ACCELERATED LIFE MODELS WHEN THE STRESS IS NOT CONSTANT

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The concept of a relation functional is defined and the accelerated life models based on the properties of this functional are described. The relations between diverse models are determined. A new test for the model of the additive accumulation of damages is considered.

1. DEFINITION OF A RELATION FUNCTIONAL AND A RESOURCE

Many items have a long life when used under normal conditions. Therefore much time is required to get sufficiently large data. To avoid this, one tests the items under high stress conditions using some models relating the life under a high stress to the life under a normal one, and then estimates the life distribution under a normal stress. Up to this moment a number of accelerated life models is proposed (see [1]–[5]). There is some eclecticism in various definitions of these models that prevents from seeing relations between them. Furthermore, many of these models are applied by intuition, without substantiation of their applicability.

Now the concept of a relation functional will be defined. The accelerated life models will be defined on the basis of the properties of this functional.

Suppose that the following objects are given:

1) The set S of the positive functions x: $[0, +\infty) \to (0, +\infty)$. Each function $x \in S$ will be called a stress.

2) $T(x)$, a non-negative random variable, the distribution of which depends on x, interpreted as a time to failure under a stress x.

Let F_x be the cumulative distribution function of $T(x)$. Suppose F_x to be a continuously differentiable increasing function on $[0, +\infty)$.

3) The non-negative functional g: $[0, +\infty) \times S \to [0, +\infty)$ such that for each

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$x \in S$ $g(t,x) = F_{x_0}^{-1}(F_x(t))$, where $x_0 \in S$ is some fixed stress, $F_{x_0}^{-1}$ is a function inverse to F_{x_0}. The value $g(t, x)$ of the functional g will be interpreted as a resource worked out up to the moment t under the stress x. The definition of g implies that

$$\mathsf{P}\{T(x) \geqq t\} = \mathsf{P}\{T(x_0) \geqq g(t, x)\},$$

i.e., the probability that an item used under a stress x would survive a moment t is equal to the probability that an item used under a stress x_0 would survive a moment $g(t, x)$.

The functional g will be called the *relation functional*, the random variable $R = f(T(x), x)$ being the *resource*. The distribution of the resource is the same under each $x \in S$. The rate of working out the resource is different under different stresses.

2. DEFINITION OF ACCELERATED LIFE MODELS ON THE BASIS OF PROPERTIES OF THE RELATION FUNCTIONAL

Model 1. There exists a positive functional $r: E \to \mathbb{R}^+$, $E \subset S$ such that for each $x \in E$ the relation functional g satisfies the differential equation

$$\partial/\partial t \, g(t, x) = r[x(t)]$$

with the initial condition $g(0, x) = 0$.

This means that the rate of working out the resource at a moment t depends only on the value of a stress at that moment. The relation functional has the form

$$g(t, x) = \int_0^t r[x(\tau)] \, d\tau$$

and hence the resource is

$$R = \int_0^{T(x)} r[x(\tau)] \, d\tau .$$

In what follows model 1 will be called the *model of additive accumulation of damages* (AAD).

Model 2. There exists a positive functional $h: \mathbb{R}^+ \times E \to \mathbb{R}^+$ such that for each $x \in E$ the relation functional satisfies the differential equation

$$\partial/\partial t \, g(t, x) = h[g(t, x), x(t)]$$

with the initial condition $g(0, x) = 0$.

Let E_1 ($E_1 \subset E$) be a set of stresses constant in time. The equation (1) for $x \in E_1$ implies

$$R = r(x) \, T(x), \quad \text{i.e.}$$

$$F_x(t) = F_{x_0}(r(x) \, t) .$$

We came to the so-called *accelerated failure time model* (see [2]). In many cases functions r and F_{x_0} are specified. Examples of different forms of $r(x)$ are the power rule law, the inverse power rule law, the Arrhenius law, the Eyring law or generalizations of these (see [2]–[4]).

Suppose that a stress has the form

$$x(\tau) = \begin{cases} x_1, & 0 \leqq t < t_1, \\ x_2, & t_1 \leqq t < t_2, \\ \dots & \dots \\ x_n, & t \geqq t_{n-1}, \end{cases}$$

where $x_1, \ldots, x_n \in E$ are constant stresses. Under the assumptions of the AAD model, the equality (1) implies that

$$R = \sum_{i=1}^{n} r(x_i)\, T_i(x)$$

where

$$T_i(x) = \begin{cases} 0, & T(x) < t_{i-1}, \\ T(x) - t_{i-1}, & t_{i-1} \leqq T(x) < t_i, \\ t_i - t_{i-1}, & T(x) \geqq t_i \end{cases}$$

for $i = 1, 2, \ldots, n-1$; $T_n(x) = \max(0, T(x) - t_n)$. For $x_i \in E$ the equality (1) implies $R = r(x_i)\, T(x_i)$ $(i = 1, 2, \ldots, n)$. Therefore

$$\sum_{i=1}^{n} r(x_i)\, \mathsf{E}\, T_i(x) = r(x_i)\, \mathsf{E}\, T(x_i) \quad (i = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} \mathsf{E}\, T_i(x)/\mathsf{E}\, T(x_i) = 1\,.$$

We have come to the so-called *Miner model* (see [5]).

Suppose that the model 2 is valid and a stress has the form

$$x(\tau) = \begin{cases} x_1(\tau), & 0 \leqq \tau \leqq t_1, \\ x_2(\tau), & \tau > t_1\,. \end{cases}$$

Then

$$f(t, x) = \begin{cases} f(t, x_1), & 0 \leqq t \leqq t_1, \\ f^*(t - t_1, x_2), & t > t_1 \end{cases}$$

where $f^*(s, x_2)$ satisfies the differential equation

$$\partial/\partial s\, f^*(s, x_2) = h(f^*(s, x_2), x_2(s))$$

with the initial condition $f^*(0, x_2) = f(t_1, x_1)$. If $x_2(\tau) = x_2$ for $\tau \geqq 0$ then

$$f^*(t - t_1, x_2) = f(t - t_1 + t_2, x_2)$$

where t_2 satisfies the equation

$$f(t_1, x_1) = f(t_2, x_2)\,.$$

Therefore,

$$F(t, x) = \begin{cases} F(t, x_1), & 0 \leqq t \leqq t_1, \\ F(t - t_1 + t_2, x_2), & t > t_1 \end{cases} \tag{2}$$

where t_2 satisfies the equation $F(t_1, x_1) = F(t_2, x_2)$ and for each $t \geqq 0$

$$\mathsf{P}\{T(x) \geqq t_1 + t \mid T(x) \geqq t_1\} = \mathsf{P}\{T(x_2) \geqq t_2 + t \mid T(x_2) \geqq t_2\} .$$

We have come to the so-called *Sediakins model* (see [5]), defined by equality (2). The more general model 2 will be also called the Sediakins model.

Suppose that the AAD model is valid. This model is a particular case of the Sediakins model. Therefore, the equality (2) is true and the moment t_2 is equal to $t_1\, r(x_1)/r(x_2)$. Let $t_p(x)$, $t_p(x_1)$ and $t_p(x_2)$ be p-quantiles of random variables $T(x)$, $T(x_1)$ and $T(x_2)$. The equality (2) implies that $t_p(x) = t_p(x_1)$ for p such that $\mathsf{P}\{T(x_1) \leqq \leqq t_1\} \geqq p$ and

$$t_p(x) = t_1 - t_2 + t_p(x_2) \tag{3}$$

for p such that $\mathsf{P}\{T(x_1) \leqq t_1\} < p$. The equality $F(t, x_i) = F_0(r(x_i)\, t)$ implies that $r(x_1)\, t_p(x_1) = r(x_2)\, t_p(x_2)$. Therefore $t_2 = t_1\, t_p(x_2)/t_p(x_1)$ and the equality (3) reduces to

$$\frac{t_1}{t_p(x_1)} + \frac{t_p(x) - t_1}{t_p(x_2)} = 1 .$$

Thus, we have come to the so-called *Stepanova-Pešes model* (see [5]).

The tests for accelerated failure time model are described in [3]. The test for the AAD model was proposed in [1]. It has a deficiency that an experiment under a normal stress is required. In the next section a new test for the AAD model is proposed.

3. TEST FOR THE AAD MODEL

Suppose that the first two moments of the failure times $T(x)$ under the stresses $x \in E$ exist. Let E be a set of stresses $x: \mathbb{R}^+ \to [a, b]$, $[a, b] \subset \mathbb{R}$ and $x^{(i)} \in E$ be stresses of the form

$$x^{(i)}(\tau) = \begin{cases} x_{n(1,i)}, & 0 \leqq \tau < t_{i1} , \\ x_{n(2,i)}, & t_{i1} \leqq \tau < t_{i2} , \\ x_{n(M,i)}, & \tau \geqq t_{i,M-1} \end{cases}$$

where $n(1, i), \ldots, n(M, i)$ are the permutations of numbers $1, 2, \ldots, M$; $x_1, \ldots, x_n \in E$ are constant stresses; $t_{ij} \in [0, +\infty)$ are the moments of switching over from one constant stress to another one ($i = 1, \ldots, N$; $j = 1, \ldots, M$). In the particular case when $t_{i1} = \infty$, the stress $x^{(i)}$ can be constant.

Suppose that N $(N > M)$ experiments are carried out and K_i items are tested under the stress $x^{(i)}$ in the ith experiment ($i = 1, 2, \ldots, N$). Let $T_{j1}^{(i)}, \ldots, T_{jK_i}^{(i)}$ be the lengths of lives of items under the constant stress x_j in the ith experiment. It is easy to show that under the assumption of AAD model the following equalities are true:

$$\sum_{j=1}^{M} r_j T_{jk}^{(i)} - 1 = \sigma R_{ik} \quad (i = 1, 2, \ldots, N)$$

where $r_j = r(x_j)/a$; $\sigma > 0$; R_{ik} are independent identically distributed random variables with means $\mathsf{E}R_{ik} = 0$ and variances $\mathsf{Var}\, R_{ik} = 1$ $(k = 1, \ldots, K_i)$. In such a case the means $\tau_j^{(i)} = \mathsf{E}T_{jk}^{(i)}$ satisfy the equations

$$\sum_{j=1}^{M} r_j \tau_j^{(i)} - 1 = 0 \quad (i = 1, 2, \ldots, N)\,.$$

It would be natural to look for the estimates of parameters r_j by minimizing the sum

$$\sum_{i=1}^{N} \sum_{k=1}^{K_i} \Big(\sum_{j=1}^{M} r_j T_{jk}^{(i)} - 1\Big)^2$$

but unfortunately, this method leads to inconsistent estimates. Therefore denote

$$\boldsymbol{T}_k^{(i)} = (T_{1k}^{(i)}, \ldots, T_{Mk}^{(i)})',\ T_{j\cdot}^{(i)} = (1/K_i) \sum_{k=1}^{K_i} T_{jk}^{(i)},$$

$$\boldsymbol{T}_{\boldsymbol{\cdot}}^{(i)} = (T_{1\cdot}^{(i)}, \ldots, T_{M\cdot}^{(i)})',\ \boldsymbol{S}_0^{(i)} = (1/K_i) \sum_{k=1}^{K_i} (\boldsymbol{T}_k^{(i)} - \boldsymbol{T}_{\cdot}^{(i)})(\boldsymbol{T}_k^{(i)} - \boldsymbol{T}_{\cdot}^{(i)})',\ \boldsymbol{\tau}^{(i)} = \mathsf{E}\boldsymbol{T}_k^{(i)}$$

and define the estimates of parameters $r = (r_1, \ldots, r_M)'$ by minimizing the sum

$$\sum_{i=1}^{N} K_i (\boldsymbol{r}^{\mathrm{T}} \boldsymbol{T}_{\cdot}^{(i)} - 1)^2\,.$$

The normal equations have the form

$$\sum_{i=1}^{N} K_i \boldsymbol{T}_{\cdot}^{(i)} \boldsymbol{T}_{\cdot}^{(i)\prime} \boldsymbol{r} = \sum_{i=1}^{N} K_i \boldsymbol{T}_{\cdot}^{(i)}\,. \tag{4}$$

Suppose that a system of vectors $\{\boldsymbol{\tau}^{(i)}, i = 1, 2, \ldots, N\}$ has a rank M. This condition is satisfied practically in all cases when $x^{(i)}$ are different stresses. It is easy to show that if the AAD model is valid on E, the solution of normal equations $\hat{\boldsymbol{r}}$ converges with probability one (as $\min K_i \to \infty$, $K_i/\max K_i \to l_i > 0$) to the parameters $\boldsymbol{r}$ satisfying a system of equations $\boldsymbol{r}'\boldsymbol{\tau}^{(i)} - 1 = 0$ $(i = 1, \ldots, N)$ and the estimate

$$\hat{\sigma}^2 = \Big(1/\Big(\sum_{i=1}^{N} K_i - N\Big)\Big) \sum_{i=1}^{N} \sum_{k=1}^{K_i} \Big[\sum_{j=1}^{M} \hat{r}_j (T_{jk}^{(i)} - T_{j\cdot}^{(i)})\Big]^2$$

converges with probability one to a parameter σ^2 satisfying an equation

$$\boldsymbol{r}'\boldsymbol{B}^{(i)}\boldsymbol{r} = \sigma^2 \quad (\boldsymbol{B}^{(i)} = \mathsf{E}\boldsymbol{S}_0^{(i)})\,.$$

Theorem. Assume:

1) the AAD model is valid on E;
2) there exist two moments of the random variables $T(x)$, $x \in E$;
3) the system of vectors $\{\boldsymbol{\tau}^{(i)}, i = 1, \ldots, N\}$ has a rank M.

Then the distribution of the statistic

$$Y^2 = (1/\hat{\sigma}^2) \sum_{i=1}^{N} K_i (\hat{\boldsymbol{r}}' T_{\cdot}^{(i)} - 1)^2$$

converges to a chi-square distribution with $N - M$ degrees of freedom with probability one as $\min K_i \to \infty$, $K_i/\max K_i \to l_i > 0$.

Proof. Denote $K = \max K_i$; $\boldsymbol{L} = (\sqrt{l_1}, \ldots, \sqrt{l_N})'$; $\boldsymbol{T} = (\sqrt{(l_i)}\, T_{j.}^{(i)})$, the $N \times M$ matrix; $\boldsymbol{Z}(\boldsymbol{r}) = \sqrt{K}\,(\boldsymbol{Tr} - \boldsymbol{L})$. In such a case

$$\sum_{i=1}^{N} K_i(\boldsymbol{r}'\boldsymbol{T}_{.}^{(i)} - 1)^2 = \boldsymbol{Z}(\boldsymbol{r})'\,\boldsymbol{Z}(\boldsymbol{r})\,.$$

Let $\boldsymbol{C}$ be an $N \times M$ matrix with elements $\sqrt{(l_i)}\,\tau_j^{(i)}$ $(i = 1, \ldots, N;\ j = 1, \ldots, M)$ From the assumption 3) of the theorem it follows that the rank of the matrix $\boldsymbol{C}$ is equal to M. The equalities rank $\boldsymbol{C}'\boldsymbol{C}$ = rank $\boldsymbol{C} = M$ implies that a random matrix $\boldsymbol{T}'\boldsymbol{T} \to \boldsymbol{C}'\boldsymbol{C}$ with probability one. Therefore, the solution of the equation (4) is the statistic

$$\hat{\boldsymbol{r}} = (\boldsymbol{T}'\boldsymbol{T})^{-1}\,\boldsymbol{T}'\boldsymbol{L}$$

provided $\min K_i$ is sufficiently large. If the AAD model is valid, the distribution of a random variable $(1/\sigma^2)\,\boldsymbol{Z}(\boldsymbol{r})'\,\boldsymbol{Z}(\boldsymbol{r})$ converges to a chi-square distribution with N degrees of freedom. Consider the limit distribution of a random variable

$$\min_{\boldsymbol{r}} \boldsymbol{Z}(\boldsymbol{r})'\,\boldsymbol{Z}(\boldsymbol{r}) = \boldsymbol{Z}(\hat{\boldsymbol{r}})'\,\boldsymbol{Z}(\hat{\boldsymbol{r}}) \tag{5}$$

It is easy to show that the random vector $\boldsymbol{Z}(\hat{\boldsymbol{r}})$ can be expressed in the form:

$$\boldsymbol{Z}(\hat{\boldsymbol{r}}) = (\boldsymbol{E} - \boldsymbol{T}(\boldsymbol{T}'\boldsymbol{T})^{-1}\,\boldsymbol{T}')\sqrt{K}(\boldsymbol{Tr} - \boldsymbol{L})$$

where $\boldsymbol{E}$ is an $N \times N$ identity matrix. The former implies that the asymptotic distributions of r.v. $\boldsymbol{Z}(\hat{\boldsymbol{r}})$ and $(\boldsymbol{E} - \boldsymbol{C}(\boldsymbol{C}'\boldsymbol{C})^{-1}\,\boldsymbol{C}')\,\boldsymbol{Z}(\boldsymbol{r})$ are the same. The matrix $\boldsymbol{A} = \boldsymbol{E} - \boldsymbol{C}(\boldsymbol{C}'\boldsymbol{C})^{-1}\,\boldsymbol{C}'$ is idempotent, i.e. $\boldsymbol{AA} = \boldsymbol{A}$. This implies that asymptotic distributions of r.v. (5) and $\boldsymbol{Z}(\boldsymbol{r})'\,\boldsymbol{A}\,\boldsymbol{Z}(\boldsymbol{r})$ are the same. The r.v. $(1/\hat{\sigma}^2)\,\boldsymbol{Z}(\boldsymbol{r})'\,\boldsymbol{A}\,\boldsymbol{Z}(\boldsymbol{r})$ has asymptotically a chi-square distribution with $N - M$ degrees of freedom. This follows from the equality rank $\boldsymbol{A}$ = Tr $(\boldsymbol{A}) = N - M$ and from the fact that $\hat{\sigma} \to \sigma$ with probability one. Thus, the proof is completed. □

Corollary. Under assumptions of the theorem the asymptotic distribution of the statistic

$$\chi^2 = \sum_{i=1}^{N} K_i(\hat{\boldsymbol{r}}'\boldsymbol{T}_{.}^{(i)} - 1)/(\hat{\boldsymbol{r}}'\boldsymbol{S}_0^{(i)}\hat{\boldsymbol{r}})$$

is chi-square with $N - M$ degrees of freedom.

If the AAD model is not valid, the estimates $\hat{\boldsymbol{r}}$ converge, on the whole, with probability one to some parameters but some of the equalities $\boldsymbol{r}_0\boldsymbol{\tau}^{(i)} - 1 = 0$ or $\boldsymbol{r}_0'\boldsymbol{B}^{(i)}\boldsymbol{r}_0 = \sigma^2$ $(i = 1, 2, \ldots, N)$ take no place. If the sequence of alternatives is such that $|v_i| <$ const, where $v_i = \sqrt{(K_i)}\,(\boldsymbol{r}_0'\boldsymbol{\tau}^{(i)} - 1)/(\boldsymbol{r}_0'\boldsymbol{B}^{(i)}\boldsymbol{r}_0)^{1/2}$, the asymptotic distribution of the statistic χ^2 is a noncentral chi-square with $N - M$ degrees of freedom and the noncentrality parameter $\sum_{i=1}^{N} v_i^2$.

The statistic χ^2 can be used as a test statistic for the AAD model when samples are sufficiently large.

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