KYBERNETIKA — VOLUME 26 (1990), NUMBER 3

SEMIRECIPROCAL WAVE PROPAGATION

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Distributed four-port network is a valid model for two coupled waves propagating in linear media. This paper deals with nonreciprocal, physically realizable wave propagating systems. In the past such systems have been applied to construction of circulators with ferrites for reaching one way propagation of signal power in microwave communication and radar systems. Now the main interest in research in nonreciprocal phenomena is caused by the needs of progress in integrated optics and in improving Fabry Perot resonators for semiconductor lasers. For optical systems it is necessary to deepen the old theory. The results will be presented. We are now able to synthesize the optimum nonreciprocal distributed parameter four-port tailored to given requirements. The methods of synthesis of *semireciprocal* wave propagating systems are described in a detailed form. The signal propagation of two wave modes with different phase and group velocities in anisotropic reciprocal or nonreciprocal media may be solved in an exact explicit form. The technologists get clear instructions for developing materials for exploring new interesting applications in photonics.

1. INTRODUCTION

The theory of two coupled homogeneous lines is well established [1]. The results have been applied to the analysis of slotted coaxial lines in GUIDAR systems [2] and for underground communications [3, 4]. Recently the theory has been improved and applied to the design of special bandpass filters with space distributed resonators [5]. All of the mentioned objects represent reciprocal systems. In this paper the concept of nonreciprocity is discussed with the aim to reach nonevanescent wave propagation in lossless media. Such condition permits the signal transfer to infinite distance. Nonreciprocity is found neither in lumped parameter networks nor in coupled transmission lines. On the other hand, the passive nonreciprocal systems are physically realizable and bring no contradiction to the laws of nature. It is well known that a broad class of interesting phenomena is based on gyromagnetic effects in anisotropic materials for integrated optics [6].

Our investigation is based on a very detailed study of the chain matrix **A** of a general linear four-port with space distributed parameters. The losslessness, reciprocity or nonreciprocity are directly reflected in the form of its secular equation. The eigenvalues determine in a unique way the type of wave propagation (evanescence or ability for space unrestricted propagation). The eigenvectors appear in various proportions in the four constituents of the matrix **A**. They contain altogether 64 complex constants. The numerical values of the entries of any constituent are mutually coupled by 72 constraints. The knowledge of all these conditions simplifies substantially the solution of a set of four first-order differential equations in a closed analytical form. Using our method we have developed procedures, algorithms and computer programs for the synthesis of reciprocal and nonreciprocal four-ports with distributed parameters.

2. GENERAL LOSSLESS NONRECIPROCAL FOUR-PORT

The admittance matrix **Y** of a linear lossless four-port fulfils the condition $\mathbf{Y} + \mathbf{Y}^+ = 0$, where asterisk denotes a Hermitean-conjugate matrix. On its main diagonal there are four imaginary entries. Above the diagonal we find 6 complex numbers $y_{ik} = g_{ik} + jm_{ik}$ $(i, k = 2, 3, 4, i > k)$. Under the diagonal there are numbers $y_{ki} = -g_{ik} + jm_{ik}$. Thus, **Y** consists of the symmetric imaginary part and of the real antisymmetric part. The chain matrix and its inverse have the form

$$\mathbf{A} = \begin{bmatrix} a_1 & r_1 & c_1 & o_1 \\ g_1 & b_1 & s_1 & d_1 \\ c_2 & o_2 & a_2 & r_2 \\ s_2 & d_2 & g_2 & b_2 \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} b_1^* & -r_1^* & d_2^* & -o_2^* \\ -g_1^* & a_1^* & -s_2^* & c_2^* \\ d_2^* & -o_2^* & b_2^* & -r_2^* \\ -s_1^* & c_1^* & -g_2^* & a_2^* \end{bmatrix} \tag{1}$$

The matrices **A** and **Y** define the following linear transformations

$$\mathbf{V} = \mathbf{A} \, . \, \mathbf{W} \, , \quad \mathbf{I} = \mathbf{Y} \, . \, \mathbf{U} \, , \tag{2}$$

where

$$\mathbf{V} = [u_1, i_1, u_2, i_2]^{\mathrm{T}}, \quad \mathbf{W} = [u_3, -i_3, u_4, -i_4]^{\mathrm{T}},$$

$$\mathbf{I} = [i_1, i_2, i_3, i_4]^{\mathrm{T}}, \quad \mathbf{U} = [u_1, u_2, u_3, u_4]^{\mathrm{T}}.$$

The physical meaning of all these variables in the two possible system descriptions of any linear four-port explain the circuit diagram in Fig. 1.

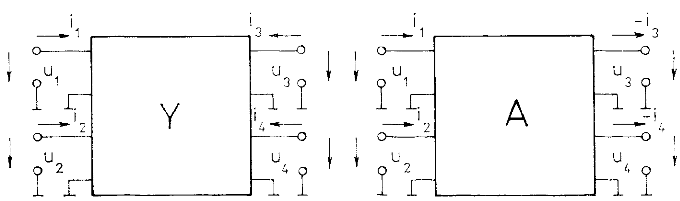

Fig. 1. Circuit diagrams for the four-port description either by the admittance matrix **Y** or by the chain matrix **Â**.

The chain matrix **A** determines two wave propagation modes. Unfortunately the constants for the losslessness are more complicated. We define the following values of subdeterminants 2 × 2

$$\begin{aligned}
\Delta_1 &= a_1 b_1 - r_1 g_1 , & \Delta_2 &= a_2 b_2 - r_2 g_2 , \\
\delta_1 &= c_1 d_1 - o_1 s_1 , & \delta_2 &= c_2 d_2 - o_2 s_2 , \\
\alpha_1 &= a_1 d_1 - g_1 o_1 , & \alpha_2 &= a_1 d_2 - g_2 o_2 , \\
\beta_1 &= b_1 c_1 - r_1 s_1 , & \beta_2 &= b_2 c_2 - r_2 s_2 , \\
R_1 &= -c_1 g_1 + a_1 s_1 , & R_2 &= -c_2 g_2 + a_2 s_2 , \\
G_1 &= -d_1 r_1 + b_1 o_1 , & G_2 &= -d_2 r_2 + b_2 o_2 .
\end{aligned} \tag{3}$$

All these variables fulfil the fundamental constraints

$$\begin{aligned}
\Delta_1 &= \Delta_2^* , \quad \delta_1 = \delta_2^* , \quad \alpha_1 = -\beta_2^* , \\
\beta_1 &= -\alpha_2^* , \quad G_1 = -G_2^* , \quad R_1 = -R_2^* .
\end{aligned} \tag{4}$$

These conditions for **A** are equivalent to the condition $\mathbf{Y} + \mathbf{Y}^+ = \mathbf{0}$ for **Y** which guarantiees the losslessness of the four-port. They are necessary but not sufficient. The consequence is that no power is absorbed by the four-port from external voltage or current sources. If no power is absorbed in the network the power flowing into the two inputs equals the power flowing out from the two outputs.

The power P_1 flowing into and the power P_2 flowing out of the four-port are expressed in the relations [8]

$$P_1 = \tfrac{1}{2}\mathbf{V}^+ . \mathbf{P} . \mathbf{V} , \quad P_2 = \tfrac{1}{2}\mathbf{W}^+ . \mathbf{P} . \mathbf{W} , \tag{5}$$

where **P** is the permutation matrix of the form

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \tag{6}$$

Using the first of the linear matrix Eqn's (1) we derive that the input power P_1 is a function of the output vector **W**. We get the relation

$$P_1 = \tfrac{1}{2}\mathbf{W}^2 . (\mathbf{A}^+ . \mathbf{P} . \mathbf{A}) . \mathbf{W} . \tag{7}$$

The matrices **P** and $\mathbf{A}^+ . \mathbf{P} . \mathbf{A}$ are Hermitean. Comparing Eqn's (4), (6) we get the condition for the equality of the input and output powers in the form

$$\mathbf{P} = \mathbf{A}^+ . \mathbf{P} . \mathbf{A} . \tag{8}$$

It is easy to prove that the entries of **A** fulfilling (7) satisfy the following constraints:

$$\begin{aligned}
a_1^* b_1 + r_1 g_1^* + c_2^* d_2 + o_2 s_2^* &= 1 \\
a_2^* b_2 + r_2 g_2^* + c_1^* d_1 + o_1 s_1^* &= 1 \\
g_1^* a_1 + g_1 a_1^* + s_2^* c_2 + s_2 c_2^* &= 0
\end{aligned}$$

$$
\begin{aligned}
g_2^* a_2 + g_2 a_2^* + s_1^* c_1 + s_1 c_1^* &= 0 \\
r_1^* b_1 + r_1 b_1^* + o_2^* d_2 + o_2 d_2^* &= 0 \\
r_2^* b_2 + r_2 b_2^* + o_1^* d_1 + o_1 d_1^* &= 0 \\
g_1^* c_1 + g_2 c_2^* + s_1 a_1^* + s_2^* a_2 &= 0 \\
a_1^* d_1 + b_2 c_2^* + r_2 s_2^* + g_1^* o_1 &= 0 \\
b_1 c_1^* + a_2^* d_2 + r_1^* s_1 + g_2 o_2^* &= 0
\end{aligned}
\tag{9}
$$

We open the question how to synthesize the matrix **A** fulfilling all the constraints specified in (4) and (9). For the lossless nonreciprocal matrix **Y**,

$$
\mathbf{Y} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \\ y_9 & y_{10} & y_{11} & y_{12} \\ y_{13} & y_{14} & y_{15} & y_{16} \end{bmatrix},
$$

which fulfils the condition $\mathbf{Y} = -\mathbf{Y}^+$, the following notation is practical:

$$
\mathbf{Y} = \begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & e_4 & e_5 \\ -e_2 & -e_4 & 0 & e_6 \\ -e_3 & -e_5 & -e_6 & 0 \end{bmatrix} + \mathrm{j} \begin{bmatrix} f_a & f_1 & f_2 & f_3 \\ f_1 & f_b & f_4 & f_5 \\ f_2 & f_4 & f_c & f_6 \\ f_3 & f_5 & f_6 & f_d \end{bmatrix}. \tag{10}
$$

The simplest way to find **A** is to select six real constants e_i, $i = 1, 2, \ldots, 6$, four diagonal constants f_a, f_b, f_c, f_d and 6 off-diagonal constants f_i.

We introduce eight 2 × 2 matrices containing all entries of **A** and **Y**:

$$
\mathbf{E} = \begin{bmatrix} a_1 & c_1 \\ c_2 & a_2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} r_1 & o_1 \\ o_2 & r_2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g_1 & s_1 \\ s_2 & g_2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} b_1 & d_1 \\ d_2 & b_2 \end{bmatrix},
$$

$$
\mathbf{Y}_1 = \begin{bmatrix} y_1 & y_2 \\ y_5 & y_6 \end{bmatrix}, \quad \mathbf{Y}_2 = \begin{bmatrix} y_3 & y_4 \\ y_7 & y_8 \end{bmatrix}, \quad \mathbf{Y}_3 = \begin{bmatrix} y_9 & y_{10} \\ y_{13} & y_{14} \end{bmatrix}, \quad \mathbf{Y}_4 = \begin{bmatrix} y_{11} & y_{12} \\ y_{15} & y_{16} \end{bmatrix}. \tag{11}
$$

Then we get the transformation $\mathbf{Y} \Rightarrow \mathbf{A}$ by simple manipulations with matrices of a low order. The resultant formulae have the form

$$
\mathbf{E} = -\mathbf{Y}_3^{-1} . \mathbf{Y}_4 , \quad \mathbf{R} = \mathbf{Y}_3^{-1} , \quad \mathbf{G} = \mathbf{Y}_2 - \mathbf{Y}_1 . \mathbf{Y}_3^{-1} . \mathbf{Y}_4 , \quad \mathbf{F} = \mathbf{Y}_1 . \tag{12}
$$

As an example we choose the values of 16 entries of **A**:

$$
\begin{aligned}
&e_1 = 13 , \quad e_2 = -3 , \quad e_3 = -7 , \quad e_4 = -5 , \quad e_5 = -11 , \quad e_6 = 1 , \\
&f_1 = 53 , \quad f_2 = 37 , \quad f_3 = 43 , \quad f_4 = -41 , \quad f_5 = -47 , \quad f_6 = 31 , \\
&f_a = -23 , \quad f_b = -29 , \quad f_c = 17 , \quad f_d = 19 .
\end{aligned}
\tag{13}
$$

Using Eqn's (12) we calculate all entries of **A** and from Eqn's (4) we determine the constraints of **A**. The numerical results are shown in Table 1. During computation the remaining energetic constraints (9) have been checked. To complete the review

Table 1. General lossless nonreciprocal four-port. Entries of the chain matrix **A** and its constraints.

$a_1 =$	$-0{\cdot}99272767 - j\ 0{\cdot}95548043$	$a_2 =$	$1{\cdot}75527959 + j\ 0{\cdot}809299943$
$b_1 =$	$4{\cdot}76703941 - j\ 0{\cdot}431720258$	$b_2 =$	$-0{\cdot}466342992 - j\ 3{\cdot}98991253$
$c_1 =$	$1{\cdot}0924063 - j\ 1{\cdot}81166417$	$c_2 =$	$-0{\cdot}780605978 - j\ 0{\cdot}976151372$
$d_1 =$	$3{\cdot}63216467 - j\ 0{\cdot}328086647$	$d_2 =$	$-0{\cdot}417001629 - j\ 5{\cdot}23035874$
$r_1 =$	$0{\cdot}118009583 - j\ 0{\cdot}079678806$	$r_2 =$	$-0{\cdot}067665392 + j\ 0{\cdot}086094976$
$g_1 =$	$42{\cdot}5505638 - j\ 2{\cdot}0432786$	$g_2 =$	$-126{\cdot}754860 - j\ 43{\cdot}568206$
$\sigma_1 =$	$-0{\cdot}09299758 + j\ 0{\cdot}07871822$	$\sigma_2 =$	$0{\cdot}0710335742 - j\ 0{\cdot}107098405$
$s_1 =$	$45{\cdot}6742913 + j\ 114{\cdot}73157$	$s_2 =$	$3{\cdot}711488290 - j\ 44{\cdot}2411094$
$\frac{1}{4}\varphi =$	$42{\cdot}8006479°$	$\lvert\mathbf{A}\rvert =$	$1{\cdot}00000004$

Constraints:

$\Delta_1 =$	$\Delta_2^* =$	$9{\cdot}91744728 + j\ 0{\cdot}584073146$
$\delta_1 =$	$\delta_2^* =$	$-9{\cdot}91744727 + j\ 0{\cdot}135693613$
$\alpha_1 =$	$-\beta_2^* =$	$7{\cdot}08852642 + j\ 0{\cdot}25663815$
$\alpha_2 =$	$-\beta_1^* =$	$8{\cdot}70505259 - j\ 1{\cdot}63717543$
$G_1 =$	$-G_2^* =$	$14{\cdot}097042 - j\ 232{\cdot}393342$
$R_1 =$	$-R_2^* =$	$0{\cdot}0383482527 + j\ 0{\cdot}156342862$

of the main properties of **A** we mention the value of its determinant, which is

$$\det(\mathbf{A}) = \exp(j\varphi)\,. \tag{14}$$

It means that the absolute value of the determinant of **A** of a linear lossless non-reciprocal four-port equals unity. In contradiction to linear two-port the condition (14) is necessary but not sufficient for the definite decision of the character of a linear four-port.

Let us assume zero value of all six constants e_i, $i = 1, 2, \ldots, 6$ in the anti-symmetric part of **Y**. Then we get a lossless reciprocal four-port. All 16 entries of **Y** are imaginary. We use transformation (12) to derive **A**. As an example we use identical parameters f_i, $i = 1, 2, \ldots, 6$ and f_a, f_b, f_c, f_d of **Y** as specified in (13). The results of computation of all entries of **A** are shown in Table 2.

We can see that the parameters $a_{1,2}$, $b_{1,2}$, $c_{1,2}$, $d_{1,2}$ from the main and second diagonal of **A** are real. The remaining parameters $r_{1,2}$, $g_{1,2}$, $o_{1,2}$, $s_{1,2}$ are imaginary. The subdeterminants of **A** relevant for the formulation of constraints (4) are defined in (3). The values of Δ_1, Δ_2, δ_1, δ_2, α_1, α_2, β_1, β_2 are real and R_1, R_2, G_1, G_2 are imaginary. The sum of the subdeterminants $\Delta_1 = \Delta_2$ and $\delta_1 = \delta_2$ of the lossless reciprocal four-port equals unity:

$$\Delta_1 + \delta_1 = 1\,, \quad \Delta_2 + \delta_2 = 1 \tag{15}$$

This is the consequence of the validity of the energy constraints defined in the first four equations in (9). In Fig. 2, the linear four-port consisting of 10 linear one-ports

Table 2. Lossless reciprocal four-port and its constraints. One half of the entries are real and the other are imaginary.

$a_1 =$	$-$ 19·66666666	$a_2 =$	26·25
$b_1 =$	49·916666	$b_2 =$	$-$ 45·8333333
$c_1 =$	28·25	$c_2 =$	$-$ 17·3333333
$d_1 =$	$-$ 42·4166666	$d_2 =$	51·83333333
$r_1 =$	$-$j 1·95833333	$r_2 =$	j 1·54166667
$g_1 =$	$-$j 429·333333	$g_2 =$	j 688·999999
$\sigma_1 =$	j 1·70833333	$\sigma_2 =$	$-$j 1·79166666
$s_1 =$	j 784·499999	$s_2 =$	$-$j 580·666666
$\frac{1}{4}\varphi =$	0°	$\lvert \mathbf{A} \rvert =$	1

Constraints:

$\Delta_1 =$	$\Delta_2 =$	$-$ 140·916666
$\delta_1 =$	$\delta_2 =$	141·916667
$\alpha_1 =$	$-\beta_2 =$	100·75
$\beta_1 =$	$-\alpha_2 =$	$-$ 126·83333
$G_1 =$	$G_2 =$	$-$j3299·83333
$R_1 =$	$R_2 =$	j2·20833331

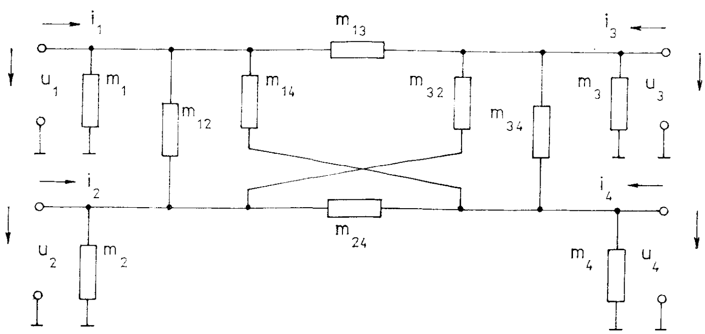

Fig. 2. The useful circuit model for a general lossless reciprocal four-port is the network consisting of 10 reactance one-ports.

is shown. The entries of its matrix $\mathbf{Y}$, defined in (10) have the values

$$y_a = m_1 + m_{12} + m_{13} + m_{14}\,, \quad y_b = m_2 + m_{12} + m_{23} + m_{24}\,,$$
$$y_c = m_3 + m_{13} + m_{23} + m_{34}\,, \quad y_d = m_4 + m_{14} + m_{24} + m_{34}\,,$$
$$y_2 = y_5 \ -m_{12}\,, \quad y_3 = y_9 \ = -m_{13}\,, \quad y_4 = y_{13} = -m_{14}\,,$$
$$y_7 = y_{10} = -m_{23}\,, \quad y_8 = y_{14} = -m_{24}\,, \quad y_6 = y_{15} = -m_{34}\,. \qquad (16)$$

If all ten one-ports with admittances m_i and m_{ik} are lossless, then m_{ik} are imaginary and the four-port is also lossless and reciprocal. Similar network representation is never possible in case of general linear nonreciprocal four-port.

The 4^{th} degree secular equation of the chain matrix $\mathbf{A}$ of a general linear passive or active four-port is a fourth-degree polynomial. Nevertheless the losslessness simplifies the solution. The equation has the form

$$\lambda^4 + K_3\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0 = 0 \tag{17}$$

where K_i, $i = 0, \ldots, 3$ are simplex constants, $K_0 = \exp(j\varphi)$. We multiply all entries of $\mathbf{A}$ by a complex constant $\exp(-j\varphi/4)$ to get a modified matrix $\mathbf{A}$. Its determinant equals unity. The form of its characteristic equation is identical to (17) but K_1, K_2, K_3 may be easily calculated. The following equations hold

$$K_2 = \Delta_1 + \Delta_2 + \delta_1 + \delta_2 + (a_1 + b_1)(a_2 + b_2) - c_1c_2 - d_1d_2 + $$
$$+ s_1o_2 + s_2o_1 ,$$
$$K_3 = -(a_1 + b_1 + a_2 + b_2), \quad K_1 = K_3^*, \quad K_0 = 1 . \tag{18}$$

where K_2 is a real constant (see (3)). Solving numerically (17) we get the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $\mathbf{A}$.

We shall find the 4 constituents $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ of the modified $\mathbf{A}$. Then the N^{th} power of the original $\mathbf{A}$ may be simply calculated using the formula

$$\mathbf{A}^N = \mathbf{A}_1 \,.\, \lambda_1^N + \mathbf{A}_2 \,.\, \lambda_2^N + \mathbf{A}_3 \,.\, \lambda_3^N + \mathbf{A}_4 \,.\, \lambda_4^N \tag{19}$$

The real continuous variable N represents the length of the line which we are modelling by a fourth-port with distributed parameters. Such a model is useful to study the properties of nonreciprocal anisotropic coupled lines which are perspective in modern optics. To reach unrestricted wave propagation for any length of N, the absolute value of all four eigenvalues λ_i, $i = 1, 2, 3, 4$ should be equal to unity. In the opposite case the evanescent wave propagating modes arise. To calculate all four constituents the knowledge of four different powers of $\mathbf{A}$ is required. We choose $N = 0, 1, -1, 2.$ and derive a set of 4 linear matrix equations with unknowns $\mathbf{A}_i$, $i = 1, 2, 3, 4$ in the form

$$\begin{aligned}
&\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \mathbf{I} ,\\
&\mathbf{A}_1 \,.\, \lambda_1 + \mathbf{A}_2 \,.\, \lambda_2 + \mathbf{A}_3 \,.\, \lambda_3 + \mathbf{A}_4 \,.\, \lambda_4 = \mathbf{A} ,\\
&\mathbf{A}_1 \,.\, \lambda_1^{-1} + \mathbf{A}_2 \,.\, \lambda_2^{-1} + \mathbf{A}_3 \,.\, \lambda_3^{-1} + \mathbf{A}_4 \,.\, \lambda_4^{-1} = \mathbf{A}^{-1} ,\\
&\mathbf{A}_1 \,.\, \lambda_1^2 + \mathbf{A}_2 \,.\, \lambda_2^2 + \mathbf{A}_3 \,.\, \lambda_3^2 + \mathbf{A}_4 \,.\, \lambda_4^2 = \mathbf{A}^2 .
\end{aligned} \tag{20}$$

where $\mathbf{I}$ is a unit matrix.

After solving this set of equations we are able to analyze the flow of waves and signals in a lossless nonreciprocal four-port with distributed parameters. Unfortunately the selection of the four-port by an arbitrary choice of 16 real parameters (13) to define the admittance matrix $\mathbf{Y}$ gives no practical chance for realizing $\mathbf{A}$ with satisfactory signal transmission. This is the reason why in the next sections we shall develop procedures for synthesizing $\mathbf{A}$ with prescribed nonevanescent wave propagation.

3. SEMIRECIPROCAL FOUR-PORT

We shall introduce a nonreciprocal network which has similar properties as reciprocal system [10]. Let us define a four-port which has the chain matrix $\mathbf{A}$ satisfying the conditions

$$a_2 = a_1^*, \quad b_2 = b_1^*, \quad c_2 = c_1^*, \quad d_2 = d_1^*, \tag{21}$$
$$r_2 = -r_1^*, \quad g_2 = -g_1^*, \quad o_2 = -o_1^*, \quad s_2 = -s_1^*.$$

After a complicated, time-consuming derivation we find the secular equation to be of the form

$$\lambda^4 - 4a_0\lambda^3 + K_2\lambda^2 - 4a_0\lambda + 1 = 0 \tag{22}$$

where

$$a_0 = (a_1 + b_1 + a_2 + b_2)/4,$$
$$K_2 = 8a_0^2 - 2a_{02},$$

and $4a_0$ and $4a_{02}$ are the traces of $\mathbf{A}$ and $\mathbf{A}^2$, respectively. The pairs of the roots λ_1, λ_2 and λ_3, λ_4 of (22) are mutually reciprocal:

$$\lambda_2 = 1/\lambda_1, \quad \lambda_4 = 1/\lambda_3. \tag{23}$$

The 4[th] degree equation (22) is the product of two quadratic equations

$$(\lambda - \lambda_1)(\lambda - 1/\lambda_1) = \lambda^2 - 2z_1\lambda + 1, \tag{24}$$
$$(\lambda - \lambda_3)(\lambda - 1/\lambda_3) = \lambda^2 - 2z_2\lambda + 1.$$

Here we have introduced two new variables z_1, z_2. They are relevant for the determination of the type of the two propagating modes. They are mutually related by the additive form

$$z_1 = a_0 + c_0, \quad z_2 = a_0 - c_0. \tag{25}$$

Comparing Eqn's (22), (24) we get the square of the new parameter c_0

$$c_0^2 = (1 + a_{02})/2 - a_0^2. \tag{26}$$

The value of c_0 may be real or imaginary depending on the sign of the right-hand side of the last equation. The propagating modes differ in these two possible cases significantly. After computation of z_1, z_2 we get all four eigenvalues from the relations

$$\begin{aligned}\lambda_1 &= z_1 + (z_1^2 - 1)^{1/2}\\ \lambda_2 &= z_1 - (z_1^2 - 1)^{1/2}\\ \lambda_3 &= z_2 + (z_2^2 - 1)^{1/2}\\ \lambda_4 &= z_2 - (z_2^2 - 1)^{1/2}\end{aligned} \tag{27}$$

Let us suppose that the following conditions hold:

$$|a_0 + c_0| < 1, \quad |a_0 - c_0| < 1. \tag{28}$$

In this case we have also $|z_1| < 1$, $|z_2| < 1$ and the absolute values of all four eigenvalues equals unity. The complex quantities λ_i^N in (19) change only the phase depending on the variable N. It represents the length of the coupled lines where two "free wave propagating modes" may exist.

In all other cases at least one of the two wave propagating modes is "evanescent". The limit of the corresponding quantity λ_i^N for $N \to \infty$ is either zero or infinity. The signals carried by the evanescent mode are rapidly attenuated in the space. If c_0 is real and either $|a_0 + c_0| > 1$ or $|a_0 - c_0| > 1$ then either $|z_1| > 1$ or $|z_2| > 1$. As a consequence, at least one pair of the eigenvalues is real. The wave propagating mode is then "exponentially evanescent". In this paper we are introducing the "oscillating evanescent wave propagating mode". It occurs if c_0 is complex (see (26)).

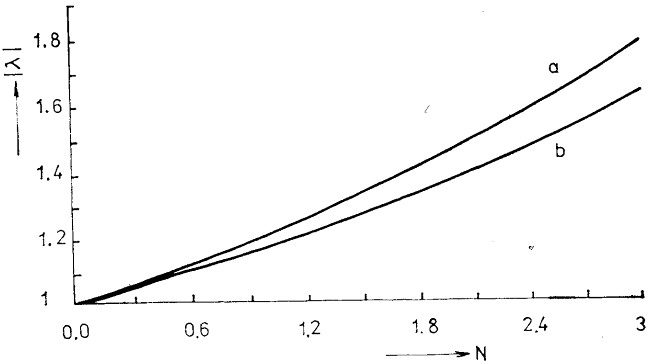

Fig. 3. Dependences of N^{th} power of the absolute value of the eigenvalues of semireciprocal four-port. Only two types of evanescent modes are shown for which $\lambda = z + (z^2 - 1)^{1/2}$, $|\lambda| = 1$. Curve a: $a_0 = 0{\cdot}97$, $c_0 = 0{\cdot}05$; curve b: $a_0 = 0{\cdot}97$, $c_0 = \mathrm{j}\,0{\cdot}05$.

In Fig. 3 the dependences of λ_1 for two types of evanescent modes are shown. The qualitative difference is apparent. Many phenomena of the evanescent modes are well known from electromagnetic theory and optics. In our analysis the explanation is based on exact methods of circuit and system theories.

Let us note that we are dealing with lossless systems. To describe also the phenomena of "leaky modes" our procedures may be modified with respect to space distributed losses.

Having the possibility of calculating the eigenvectors and mode variables z_1, z_2 in analytically simple and closed form, we are able to derive similar procedures for calculating the matrix constituents $\mathbf{A}_i$, $i = 1, 2, 3, 4$. We choose low powers of $\mathbf{A}$ again. The following equations hold

$$
\begin{aligned}
&\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \mathbf{I}\,,\\
&\mathbf{A}_1\lambda_1 + \mathbf{A}_2/\lambda_1 + \mathbf{A}_3\lambda_3 + \mathbf{A}_4/\lambda_3 = \mathbf{A}\,,\\
&\mathbf{A}_1/\lambda_1 + \mathbf{A}_2\lambda_1 + \mathbf{A}_3/\lambda_3 + \mathbf{A}_4\lambda_3 = \mathbf{A}^{-1}\,,\\
&\mathbf{A}_1\lambda_1^2 + \mathbf{A}_2/\lambda_1^2 + \mathbf{A}_3\lambda_3^2 + \mathbf{A}_4/\lambda_3^2 = \mathbf{A}^2\,,\\
&\mathbf{A}_1/\lambda_1^2 + \mathbf{A}_2\lambda_1^2 + \mathbf{A}_3/\lambda_3^2 + \mathbf{A}_4\lambda_3^2 = \mathbf{A}^{-2}\,.
\end{aligned}
\tag{29}
$$

Solving this set of linear matrix equations in variables $\mathbf{A}_1 + \mathbf{A}_2$ and $\mathbf{A}_3 + \mathbf{A}_4$ and using (25) we get the expressions

$$\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I} \,.\, z_2/(z_2 - z_1) + (\mathbf{A} + \mathbf{A}^{-1})/(2z_2 - 2z_1)\,, \tag{30}$$
$$\mathbf{A}_3 + \mathbf{A}_4 = \mathbf{I} \,.\, z_1/(z_1 - z_2) - (\mathbf{A} + \mathbf{A}^{-1})/(2z_2 - 2z_1)\,.$$

We introduce new supporting variables

$$q_1 = (z_1^2 - 1)^{1/2}\,, \quad q_2 = (z_2^2 - 1)^{1/2}\,. \tag{31}$$

For $N = 2$ the following relations hold

$$(\lambda_1^2 + 1/\lambda_1^2)/2 = 2z_1^2 - 1\,, \quad (\lambda_3^2 + 1/\lambda_3^2)/2 = 2z_2^2 - 1\,, \tag{32}$$
$$(\lambda_1^2 - 1/\lambda_1^2)/2 = 2q_1 z_1\,, \quad (\lambda_3^2 - 1/\lambda_3^2)/2 = 2q_2 z_2\,.$$

We may write another set of two linear matrix equations. To this end we subtract the second and the third equation in (29), and use similar relations for $N = \pm 2$

$$(\mathbf{A}_1 - \mathbf{A}_2)\, q_1 + (\mathbf{A}_3 - \mathbf{A}_4)\, q_2 = \mathbf{M}\,, \tag{33}$$
$$(\mathbf{A}_1 - \mathbf{A}_2)\, q_1 z_1 + (\mathbf{A}_3 - \mathbf{A}_4)\, q_2 z_2 = \mathbf{L}\,,$$

where $\mathbf{M} = (\mathbf{A} - \mathbf{A}^{-1})/2\,, \quad \mathbf{L} = (\mathbf{A}^2 - \mathbf{A}^{-2})/4\,.$

We substitute the parameters a_0, c_0 into the mode variables z_1, z_2. The solution is of the form

$$\mathbf{A}_1 - \mathbf{A}_2 = [-\mathbf{M}(a_0 - c_0) + \mathbf{L}]/2q_1 c_0\,, \tag{34}$$
$$\mathbf{A}_3 - \mathbf{A}_4 = [\;\mathbf{M}(a_0 + c_0) - \mathbf{L}]/2q_2 c_0\,.$$

Now we shall introduce the matrix $\mathbf{C}$ and decompose $\mathbf{L}$ defined in (33):

$$\mathbf{C} = (\mathbf{A} + \mathbf{A}^{-1})/2\,, \quad \mathbf{L} = \mathbf{C} \,.\, \mathbf{M}\,. \tag{35}$$

Using parameters a_0 and c_0 defined in (25) instead of mode variables z_1, z_2 and solving Eqn's (30), (33) we get the 4 constituents multiplied by the scalar number $4c_0$:

$$4c_0\mathbf{A}_1 = \mathbf{C}_1 + \mathbf{C}_1 \,.\, \mathbf{M}/q_1\,, \quad 4c_0\mathbf{A}_2 = \mathbf{C}_1 - \mathbf{C}_1 \,.\, \mathbf{M}/q_1\,, \tag{36}$$
$$4c_0\mathbf{A}_3 = \mathbf{C}_2 + \mathbf{C}_2 \,.\, \mathbf{M}/q_2\,, \quad 4c_0\mathbf{A}_3 = \mathbf{C}_2 - \mathbf{C}_2 \,.\, \mathbf{M}/q_2\,,$$

where $\mathbf{C}_{1,2} = \mathbf{I} \,.\, a_0 \pm (\mathbf{I} \,.\, c_0 - \mathbf{C})\,.$

In diagonal entries of $\mathbf{C}$ also appears the parameter a_0 which will be subtracted from the elements of $\mathbf{I} \,.\, a_0$. On the diagonals $\mathbf{C}_{1,2}$ remain the linear functions of the wave parameter c_0. We introduce them in the form

$$\Phi = c_0 + \mathrm{j}\varkappa\,, \quad \Psi = c_0 - \mathrm{j}\varkappa\,, \tag{37}$$

where $\varkappa = [(a_1 - a_1^*) - (b_1 - b_1^*)]/2\mathrm{j}$. Using these expressions and the definition of $\mathbf{C}$ in (35) we get the simple description of

$$\mathbf{C}_1 = \begin{bmatrix} \Phi & R & v & 0 \\ G & \Psi & 0 & v \\ v^* & 0 & \Phi & -R \\ 0 & v^* & -G & \Psi \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} \Psi & -R & -v & 0 \\ -G & \Phi & 0 & -v \\ -v^* & 0 & \Psi & R \\ 0 & -v^* & G & \Phi \end{bmatrix} \tag{38}$$

where

$$R = (r_1 + r_1^*)/2\,, \quad G = (g_1 + g_1^*)/2\,,$$
$$v = (c_1 + d_2^*)/2 = (c_1 + d_1)/2\,.$$

The final form of matrix constituents is

$$4c_0\mathbf{A}_{1,2} = \mathbf{C}_1 \pm \mathbf{B}_1/q_1\,, \tag{39}$$
$$4c_0\mathbf{A}_{3,4} = \mathbf{C}_2 \pm \mathbf{B}_2/q_2\,,$$

where $\mathbf{B}_1 = \mathbf{C}_1 \,.\, \mathbf{M}$, $\mathbf{B}_2 = \mathbf{C}_2 \,.\, \mathbf{M}$ and $\mathbf{M}$ is defined in (33). The last equations may be used directly for computation of the N^{th} power of a given semireciprocal matrix $\mathbf{A}$. In synthesizing $\mathbf{A}$ the matrix multiplications $\mathbf{C}_1 \,.\, \mathbf{M}$ and $\mathbf{C}_2 \,.\, \mathbf{M}$ are too complicated to find an efficient design algorithm. Nevertheless these multiplications show the simple structure of the matrices $\mathbf{B}_{1,2}$ belonging to both wave propagating modes. They contain only 10 different complex parameters. The following relations hold

$$b_{44} = -b_{11}\,, \quad b_{33} = -b_{22}\,, \quad b_{34} = b_{12}\,, \tag{40}$$
$$b_{43} = b_{21}\,, \quad b_{24} = -b_{13}\,, \quad b_{42} = -b_{13}\,.$$

We shall now specify the procedures and constraints for selecting the entries of both matrices $\mathbf{B}_1$, $\mathbf{B}_2$ as a base for synthesizing $\mathbf{A}$ of a semireciprocal four-port.

All four constituents are composed of the four eigenvectors of $\mathbf{A}$ multiplied by different scalar parameters. The matrices $\mathbf{A}_k$ are singular since their determinants are equal to zero. All subdeterminants 2×2 of $\mathbf{A}_k$, $k = 1, 2, 3, 4$ should be zero, too. As an example we show the expression for the subdeterminant Δ_{11} of $\mathbf{A}_1$ in (39)

$$\Delta_{11} = (\Phi + b_{11}/q_1)(\Psi + b_{22}/q_1) - (R + b_{12}/q_1)(G + b_{21}/q_1) \tag{41}$$

The mode parameters $q_{1,2}$ are defined in (31). They may be either real or complex. To guarantee that $\Delta_{11} = 0$ we have to fulfil two conditions

$$(\Phi\Psi - RG)\,q^2 + b_{11}b_{22} - b_{12}b_{21} = 0 \tag{42}$$
$$\Phi b_{22} + \Psi b_{11} - Rb_{21} - Gb_{12} = 0$$

Similar consideration leads to 36 constraints among entries of any $\mathbf{A}_k$. In 21 of them the wave propagation variable $q^2 = z^2 - 1$ (see (31)) plays its role. In 15 of these conditions only the relations between the elements of the matrices (38) take place. Fortunately the wisdom of nature causes that the validity of a restricted number of conditions guarantees the validity of the remaining ones. The method simplifies the synthesis substantially. The constraints are formulated in Tables 3 and 4.

Table 3. 21 conditions for the entries of the constituents $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ of the chain matrix $\mathbf{A}$ of a lossless semireciprocal four-port; q is the mode variable which depends on the eigenvalues of $\mathbf{A}$ according to (31).

$$\begin{aligned}
(\Phi\Psi - RG)\, q^2 + b_{11}b_{22} - b_{12}b_{21} &= 0\\
(\Phi\Psi - v\bar{v})\, q^2 - b_{11}b_{22} - b_{13}b_{31} &= 0\\
(GR + v\bar{v})\, q^2 - b_{12}b_{21} - b_{13}b_{31} &= 0\\
v\Phi q^2 - b_{11}b_{13} - b_{21}b_{14} &= 0\\
\bar{v}\Phi q^2 - b_{11}b_{31} - b_{12}b_{41} &= 0\\
v\Psi q^2 + b_{22}b_{13} - b_{12}b_{23} &= 0\\
\bar{v}\Psi q^2 + b_{22}b_{31} - b_{21}b_{32} &= 0\\
\Phi^2 q^2 - b_{11}^2 - b_{14}b_{41} &= 0\\
\bar{v}^2 q^2 - b_{31}^2 - b_{32}b_{41} &= 0\\
\Psi^2 q^2 - b_{22}^2 - b_{32}b_{23} &= 0\\
v^2 q^2 - b_{13}^2 - b_{14}b_{23} &= 0\\
vGq^2 - b_{11}b_{23} + b_{13}b_{21} &= 0\\
\bar{v}Gq^2 - b_{22}b_{41} - b_{31}b_{21} &= 0\\
\Phi Gq^2 + b_{13}b_{41} - b_{11}b_{21} &= 0\\
\Psi Gq^2 - b_{32}b_{23} - b_{22}b_{21} &= 0\\
G^2 q^2 - b_{21}^2 + b_{21}b_{41} &= 0\\
vRq^2 - b_{22}b_{14} - b_{13}b_{12} &= 0\\
\bar{v}Rq^2 - b_{11}b_{32} + b_{31}b_{12} &= 0\\
\Phi Rq^2 + b_{31}b_{14} - b_{11}b_{12} &= 0\\
\Psi Rq^2 - b_{13}b_{32} - b_{22}b_{12} &= 0\\
R^2 q^2 - b_{12}^2 - b_{14}b_{32} &= 0
\end{aligned}$$

Table 4. 15 conditions for the entries of the constituents $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$. There is no dependence on the eigenvalues here.

$$\begin{aligned}
\Psi b_{11} + b_{22} - Rb_{21} - Gb_{12} &= 0\\
\Psi b_{11} - b_{22} - b_{31} - \bar{v}b_{13} &= 0\\
Gb_{12} - Rb_{21} - b_{31} + \bar{v}b_{13} &= 0\\
vb_{11} - \Phi b_{13} - Gb_{14} &= 0\\
\bar{v}b_{11} - \Phi b_{31} - Rb_{41} &= 0\\
vb_{22} + \Psi b_{13} - Rb_{23} &= 0\\
\bar{v}b_{22} + \Psi b_{31} - Gb_{32} &= 0\\
vb_{21} - \Phi b_{23} + Gb_{13} &= 0\\
\bar{v}b_{21} - \Phi b_{41} - Gb_{31} &= 0\\
vb_{41} - \Phi b_{21} + Gb_{11} &= 0\\
\bar{v}b_{23} - \Psi b_{21} + Gb_{22} &= 0\\
vb_{12} - \Psi b_{14} - Rb_{13} &= 0\\
\bar{v}b_{12} - \Phi b_{32} + Rb_{31} &= 0\\
\bar{v}b_{14} - \Phi b_{12} + Rb_{11} &= 0\\
vb_{32} - \Psi b_{12} + Rb_{22} &= 0
\end{aligned}$$

4. SYNTHESIS OF THE CHAIN MATRIX OF SEMIRECIPROCAL FOUR-PORT

First of all we choose the constants a_0, c_0 to determine the type of wave propagation. Doing this, the mode variables z_1, z_2 and all 4 eigenvalues are determined. Then we choose 8 remaining "free" parameters to get semireciprocal or reciprocal four-port. Let us go back to the method of designing a general nonreciprocal four-port explained in Section 2. We had the freedom to choose 16 real parameters specified in (10). If the four-port should be semireciprocal the choice is restricted to only 10 parameters. The following relations hold

$$
\begin{aligned}
&e_5 = -e_2\,, \quad e_4 = -e_3\,, \quad f_5 = f_2\,, \\
&f_4 = f_3\,, \quad f_b = f_a\,, \quad f_d = f_c\,.
\end{aligned}
\tag{43}
$$

The validity of such conditions ensures the symmetry of the entries of **A**, expressed in the next equations

$$
\begin{aligned}
&a_2 = a_1^*\,, \quad b_2 = b_1^*\,, \quad c_2 = c_1^*\,, \quad d_2 = d_1^*\,, \\
&r_2 = -r_1^*\,, \quad g_2 = -g_1^*\,, \quad o_2 = -o_1^*\,, \quad s_2 = -s_1^*\,.
\end{aligned}
\tag{44}
$$

These conditions are not satisfied for the lossless reciprocal four-port with entries shown in Table 2. The reason is that the conditions $f_a = f_b$, $f_5 = f_2$, $f_4 = f_3$ have not been respected. It means that not all reciprocal four-ports belong to the class of semireciprocal networks.

Our first task will be to synthesize a semireciprocal four-port with prescribed wave propagating parameters. We select 8 quantities a_0, c_0, R, G, α, $\varkappa$, b_{14} for modes 1, 2. The first 6 are real, b_{14} may be complex. Altogether it represents 10 real numbers. It is equivalent to the design based on selection of the constants necessary for construction of **Y**. Let us remind that the specification of c_0 and $\varkappa$ determines the value of Φ, Ψ. We have to respect the other constraint which binds the entries of **C** in (37)

$$
\Phi\Psi - wvv^* - RG = 0\,, \tag{45}
$$

where $w = 1$ for c_0 real, $w = -1$ for c_0 imaginary (see (26)), $v = (vv^* \exp(j\alpha))^{1/2}$ and α is one of 10 starting parameters. Using Table 4 we get the set of 3 linear equations with unknown variables b_{11}, b_{22} b_{13} written in the matrix form

$$
\begin{bmatrix} v & 0 & \Phi \\ 0 & v^* & 0 \\ \Psi & \Phi & -v^* \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{22} \\ b_{13} \end{bmatrix} = \begin{bmatrix} G\ b_{14} \\ -G\ b_{32} \quad -\Psi\ b_{31} \\ v\ b_{31} \end{bmatrix} \tag{46}
$$

For inversion of the coupling matrix we have used with advantage the constraint (45). The final solution has the form

$$
\begin{aligned}
b_{11} &= m_1 + k_1 b_{31}\,, \\
b_{22} &= m_2 + k_2 b_{31}\,, \\
b_{13} &= m_3 + k_3 b_{31}\,,
\end{aligned}
\tag{47}
$$

The complex constants are of the form

$$m_1 = (\Phi^2 G b_{32} - v^{*2} G b_{14})/v^* R G\,, \quad k_1 = \Phi/v^*\,,$$
$$m_2 = G b_{32}/v^*\,, \quad k_2 = -\Psi/v^*\,, \tag{48}$$
$$m_3 = -(\Psi G b_{14} + v G b_{32})/v^* R G\,, \quad k_3 = -v/v^*\,.$$

To complete the solution we use the first equation in Table 3 to find directly the value of b_{31}. We get a quadratic equation

$$b_{31}^2 - 2b b_{31} + c = 0 \tag{49}$$

where

$$b = (m_1 k_2 + m_2 k_1 + m_3)/m_4\,, \quad c = (RGq^2 - m_1 m_2)/m_4\,,$$
$$m_4 = k_3 - k_1 k_2\,.$$

After solving (50) the remaining entries of **B** are computed from other relations in Table 4. We get the results

$$b_{14} = (v b_{11} - \Phi b_{13})/G\,, \quad b_{32} = -b_{13}^*\,,$$
$$b_{23} = (v b_{22} + \Psi b_{13})/R\,, \quad b_{41} = -b_{23}^*\,, \tag{50}$$
$$b_{21} = (\Phi b_{23} - G b_{13})/v\,, \quad b_{43} = -b_{21}^*\,,$$
$$b_{12} = (\Psi b_{14} + R b_{13})/v\,, \quad b_{34} = -b_{12}^*\,.$$

Performing all these computations for both mode variables q_1, q_2 introduced in (25), (31) open the way to find $\mathbf{B}_1$, $\mathbf{B}_2$ and finally, all powers N of the chain matrix $\mathbf{A}$ from (42) and (19).

The proposed design does not work if the matrix in (46) is singular. In this case it happens that either $R = 0$ or $G = 0$. We restrict our explanation to the most important case when $R = 0$ and $G = 0$. The remaining starting parameters are a_0, c_0, v, $\varkappa$, b_{12}, b_{21} for mode 1 and 2. We suppose that c_0 is real and b_{12}, b_{21} are imaginary $[\mathrm{Re}(b_{12}) = 0,\ \mathrm{Re}(b_{21}) = 0]$ for both modes. We have the freedom of selecting only 8 constants at the beginning. If we satisfy (28) than two nonevanescent wave modes may propagate to unlimited distance in space. According to our choice $w = 1$ it follows from (47) that

$$\Phi\Psi = v v^* \tag{51}$$

We use Table 3 and 4 to derive formulae for computation of all entries of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$ in (42). They have the form

$$b_{11} = \pm\Phi k\,. \quad b_{22} = \pm\Psi k\,, \quad b_{13} = \pm v k\,, \quad b_{31} = \pm v^* k\,,$$
$$b_{14} = b_{12} v^*/\Psi\,, \quad b_{32} = b_{12} v^*/\Phi\,, \quad b_{23} = b_{21}\Psi/v^*\,, \tag{52}$$
$$b_{41} = b_{21}\Phi/v^*\,, \quad k = (q^2 - b_{12} b_{21}/v v^*)^{1/2}\,.$$

If $R = 0$ and $G = 0$ then the constants m_1, m_2, m_3 in (48) equal to zero that may be easy proved. This fact explains the relations of proportionality between the entries on the main and second diagonal of $\mathbf{B}_1$, $\mathbf{B}_2$.

Now we are able to design chain matrices of various semireciprocal four-ports. Let us turn our attention to the problem of synthesis of chain matrices of the semireciprocal four-port with space distributed parameters based on the specification of some restricted number of immitance parameters of $\mathbf{B}_1$, $\mathbf{B}_2$, e.g. $r_{1,2}$, $g_{1,2}$, $o_{1,2}$, $s_{1,2}$. It represents 16 constants. But we have at most 10 degrees of freedom to design a physically realized semireciprocal $\mathbf{A}$. It was shown in the last example that if $G = 0$ and $R = 0$ the number of free parameters is reduced to 8. We choose more, nine parameters c_0, complex b_{12}, b_{21}, o_1, s_1 for one mode only. The last constant a_0 which together with c_0 determines the character of both wave propagating modes will complete the choice. The synthesis procedures are different for c_0 real and c_0 imaginary. Let us begin with the design of $\mathbf{A}$ with c_0 imaginary. We shall use the symbol jc_0 for c_0 imaginary. Using Table 4 we are able to write the following equations

$$\begin{aligned} b_{13} &= b_{11}b_{23}/b_{21}\,, \qquad b_{22} = b_{11}/h\,, \qquad b_{31} = b_{11}b_{12}/b_{14}\,, \\ b_{41} &= b_{21}/b_{23}\,, \qquad b_{32} = b_{12}/b_{14}\,, \quad \Phi/\Psi = -h\,, \\ h &= b_{21}b_{14}/b_{12}b_{23}\,, \end{aligned} \tag{53}$$

The key to the solution is the knowledge of the ratio Φ/Ψ. Let us suppose that if h belongs to the mode 1 then $1/h$ belongs to the mode 2. Since the change of modes causes mutual change of Φ and Ψ, the following equations hold

$$(c_0 + \varkappa)/(c_0 - \mathbf{x}) = h\,. \quad (c_0 - \varkappa)/(c_0 + \varkappa) = 1/h\,. \tag{54}$$

These equations are identical. Using one of them we derive the expression

$$\varkappa = c_0(h - 1)/(h + 1)\,. \tag{55}$$

For h we get $\varkappa$, for $1/h - \varkappa$. It confirms that our supposition was correct. The signs of $\varkappa$ in Φ and Ψ in $\mathbf{C}$ in (37) are different for modes 1, 2. If the quantities c_0, $\varkappa$ should be real then h has to be real, too. It implicates the new constraint on the phase of complex quantities b_{14}, b_{23}, b_{12}, b_{21}.

$$\varphi_{21} + \varphi_{14} - \varphi_{12} - \varphi_{23} = 0\,. \tag{56}$$

It reduces the number of free parameters to 8. Using Table 4 and (55) we get

$$\begin{aligned} &v = \Phi b_{23}/b_{21}\,, \quad \bar{v} = \Psi b_{21}/b_{23}\,, \\ &b_{22} = -b_{11}\Psi/\Phi\,, \quad b_{13} = b_{11}v/\Phi\,, \quad b_{31} = b_{11}\bar{v}/\Phi\,. \end{aligned} \tag{57}$$

Using these relations and Table 3 we derive the quadratic equation

$$v\bar{v}q^2 - b_{11}^2\Psi/\Phi - b_{12}b_{21} = 0 \tag{58}$$

where $q^2 = a_0^2 - (jc_0)^2 - 1 \pm 2\,ja_0c_0$, (see (31)), which enables to determine the value of b_{11} for both modes and with the help of (57) the remaining quantities b_{22}, b_{13}, b_{31}. The problem is solved up. Let us remind that after a simple derivation from (22), (23), (31) the eigenvalues may be expressed in the form

$$\lambda_{1,2} = z_1 \pm q_1\,, \quad \lambda_{3,4} = z_2 \pm q_2\,. \tag{59}$$

Using (42), (19) for $N = 1$ we express the chain matrix in the explicit form

$$\mathbf{A} = \mathbf{I}(c_0 \pm j\varkappa) + \mathbf{D} + (\mathbf{B}_1 + \mathbf{B}_2)/2c_0 \tag{60}$$

where $\mathbf{D}$ is sparse and contains nonzero parameters v, $\bar{v}$ on its second diagonal only, and $\mathbf{B}_1$, $\mathbf{B}_2$ are constituent matrices for modes 1, 2 we have just derived.

For c_0 imaginary the entries of $\mathbf{B}_1$, $\mathbf{B}_2$ are mutually coupled by the relations

$$\begin{aligned} b_{11(2)} &= b^*_{22(1)}, & b_{22(2)} &= b^*_{11(1)}, \\ b_{13(2)} &= -b^*_{31(1)}, & b_{31(2)} &= -b^*_{13(1)}, \\ b_{12(2)} &= b^*_{12(1)}, & b_{21(2)} &= b^*_{21(1)}, \\ b_{14(2)} &= b^*_{32(1)}, & b_{32(2)} &= b^*_{14(1)}, \\ b_{23(2)} &= b^*_{41(1)}, & b_{41(2)} &= b^*_{23(1)}, \end{aligned} \tag{61}$$

where the number in brackets corresponds to the modes 1, 2. Let us turn our attention to the case when c_0 is real. If the conditions (28) holds we get two free propagating modes. We choose 7 starting parameters a_0, c_0, $|b_{21(1)}/b_{23(1)}|$, and φ_{12}, φ_{14}, φ_{21}, φ_{23} (i.e., 4 phases of b_{12}, b_{14}, b_{21}, b_{23}). We satisfy the condition

$$|b_{12(1)}/b_{14(1)}| = |b_{21(1)}/b_{23(1)}| \tag{62}$$

In the consequence, the module of h introduced in (53) equals unity. If c_0 was imaginary the h was real. Now c_0 is real and h is complex. It is of the value

$$h = \exp(j\vartheta) \tag{63}$$

where $\vartheta = \frac{1}{2}(\varphi_{21} + \varphi_{14} - \varphi_{12} - \varphi_{23})$.

Using (55) we derive the equation

$$\varkappa = c_0 \sin(\vartheta)/\cos(\vartheta) \tag{64}$$

The change of the mode causes the change of the sign of $\varkappa$. As a consequence the argument ϑ of the exponential function in (63) should change the sign at the time. Therefore the relevant entries from the matrix $\mathbf{C}$ in (37) are already known. Their value are

$$\begin{aligned} \Phi &= c_0 + j\varkappa, & \Psi &= c_0 - j\varkappa, \\ v &= \Phi b_{23}/b_{21}, & \bar{v} &= \Psi b_{21}/b_{23}, \\ \bar{v} &= v^*, & v\bar{v} &= c_0^2 + \varkappa^2. \end{aligned} \tag{65}$$

After solving (58) we get b_{11}, b_{22}, b_{13}, b_{31} with the help of (57). The constant q^2 in (58) are real and have the values

$$q^2 = a_0^2 + c_0^2 \pm 2a_0c_0 - 1. \tag{66}$$

The entries of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$ in (60) are not coupled by the relations similar to (61). If we already know $\mathbf{B}_1$ we have to repeat the procedure for mode 2. For the entries b_{12}, b_{21} the following relations hold

$$b_{21(2)} = -b^*_{21(1)}, \quad b_{12(2)} = -b^*_{12(1)}, \tag{67}$$

The conditions for b_{12}, b_{21} guarantee that a_{12}, a_{34}, a_{21}, a_{43} of $\mathbf{A}$ are imaginary ($R = 0$, $G = 0$, see (60)). For the calculation of the remaining immittance parameters the knowledge of b_{12}, b_{21}, b_{11}, b_{22}, b_{13}, b_{31}, is required.

The calculations for both modes are based on the equations in the Table 3. The expressions are

$$b_{23} = b_{13}b_{21}/b_{11}\,, \quad b_{14} = -b_{13}b_{12}/b_{22}\,, \qquad (68)$$
$$b_{41} = b_{21}^2/b_{23}\,, \quad b_{32} = b_{12}^2/b_{14}\,.$$

Now we are ready to compute all the entries of $\mathbf{A}$ using (60) and to check its constrains (4) and (9). A more detailed explanation and description of the software may be found in the report [11].

Let us show one possible application of the synthesis procedure described for the solution of the set of partial differential equations describing the wave propagation in reciprocal or nonreciprocal anisotropic media [9, 10]. In most cases the anisotropy may be described by permittivity and permeability tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. In some cases they have in addition to the diagonal elements only one pair of nonzero off-diagonal elements. In a stratified geometry these represent one of three orientations, namely polar, equatorial and longitudinal. It depends on whether the optical axis lies in the plane of the interface, incidence or is perpendicular to both, respectively. The tensors have the form

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & \varepsilon_{yz} \\ 0 & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & \mu_{yz} \\ 0 & \mu_{zy} & \mu_{zz} \end{bmatrix}. \qquad (69)$$

The only restriction placed on the elements of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ is caused by the requirement that the material is passive and lossless. Then all diagonal elements should be real and positive, the off-diagonal elements have to be mutually complex conjugate.

Consider a homogeneous slab geometry in which the interface lies in the $y - z$ plane. The field components contain the common phase factor

$$\exp\left[\mathrm{j}(\omega t - \beta_y y - \beta_z z)\right].$$

The propagation constants β_y, β_z are normalized to the vacuum constant $\beta = \omega(\varepsilon_0\mu_0)^{1/2}$. Using four of the six Maxwell's equations the set of four coupled differential equations was derived in the form

$$\mathrm{d}\boldsymbol{U}/\mathrm{d}x = -\mathbf{K}\,.\,\boldsymbol{U} \qquad (70)$$

where $\boldsymbol{U} = [y_0\, E_y(x)\;\; z_0\, H_z(x)\;\; y_0\, E_z(x)\;\; -z_0\, H_y(x)]^{\mathrm{T}}$, $z_0 = (Z_0)^{1/2}$, $y_0 = (Y_0)^{1/2}$, $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$, $Y_0 = 1/Z_0$, and

$$\mathbf{K} = \begin{bmatrix} 0 & r_1 & 0 & o_1 \\ g_1 & 0 & s_1 & 0 \\ 0 & o_2 & 0 & r_2 \\ s_2 & 0 & g_2 & 0 \end{bmatrix}. \qquad (71)$$

The entries of the incremental matrix $\mathbf{K}$ have the values

$$\begin{aligned} r_1 &= \mathrm{j}\left(-\mu_{zz} + \beta_y^2/\varepsilon_{xx}\right), & r_2 &= \mathrm{j}\left(-\mu_{yy} + \beta_z^2/\varepsilon_{xx}\right), \\ g_1 &= \mathrm{j}\left(-\varepsilon_{yy} + \beta_z^2/\mu_{xx}\right), & g_2 &= \mathrm{j}\left(-\varepsilon_{zz} + \beta_y^2/\mu_{xx}\right), \\ o_1 &= \mathrm{j}\left(\mu_{zy} + \beta_y\beta_z/\varepsilon_{xx}\right), & o_2 &= \mathrm{j}\left(\mu_{yz} + \beta_y\beta_z/\varepsilon_{xx}\right), \\ s_1 &= \mathrm{j}\left(-\varepsilon_{yz} - \beta_y\beta_z/\mu_{xx}\right), & s_2 &= \mathrm{j}\left(-\varepsilon_{zy} - \beta_y\beta_z/\mu_{xx}\right). \end{aligned} \tag{72}$$

The final solution of the set of linear differential equations (72) should have the form

$$\boldsymbol{U}_1 = \mathbf{A} \,.\, \boldsymbol{U}_2 \tag{73}$$

where

$$\begin{aligned} \boldsymbol{U}_1 &= \left[y_0\, E_y(x_1) \quad z_0\, H_z(x_1) \quad y_0\, E_z(x_1) \quad z_0\, H_y(x_1)\right]^{\mathrm{T}}, \\ \boldsymbol{U}_2 &= \left[y_0\, E_y(x_2) \quad z_0\, H_z(x_2) \quad y_0\, E_z(x_2) \quad z_0\, H_y(x_2)\right]^{\mathrm{T}}. \end{aligned}$$

If all diagonal elements of the tensors (69) are real and positive, the off-diagonal ones are real and the tensors are symmetrical ($\varepsilon_{xy} = \varepsilon_{yx}$, $\mu_{xy} = \mu_{yx}$). Then all elements of $\mathbf{K}$ are imaginary and therefore the system is LOSSLESS and RECIPROCAL.

If all diagonal entries of the tensors (69) are real and positive, the off-diagonal ones are complex conjugate: $\varepsilon_{xy} = \varepsilon_{yx}^*$, $\mu_{xy} = \mu_{yx}^*$. Then the elements of $\mathbf{K}$ satisfy the constraints (22). The system is LOSSLESS and SEMIRECIPROCAL for all real β_y, β_z.

Let us consider the differential $\mathrm{d}x$ in (70) as small but definite difference δx. We get the difference equations ($\boldsymbol{U}_2 = \boldsymbol{U}_1 + \boldsymbol{U}$)

$$\boldsymbol{U}_2 = \boldsymbol{U}_1 - \mathbf{K} \,.\, \boldsymbol{U}_2 \,, \quad \boldsymbol{U}_1 = (\mathbf{I} + \mathbf{K}) \,.\, \boldsymbol{U}_2 \,. \tag{74}$$

Comparing Eqn's (73), (74) we may write the identity

$$\mathbf{A} = \mathbf{I} + \mathbf{K} \,. \tag{75}$$

The off-diagonal elements of $\mathbf{K}$ in (71) are zeros. In this case we may synthesize the remaining entries of $\mathbf{A}$ by the methods described in the last section. We want to obtain two nonevanescent wave propagating modes (c_0 should be real). For the beginning we know 6 real parameters $r_1 = r_2$ (imaginary), o_1 (complex), s_1 (complex). To realize $\mathbf{A}$ we need 16 parameters of $\mathbf{B}_1$, $\mathbf{B}_2$ defined in (39) and we can use the freedom of choosing two arbitrary real parameters, completing the number of starting parameters to eight ($R = 0$, $G = 0$). The freedom is not absolute if we want to reach two nonevanescent wave propagating modes.

The equation (62) is valid not only for the elements of partial constituents $\mathbf{B}_1$, $\mathbf{B}_2$ but also for the chain matrix $\mathbf{A}$

$$\left|r_1 o_1/g_1 s_1\right| = \left|a_{21}a_{14}/a_{12}a_{23}\right| = 1 \,. \tag{76}$$

It follows from equation (62) that the parameters $h_{(1)}$, $h_{(2)}$ introduced in (53) and corresponding to modes 1, 2 have the values

$$\begin{aligned} h_{(1)} &= b_{21(1)}b_{14(1)}/b_{12(1)}b_{23(1)}, \\ h_{(2)} &= \left(a_{21} - b_{21(1)}\right)\left(a_{14} - b_{14(1)}\right)/\left(a_{12} - b_{12(1)}\right)\left(a_{23} - b_{23(1)}\right). \end{aligned} \tag{77}$$

The $\varkappa$ in (66) calculated for one mode changes to $-\varkappa$ for the other one. To fulfil this requirement the phases of the complex quantities $h_{(1)}$. $h_{(1)}$ should have opposite signs. After solving two quadratic equations we derive the values of the unknown $b_{14(1,2)}$. $b_{24(1,2)}$ as functions of $b_{12(1,2)}$, $b_{21(1,2)}$. They have the form

$$b_{14(1,2)} = a_{14}(1 \pm (1-4b_{12(1)}b_{12(2)}/a_{14}^2)^{1/2})/2\,, \qquad (78)$$
$$b_{23(1,2)} = a_{23}(1 \pm (1-4b_{21(1)}b_{21(2)}/a_{23}^2)^{1/2})/2\,.$$

We choose an arbitrary value of the real wave parameter $c_0 < 1$. The entries $r_1 = a_{12}$, $g_1 = a_{21}$ are imaginary $(a_{12} = \mathrm{j}r_\mathrm{m},\ a_{21} = \mathrm{j}g_\mathrm{m})$. Using (62) we suppose that the primary entries of $\mathbf{B}_1$, $\mathbf{B}_2$ have the values

$$b_{12(1)} = r + \mathrm{j}c_0 r_\mathrm{m}\,, \quad b_{12(2)} = -r + \mathrm{j}c_0 r_\mathrm{m}\,, \qquad (79)$$
$$b_{21(1)} = g + \mathrm{j}c_0 g_\mathrm{m}\,, \quad b_{21(2)} = -g + \mathrm{j}c_0 g_\mathrm{m}\,.$$

The arbitrarily chosen real parameters r, g have no influence on the values a_{12}, a_{21} but guarantee the correct values and signs of phase angles of $h_{(1)}$, $h_{(2)}$. Also the condition $h_{(1)}h_{(2)} = 1$ holds. But we need more to satisfy the condition $h_{(1)} = h^*_{(2)}$. To reach this goal we are allowed to choose one parameter of the couple of real parameters r, g only. The other one is determined by a simple numerical evaluation.

Let us turn our attention to the question how to satisfy (76) for the system with parameters (72). To fulfil equations (21) valid for any semireciprocal four-port the elements of the tensors and the propagation constants β_y, β_z have to fulfil the conditions

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_2 = \varepsilon_3\,, \quad \mu_{yy} = \mu_{zz} = \mu_2 = \mu_3\,, \quad \beta_y = \beta_z = \beta\,. \qquad (80)$$

If the absolute values of $h_{(1)}$, $h_{(2)}$ in (77) equals one then the bicubic equation in variable β should hold:

$$\beta^6 + \beta^4 C_2 + \beta^2 C_1 + C_0 = 0\,, \qquad (81)$$

where

$$C_2 = (A_2 - B_2)/(A_3 - B_3)\,, \quad C_1 = (A_1 - B_1)/(A_3 - B_3)\,,$$
$$C_0 = (A_0 - B_0)/(A_3 - B_3)\,,$$
$$A_3 = 2[\varepsilon_1 \operatorname{Re}(\mu_4) - \mu_1\varepsilon_2]\,, \quad B_3 = 2[\mu_1 \operatorname{Re}(\varepsilon_4) - \varepsilon_1\mu_2]\,,$$
$$A_2 = \mu_1\varepsilon_2 + \varepsilon_1[\operatorname{Re}(\mu_4)^2 + \operatorname{Im}(\mu_4)^2] - 4\mu_1\varepsilon_1\varepsilon_2 \operatorname{Re}(\mu_4)\,,$$
$$B_2 = \varepsilon_1^2\mu_2^2 + \mu_1[\operatorname{Re}(\varepsilon_4)^2 + \operatorname{Im}(\varepsilon_4)^2] - 4\varepsilon_1\mu_1\mu_2 \operatorname{Re}(\varepsilon_4)\,,$$
$$A_1 = 2\varepsilon_1\mu_1\varepsilon_2\{\mu_1\varepsilon_2 \operatorname{Re}(\mu_4) - \varepsilon_1[\operatorname{Re}(\mu_4)^2 + \operatorname{Im}(\mu_4)^2]\}\,,$$
$$B_1 = 2\mu_1\varepsilon_2\mu_2\{\varepsilon_1\mu_2 \operatorname{Re}(\varepsilon_4) - \mu_1[\operatorname{Re}(\varepsilon_4)^2 + \operatorname{Im}(\varepsilon_4)^2]\}\,,$$
$$A_0 = (\mu_1\varepsilon_1\varepsilon_2)^2\,[\operatorname{Re}(\mu_4)^2 + \operatorname{Im}(\mu_4)^2]\,,$$
$$B_0 = (\varepsilon_1\mu_1\mu_2)^2\,[\operatorname{Re}(\varepsilon_4)^2 + \operatorname{Im}(\varepsilon_4)^2]\,.$$

After we have found at least one real zero β^2 of (81) we are able to calculate all entries of $\mathbf{B}_1$, $\mathbf{B}_2$ with the help of (78), (79). Before that step we determine $\varkappa$ from

(55), and Φ, Ψ from (37). In Table 5 one numerical example is shown. The final step is to find all entries of **A** specified in (60). The wave propagation constant a_0 will be determined on the base of numerical evaluation of the first or second equation (9).

Table 5. The relevant steps to the synthesis of the matrix **A** for nonevanescent plane wave propagation in linear reciprocal or semireciprocal media, described by permittivity tensor ε and permeability tensor μ.

$\varepsilon_1 = 1{\cdot}5$	$\varepsilon_2 = \varepsilon_3 = 3$	$\varepsilon_3 = 0{\cdot}5 + \mathrm{j}\,0{\cdot}3$
$\mu_1 = 2$	$\mu_2 = \mu_3 = 2{\cdot}2$	$\mu_4 = 1{\cdot}4 - \mathrm{j}\,0{\cdot}21$
$C_0 = -46{\cdot}1005314$	$C_1 = -26{\cdot}3391562$	$C_2 = 2{\cdot}79399218$
	$\beta_y = \beta_z = 2{\cdot}1828143$	

$a_{12} = \mathrm{j}\,0{\cdot}976452161$	$a_{21} = -\mathrm{j}\,0{\cdot}617660878$
$a_{14} = 0{\cdot}21 + \mathrm{j}\,4{\cdot}57645216$	$a_{23} = -0{\cdot}3 - \mathrm{j}\,2{\cdot}88233912$
Matrix $\mathbf{B}_1$	
$b_{12} = 2{\cdot}3 + \mathrm{j}\,0{\cdot}390580865$	$b_{21} = 4{\cdot}29724854 - \mathrm{j}\,0{\cdot}370596527$
$b_{14} = 0{\cdot}647872668 + \mathrm{j}\,2{\cdot}31710899$	$b_{23} = -4{\cdot}21246019 - \mathrm{j}\,1{\cdot}42904667$
$\lvert h_1\rvert = 0{\cdot}999999999$	$\mathrm{Arg}(h_1) = 41{\cdot}0727538°$
Matrix $\mathbf{B}_2$	
$b_{12} = -2{\cdot}3 + \mathrm{j}\,0{\cdot}585871297$	$b_{21} = -\,4{\cdot}29724854 - \mathrm{j}\,0{\cdot}247064351$
$b_{14} = -0{\cdot}437872668 + \mathrm{j}\,2{\cdot}25934317$	$b_{23} = 3{\cdot}91246019 - \mathrm{j}\,1{\cdot}45329244$
$\lvert h_2\rvert = 1{\cdot}00000004$	$\mathrm{Arg}(h_2) = -41{\cdot}0727542°$

In other directions $\beta_y \neq \beta_z$ the system becomes general lossless nonreciprocal four-port described in Section 2. If the tensor elements ε_4, μ_4 are real, then the four-port is lossless and reciprocal. In the last case similar method of design may be derived and the solution will indicate the whole range of directions (β_y, β_z), where non-evanescent wave propagation may exist [11].

5. CONCLUSION

In this paper the theory of a general lossless nonreciprocal four-port is presented. The method enables to synthesize various systems with prescribed nonevanescent wave propagation modes. For the description of the system behaviour the chain matrix representation is used. The advantage is the easy solution of the wave propagation in space in the form of the Nth power of the matrix **A**. The transformation to the scattering parameter representation may be easily found by simple linear transformation as a final step of the procedure described. For the design of the most general system we have the freedom of choosing 16 starting real parameters. Great attention has been paid to the semireciprocal system introduced in [10], which is fully determined by 10 ($R \neq 0$, $G \neq 0$) or 8 ($R = 0$, $G = 0$) real parameters.

(Received December 1, 1989.)

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*) Report and 5·25″ diskette for IBM-PC is available on request from the first author.

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