16) hold and let

$$\bar{a}(y) = \left(\int_{-\infty}^{y} e^{s}\, \theta(s, k_0)\, ds\right)^{-1} \int_{-\infty}^{y} e^{s}\, \theta(s, k_0)\, a(s)\, ds \tag{18}$$

where $\theta(y, k) = (\text{trace}\,(v(a(y), k) f_i' l f_j))$ and $i, j \in \{1, \ldots, q\}$. For $\lambda \downarrow 0$

$$(\alpha^*(T/\lambda) - \bar{a}(T))/\sqrt{\lambda}$$

has asymptotically a normal distribution with zero mean and covariance matrix

$$V(T) = \bar{\theta}^{-1}(T) \int_{-\infty}^{T} e^{2s}\, \Delta(T, s)\, ds\, \bar{\theta}^{-1}(T) \tag{19}$$

where

$$\bar{\theta}(T) = \int_{-\infty}^{T} e^{s}\, \theta(s, k_0)\, ds\,, \tag{20}$$

$$\Delta(T, s) = (\text{trace}\,(b_i'(T, s)\, h\, b_j(T, s)\, v(a(s), k_0))) \tag{21}$$

for $i, j \in \{1, \ldots, q\}$ and

$$b_i(T, s) = 2 \sum_{j=1}^{q} (a^j(s) - \bar{a}^j(T))\, w_{ij}(s) + l f_i \tag{22}$$

$$w_{ij}(s)\,(f(a(s)) + g k_0) + (f(a(s)) + g k_0)'\, w_{ij}(s) + f_i' l f_j = 0\,. \tag{23}$$

Let $(F(t, s), (t,s) \in \mathbb{R}^2)$ be the fundamental solution of the matrix equation

$$\frac{d}{dt} F(t, s) = (f(a(t)) + g k_0)\, F(t, s)\,, \quad F(s, s) = I\,.$$

The solution of (1) at time t, $X(t)$, in the periodic state has a normal distribution with zero mean and covariance matrix

$$\mathsf{E} X(t)\, X'(t) = \int_{-\infty}^{t} F(t, s)\, h\, F'(t, s)\, ds\,.$$

By Assumption 1 it is well known that

$$\|F(t, s)\| \leqq c\, e^{-\gamma(t-s)}$$

where $\gamma > 0$ and $c \in \mathbb{R}_+$. The symbol c will be used subsequently as a generic finite positive constant.

Proposition 2. There is an $\bar{\varepsilon} > 0$ such that for $\varepsilon \in (0, \bar{\varepsilon})$ in (17) the following are satisfied:

i) There is a unique periodic solution $\bar{a}(\cdot)$ of

$$\bar{a}(y) = \left(\int_{-\infty}^{y} e^{s}\, \theta(s, k(\bar{a}(s)))\, ds\right)^{-1} \int_{-\infty}^{y} e^{s}\, \theta(s, k(\bar{a}(s)))\, a(s)\, ds \tag{24}$$

for $y \in \mathbb{R}$ and

ii) For $\bar{k}(y) = k(\bar{a}(y))$ and the discount rate $\lambda > 0$ sufficiently small the inequality

$$\mathsf{E}|k(\alpha^*(T)) - \bar{k}(\lambda T)|^2 \leqq c\lambda \tag{25}$$

is satisfied.

To evaluate the accuracy of the least squares estimate when the parameters are constants, that is, $a(y) \equiv \alpha$ the following proposition is useful.

Proposition 3. Let $a(y) \equiv \alpha$. Then there is an $\bar{\varepsilon} > 0$ such that for $\varepsilon \in (0, \bar{\varepsilon})$ in (17)

$$(\alpha^*(T/\lambda) - \alpha)/\sqrt{\lambda}$$

has asymptotically as $\lambda \downarrow 0$ a normal distribution with zero mean and covariance matrix

$$V = \bar{\Theta}^{-1}\bar{\Delta}\bar{\Theta}^{-1} \tag{26}$$

where

$$\bar{\Theta} = (\operatorname{trace}(v(\alpha, k(\alpha)) f_i' l f_j)) \tag{27}$$

$$\bar{\Delta} = \tfrac{1}{2}(\operatorname{trace}(v(\alpha, k(\alpha)) f_i' l h l f_j)) \tag{28}$$

for $i, j \in \{1, \ldots, q\}$.

For time varying parameters Proposition 2 allows one also to estimate the quadratic cost. Consider the average cost over one period which is

$$C(\lambda) = \frac{\lambda}{\tau} \int_0^{\tau/\lambda} (X'(t)\, r\, X(t) + |U(t)|^2)\, \mathrm{d}t \tag{29}$$

and define $\gamma(y)$ by the equation

$$\gamma(y) = \operatorname{trace}(v(a(y), \bar{k}(y))(r + \bar{k}(y)\, \bar{k}'(y)))$$

Proposition 4. There is an $\bar{\varepsilon} > 0$ such that for $\varepsilon \in (0, \bar{\varepsilon})$ in (17) the following inequality is satisfied

$$\mathsf{E}\left|C(\lambda) - \frac{1}{\tau} \int_0^\tau \gamma(y)\, \mathrm{d}y\right| \leqq c\sqrt{\lambda}\,. \tag{30}$$

4. PROOFS

Lemma 1. For $p > 0$ and the discount rate $\lambda > 0$ sufficiently small the following inequalities are satisfied

$$\mathsf{E}|X(0)|^p \leqq c/\lambda \tag{31}$$

$$\int_{-\infty}^0 \mathrm{e}^{\lambda t}\, \mathsf{E}|X(t)|^p\, \mathrm{d}t \leqq c/\lambda\,. \tag{32}$$

Proof. Let z be the solution of (3) as mentioned in Assumption 1. By the change of variables formula of K. Itô it follows that for $p \geqq 1$, $p \in \mathbb{N}$

$$\int_S^0 \mathrm{d}(\mathrm{e}^{\lambda t}(X'zX)^p) = p\int_S^0 (\mathrm{e}^{\lambda t}(X'zX)^{p-1}\, 2X'z(f(\alpha(t)) + gk(\alpha^*(t))\, X\, \mathrm{d}t + \mathrm{d}W) +$$

$$+ \lambda \int_S^0 e^{\lambda t}(X'zX)^p \, dt + 2p(p-1) \int_S^0 e^{\lambda t}(X'zX)^{p-2} X'zhzX \, dt$$
$$+ p \operatorname{trace}(zh) \int_S^0 e^{\lambda t}(X'zX)^{p-1} \, dt \,. \tag{33}$$

Apply expectation to (33), use (3) and let $S \to -\infty$ to obtain

$$\mathsf{E}(X(0)' zX(0))^p + \left(\frac{p}{\operatorname{trace} z} - \lambda\right) \int_{-\infty}^0 e^{\lambda t} \, \mathsf{E}(X'zX)^p \, dt \leqq$$
$$\leqq (2p(p-1)\,|hz| + p \operatorname{trace}(zh)) \int_{-\infty}^0 e^{\lambda t} \, \mathsf{E}(X'zX)^{p-1} \, dt \,. \tag{34}$$

The inequalities (31), (32) follow by iteration with respect to p on the inequality (34). □

Corollary. If $r(t) \in L(\mathbb{R}^n, \mathbb{R}^n)$ is uniformly bounded for $t \in \mathbb{R}$ then

$$\mathsf{E}(\sqrt{(\lambda)} \int_{-\infty}^T e^{\lambda(t-T)} X'(t)\, r(t) \, dW(t))^8 \leqq c \tag{35}$$

Proof. The verification of (35) follows from Lemma 1 and the inequality

$$\mathsf{E}(\int_{-\infty}^T e^{\lambda t} X'r \, dW)^8 \leqq \frac{c}{\lambda^3} \int_{-\infty}^T e^{2\lambda t} \, \mathsf{E}(X'rhrX)^4 \, dt \tag{36}$$

which is obtained by integrating the differential of the left hand side of (36) and using the Hölder inequality. □

Lemma 2. For $\delta > 0$ sufficiently small, the inequality

$$\mathsf{E} \exp\left[\delta \int_{-T}^0 |X|^2 \, dt\right] \leqq c \, e^{\delta c T} \tag{37}$$

is satisfied for all $T \geqq 0$.

Proof. Using the uniform stability in Assumption 1 and the Itô formula as in the proof of Lemma 1 we have

$$X'(0)\, zX(0) - X'(-T)\, zX(-T) \leqq$$
$$\leqq (\operatorname{trace} zh)\, T - \int_{-T}^0 |X|^2 \, dt + 2 \int_{-T}^0 X'z \, dW \,. \tag{38}$$

Thus exponentiating (38)

$$\exp\left[\delta \int_{-T}^0 |X|^2 \, dt - 4\delta^2 \int_{-T}^0 X'zhzX \, dt\right] \leqq$$
$$\exp\left[\delta(\operatorname{trace} zh)\, T + \delta X'(-T)\, zX(-T) + 2\delta \int_{-T}^0 X'z \, dW -\right.$$
$$\left. - 4\delta^2 \int_{-T}^0 X'zhzX \, dt\right] .$$

By the Schwarz inequality we obtain

$$\mathsf{E} \exp\left[(\delta - 4\delta^2 |z|^2 \, |h|) \int_{-T}^0 |X|^2 \, dt\right] \leqq$$
$$\leqq \exp\left[\delta(\operatorname{trace} zh)\, T\right] \left[\mathsf{E} \exp\left(2\delta X'(-T)\, zX(-T)\right)\right]^{1/2} .$$
$$. \left[\mathsf{E} \exp\left(4\delta \int_{-T}^0 X'z \, dW - 8\delta^2 \int_{-T}^0 X'zhzX \, dt\right]^{1/2} . \tag{39}$$

The second expectation on the right hand side is $\leqq 1$ by a known property of stochastic integrals. The other expectation on the right hand side is bounded for $\delta > 0$ sufficiently small by Assumption 4. Since $\delta - 4\delta^2|z|^2\,|h| > \frac{1}{2}\delta$ for $\delta > 0$ sufficiently small the inequality (37) follows from (39). □

Lemma 3. There is a $\vartheta > 0$ such that

$$\theta(y, k) \geqq \vartheta I \tag{40}$$

for $k \in \mathscr{K}$ and $y \in \mathbb{R}$ where θ is defined in Proposition 1.

Proof. Let $d(\cdot, \cdot)$ be defined by the equation

$$d(a, k) = (\operatorname{trace}(v(a, k) f_i' l f_j))$$

for $i, j \in \{1, \ldots, q\}$ where v is the solution of (4). For $\mu \in \mathbb{R}^q$ let $q(\mu) \in L(\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$q(\mu) = \sum_{i,j} \mu_i f_i' l f_j \mu_j \,.$$

It is clear that

$$\mu' d(a, k)\, \mu = \operatorname{trace}(q(\mu)\, v(a, k)) = \int_0^\infty \operatorname{trace}(q(\mu)\, F'(t) h F(t))\, \mathrm{d}t$$

where

$$F(t) = \exp\{t(f(a) + gk)\}\,.$$

The linear independence condition in Assumption 2 implies that $\operatorname{trace}(q(\mu) h) > 0$. Thus

$$\mu' d(a, k)\, \mu > 0$$

for each $a \in \mathscr{A}$ and $k \in \mathscr{K}$. The continuity of v as a function of $a \in \mathscr{A}$ and $k \in \mathscr{K}$ and the compactness of $\mathscr{A}$ and $\mathscr{K}$ imply that there is a $\vartheta > 0$ such that

$$d(a, k) \geqq \vartheta I$$

for all $a \in \mathscr{A}$ and $k \in \mathscr{K}$. Since $a(\cdot)$ takes values in $\mathscr{A}$, (40) is satisfied. □

Proof of Proposition 2. To avoid complicated notation the proof is given for $q = 1$ and $a(\cdot)$ continuously differentiable. Indices are omitted where superfluous.

To establish the existence and uniqueness of a solution to (24) let $F(y, \bar{a})$ be the right hand side of (24). Using Lemma 2 and the Lipschitz continuity of $j(\cdot)$ in (17) it follows that

$$|F(y, \bar{a}) - F(y, \bar{b})| \leqq c\varepsilon \int_{-\infty}^{y} \mathrm{e}^{s-y} |\bar{a}(s) - \bar{b}(s)|\, \mathrm{d}s\,.$$

If $\varepsilon > 0$ in (17) is sufficiently small then

$$|F(y, \bar{a}) - F(y, \bar{b})| \leqq d \sup_y |\bar{a}(y) - \bar{b}(y)|$$

where $d < 1$. The existence and uniqueness of a periodic solution of (24) follows by successive approximation.

To verify (25) let $T = 0$ for notational simplicity. Let $w(s)$ be the solution of (23). Apply the Itô formula to $\int_{-\infty}^{0} \mathrm{d}(\mathrm{e}^{\lambda t} X' w(\lambda t) X)$ and use (4) and θ from Proposition 1 to obtain

$$\int_{-\infty}^{0} \mathrm{e}^{\lambda t}\, Q(t)\, \mathrm{d}t = \int_{-\infty}^{0} \mathrm{e}^{\lambda t}\, \theta(t, \lambda, \bar{k}(t\lambda))\, \mathrm{d}t - X'(0)\, w(0)\, X(0) +$$

$$+ \lambda \int_{-\infty}^{0} e^{\lambda t} X'(t) (w(t\lambda) + \dot{w}(t\lambda)) X(t) \, dt +$$
$$+ 2 \int_{-\infty}^{0} e^{\lambda t} X'(t) w(t\lambda) g(k(\alpha^*(t)) - \bar{k}(t\lambda)) X(t) \, dt +$$
$$+ 2 \int_{-\infty}^{0} e^{\lambda t} X'(t) w(t\lambda) \, dW(t) . \tag{41}$$

Using Lemma 1 and its Corollary we have

$$\int_{-\infty}^{0} e^{\lambda t} Q(t) \, dt = \int_{-\infty}^{0} e^{\lambda t} \theta \, dt + 2 \int_{-\infty}^{0} e^{\lambda t} X' w g(k - \bar{k}) X \, dt + \frac{1}{\sqrt{\lambda}} R_0 \tag{42}$$

where

$$\mathsf{E}|R_0|^8 \leqq c .$$

In a similar fashion for the other term on the left hand side of (12) we have

$$\int_{-\infty}^{0} e^{\lambda t} Q\alpha \, dt = \int_{-\infty}^{0} e^{\lambda t} \theta\alpha \, dt + 2 \int_{-\infty}^{0} e^{\lambda t} X' w g(k - \bar{k}) X\alpha \, dt + \frac{1}{\sqrt{\lambda}} S_0 . \tag{43}$$

Finally the right hand side of (12) can be bounded to obtain

$$\alpha^*(0) \int_{-\infty}^{0} e^{\lambda t} Q \, dt = \int_{-\infty}^{0} e^{\lambda t} Q\alpha \, dt + \frac{1}{\sqrt{\lambda}} Z_0 \tag{44}$$

where

$$\mathsf{E}|S_0|^8 \leqq c$$
$$\mathsf{E}|Z_0|^8 \leqq c .$$

Using (18) we obtain from (42), (43), (44) that

$$(\alpha^*(0) - \bar{a}(0)) \int_{-\infty}^{0} e^{\lambda t} Q \, dt = 2 \int_{-\infty}^{0} e^{\lambda t} X' w g(k - \bar{k}) X(\alpha - \bar{a}(0)) \, dt +$$
$$+ \frac{1}{\sqrt{\lambda}} (S_0 - \bar{a}(0) R_0 + Z_0) . \tag{45}$$

Now the special form of $k(\cdot)$ is used. Since $j(\cdot)$ is Lipschitz continuous we have

$$|j(\alpha^*(0)) - j(\bar{a}(0))| \leqq L|\alpha^*(0) - \bar{a}(0)| . \tag{46}$$

From (45), (46) we have

$$|j(\alpha^*(0)) - j(\bar{a}(0))| (\lambda \int_{-\infty}^{0} e^{\lambda t} \theta \, dt - 2\lambda \int_{-\infty}^{0} e^{\lambda t} |X|^2 \, |g| \, |k - \bar{k}| \, dt -$$
$$- \sqrt{(\lambda)} \, |R_0|) \leqq L(2\lambda \int_{-\infty}^{0} e^{\lambda t} |X|^2 \, |w| \, |g| \, |k - \bar{k}| \, |\alpha - \bar{a}(0)| \, dt +$$
$$+ \sqrt{(\lambda)} \, |S_0 - \bar{a}(0) R_0 + Z_0|) .$$

Using (17) and evaluating the terms at T instead of 0 we have

$$|j(\alpha^*(T)) - j(\bar{a}(T)| \leqq \varepsilon c \lambda \int_{-\infty}^{T} e^{\lambda(t-T)} \, |X(t)|^2 \, |j(\alpha^*(t)) - j(\bar{a}(\lambda t))| \, dt +$$
$$+ \sqrt{(\lambda)} \, |M(T)| \tag{47}$$

where

$$\mathsf{E}|M(T)|^8 \leqq c .$$

Let Ψ be defined by the equation

$$\Psi(T) = e^{\lambda t} |j(\alpha^*(T)) - j(\bar{a}(\lambda T))| .$$

Thus (47) can be written as

$$\Psi(T) \leqq \varepsilon c \lambda \int_{-\infty}^{T} |X|^2 \Psi \, dt + \sqrt{(\lambda)} \, e^{\lambda t} |M(T)| .$$

The Gronwall inequality yields

$$\Psi(T) \leqq \sqrt{(\lambda)} \, e^{\lambda t} |M(T)| + \\ + \varepsilon c \lambda \int_{-\infty}^{T} |X|^2 \sqrt{(\lambda)} \, e^{\lambda t} |M(t)| \exp \left[\varepsilon c \lambda \int_t^T |X|^2 \, ds \right] dt . \tag{48}$$

Applying the Schwarz inequality a few times to terms in the square of (48) we obtain

$$\mathsf{E} \Psi^2(T) \leqq 2\lambda \, e^{2\lambda T} \, \mathsf{E} |M(T)|^2 + \\ + 2\varepsilon^2 c^2 \lambda^2 (e^{\lambda T} \int_{-\infty}^{T} e^{\lambda t} \, \mathsf{E} |X|^{16} \, dt \, e^{\lambda T} \int_{-\infty}^{T} e^{\lambda t} \, \mathsf{E} |M|^8 \, dt)^{1/2} . \\ . (e^{\lambda T} \int_{-\infty}^{T} e^{\lambda t} \, \mathsf{E} \exp \left[4 \varepsilon c \lambda \int_t^T |X|^2 \, ds \right] dt)^{1/2} .$$

Thus

$$\mathsf{E} |j(\alpha^*(T)) - j(\bar{a}(\lambda T))|^2 \leqq 2\lambda \, \mathsf{E} |M(T)|^2 + \\ + 2\varepsilon^2 c^2 \lambda (\lambda \int_{-\infty}^{T} e^{\lambda(t-T)} \, \mathsf{E} |X|^{16} \, dt \, \lambda \int_{-\infty}^{T} e^{\lambda(t-T)} \, \mathsf{E} |M|^8 \, dt)^{1/4} . \\ . (\lambda \int_{-\infty}^{T} e^{\lambda(t-T)} \, \mathsf{E} \exp \left[4 \varepsilon c \lambda \int_t^T |X|^2 \, ds \right] dt)^{1/2} . \tag{49}$$

The last term in the parentheses on the right hand side of (49) is bounded for $\varepsilon > 0$ sufficiently small by Lemma 2. Since the other coefficients of λ on the right hand side of (49) are also bounded, the equation (25) follows. □

Proof of Propositions 1 and 3. Since in (45)

$$S_0 - \bar{a}(0) R_0 + Z_0 = \sqrt{(\lambda)} \int_{-\infty}^{0} e^{\lambda t} X'(2w(\alpha - \bar{a}(0)) + f_1' l) \, dW + o_p(1)$$

we have that

$$\frac{\alpha^*(0) - \bar{a}(0)}{\sqrt{\lambda}} \lambda \int_{-\infty}^{0} e^{\lambda t} Q \, dt = \sqrt{(\lambda)} \int_{-\infty}^{0} e^{\lambda t} X'(2w(\alpha - \bar{a}(0)) + f_1' l) \, dW + o_p(1) \tag{50}$$

where $o_p(1) \to 0$ in probability as $\lambda \downarrow 0$.

From the uniform stability of (1) and the self tuning property in Proposition 2 we have that

$$\operatorname*{p\,lim}_{\lambda \to 0} \lambda \int_{-\infty}^{0} e^{\lambda t} Q \, dt = \int_{-\infty}^{0} \theta(y) \, dy \tag{51}$$

where $\theta(y)$ is $\theta(y, \bar{k}(y))$ and $\theta(y, k_0)$ for Proposition 1 and Proposition 3 respectively. The stochastic integral in (50) can be obtained from a time change of a Wiener process as

$$\mathscr{W}(\int_{-\infty}^{0} \lambda \, e^{2\lambda t} |X'(2w(\alpha - \bar{a}(0)) + f_1' l) \sqrt{(h)} \,|^2 \, dt) . \tag{52}$$

Analogously as in (51) the integral in (52) converges to

$$\int_{-\infty}^{0} e^{-2s} \Delta(0, s) \, ds \tag{53}$$

where $\Delta(0, s)$ is given by (20). Under the hypotheses of Proposition 3, (53) reduces to (28). Hence we obtain the desired asymptotic distribution in the two cases. □

Proof of Proposition 4. Let $x(\cdot)$ be the solution of

$$x(y)(f(\bar{a}(y)) + g\bar{k}(y)) + (f(\bar{a}(y) + g\bar{k}(y))' x(y) + r + \bar{k}(y)\bar{k}(y)' = 0 .$$

Then

$$\gamma(y) = \text{trace}(x(y) h) .$$

Apply the Itô formula to obtain

$$\tau C(\lambda) - \lambda \int_0^{\tau/\lambda} \gamma(t\lambda) \, dt = \lambda \int_0^{\tau/\lambda} X'(k + \bar{k} + 2g'w)'(k - \bar{k}) X \, dt + \\ + \lambda(X'(0) x(0) X(0) - X'(\tau/\lambda) x(\tau/\lambda) X(\tau/\lambda)) + \\ + \lambda^2 \int_0^{\tau/\lambda} X'\dot{x}X \, dt + 2\lambda \int_0^{\tau/\lambda} X'x \, dW. \tag{51}$$

The expected absolute value of the first term on the right hand side of (51) is majorized, using Proposition 2, by

$$\lambda(\int_0^{\tau/\lambda} \mathsf{E}|X|^4 |k + \bar{k} + 2g'x|^2 \, dt)^{1/2} (\int_0^{\tau/\lambda} \mathsf{E}|k - \bar{k}|^2 \, dt)^{1/2} \leqq c\sqrt{\lambda} .$$

Similar estimates of the remaining terms on the right hand side of (51) have been used in the preceding proofs. □

5. EXAMPLES AND REMARKS

1. If $f(a) = af_1$ and $a(\cdot) > 0$ then under Assumption 2

$$\bar{a}(y) = (\int_{-\infty}^{y} e^{(s-y)} a^{-1}(s) \, ds)^{-1}$$

is a weighted harmonic mean of $a(\cdot)$.

2. To determine the response of the system to slow or fast variations of the parameters let $a_c(\cdot)$ be defined by

$$a_c(y) = a(cy)$$

for $c > 0$. From (24) it follows that

$$\bar{a}_c(y/c) = (\int_{-\infty}^{y} e^{s/c} \theta(a(s), k(\bar{a}_c(s/c))) \, ds)^{-1} . \\ . \int_{-\infty}^{y} e^{s/c} \theta(a(s), k(\bar{a}_c(s/c))) a(s) \, ds .$$

From this equation it can be deduced that for $\varepsilon \in (0, \bar{\varepsilon})$ in (17) as $c \downarrow 0$

$$\bar{a}_c(y/c) \to a(y)$$

at all points of continuity of $a(\cdot)$ and as $c \to \infty$

$$\bar{a}_c(y/c) \to a_\infty$$

where

$$a_\infty = (\int_0^\tau \theta(a(s), k(a(s))) \, ds)^{-1} . \int_0^\tau \theta(a(s), k(a(s))) a(s) \, ds .$$

3. Proposition 2 can be extended to some cases where $a(\cdot)$ is a stochastic process.

In particular if $a(\cdot)$ assumes two values with independent exponential holding times, the so-called random telegraph signal, and the process $(X(t), \alpha^*(t), Y(t), \alpha(t); t \in \mathbb{R})$ is assumed to be stationary, then $(\bar{a}(t), t \in \mathbb{R})$ is a stationary process.

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