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ON THE SYNTACTICO-SEMANTICAL COMPLETENESS OF FIRST-ORDER FUZZY LOGIC

Part I. Syntax and Semantics

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This is the first part of the extensive paper which presents the syntax and semantics of first-order fuzzy logic. We introduce the structure of truth values and present some main properties of its. Then the language of first-order fuzzy logic and its syntax and semantics are defined, and proved many theorems demonstrating their good properties. In Sections 6.1 and 6.2, the concept of a fuzzy theory is defined and the main properties of fuzzy theories are presented including the problem of their consistency and completeness.

1. INTRODUCTION

This is the first part of the extensive paper which present first-order fuzzy logic. Many theorems describing the properties of its syntax and semantics are proved and the connection between them is demonstrated. The most important result is the completeness theorem which is based on deep algebraic properties of the set of formulae. Some other important theorems, especially closure and deduction ones are also proved.

The paper stems from the results of J. Pavelka [9]. From the point of the theory of continuous models [2], fuzzy logic is a special case of continuous logic.

In this part of the paper, we introduce all the necessary concepts and notation and prove various lemmas and theorems concerning the behaviour of fuzzy logic. The main results are contained in the second part [8].

Recall that a *fuzzy set* $A \subsetneq U$ in the universe U is a function $A: U \to L$ where L is the lattice of membership grades. The grade of membership of $x \in U$ in A is denoted by Ax, $Ax \in L$. Fuzzy set theory is explained in detail in [6].

2. TRUTH VALUES, OPERATIONS AND GENERALISED FUNCTIONS

We assume that truth values form a complete, infinitely distributive, residuated lattice

$$\mathscr{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{1}, \mathbf{0} \rangle$$

where $\mathbf{0}, \mathbf{1}$ are the smallest and the greatest elements respectively, and $\otimes$, $\rightarrow$ are binary operations of (bold) multiplication and residuation respectively with the following properties:

(a) $\langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid.

(b) The operation $\otimes$ is isotone in both variables and $\rightarrow$ is antitone in the first variable and isotone in the second one.

(c) The adjunction property

$$a \otimes b \leqq c \quad \textit{iff} \quad a \leqq b \rightarrow c$$

holds for every $a, b, c \in L$.

We moreover assume that L is either the interval $\langle 0, 1 \rangle$ or a finite chain $L = \{\mathbf{0} = a_0 \leqq \ldots \leqq a_m = \mathbf{1}\}$ and put

$$a \otimes b = 0 \vee (a + b - 1) \tag{1}$$

$$a \rightarrow b = 1 \wedge (1 - a + b) \tag{2}$$

if $L = \langle 0, 1 \rangle$ and

$$a_k \otimes a_p = a_{\max(0, k+p-m)} \tag{3}$$

$$a_k \rightarrow a_p = a_{\min(m, m-k+p)} \tag{4}$$

if L is a finite chain where $0 \leqq k, p \leqq m$. The reason for using this kind of structure have been extensively discussed in [5, 6, 7, 9]. Among them, the following reasons are most important. Let $L = \langle 0, 1 \rangle$. Then the following holds:

- if the operation $\rightarrow$ is not continuous in both variables then it is not possible to construct fuzzy logic so that the completeness theorem holds. This is not also possible in the case of L being a countably infinite chain.
- every residuated lattice with the continuous operation $\rightarrow$ is isomorphic with the above defined one.

The operations (3) and (4) represent a finite counterpart to the respective operations (1) and (2). In [9] it is proved that in a finite L, the completeness theorem holds for any adjoint couple of operations $\otimes$, $\rightarrow$. For the sake of simplicity, we will consider only (3) and (4) in the sequel.

We will use the following symbols:

$$a \longleftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$$

$$\neg\, a := a \rightarrow \mathbf{0}$$

($:=$ stands for "is defined as").

Lemma 1. Let $a, b, c, \in L$ and $K \subseteq L$. Then

(a) $a \otimes \bigvee_{b \in K} b = \bigvee_{b \in K} (a \otimes b)$

(b) $a \vee b = (a \to b) \to b$

(c) $\neg(\neg a) = a$

(d) $\bigwedge_{a \in K} (a \otimes b) = \bigwedge_{a \in K} a \otimes b$

(e) $a \wedge b = \neg(\neg a \vee \neg b)$

(f) $a \to b = \neg(a \otimes \neg b)$

(g) $a \to b = \neg b \to \neg a$.

The proof follows from the definition of the residuated lattice and, in some cases, also from the assumption that L is a finite chain or the interval $\langle 0, 1 \rangle$.

Many other properties of the operations in residuated lattices have been proved in [5, 6, 9].

It is possible to enrich the lattice $\mathscr{L}$ by additional n-ary operations $o: L^n \to L$ and also by generalised operations Q: $P(L) \to L$ (cf. [6, 7, 9]). We leave this problem to another paper. We will consider only the generalised operations $\bigvee$, $\bigwedge$ since $\mathscr{L}$ is, by the assumption, complete and infinitely distributive lattice.

3. LANGUAGE, TERMS AND FORMULAE

In this paper, we consider only the *basic language* of first-order fuzzy logic which consists of:

(i) Variables $x, y, \ldots$

(ii) Constants $\mathbf{c}, \mathbf{d}, \mathbf{r}, \ldots$

(iii) Symbols for truth values $\{\boldsymbol{a}; a \in L\}$.

(iv) n-ary functional symbols $f, g, \ldots$

(v) n-ary predicate symbols $p, q, \ldots$

(vi) A binary connective $\Rightarrow$.

(vii) A symbol for a general quantifier $\forall$.

(viii) Auxiliary symbols.

Terms are defined in the same way as in classical logic.

Formulae

(a) A symbol $\boldsymbol{a}$ for a truth value $a \in L$ is a (atomic) formula.

(b) If $t_1, \ldots, t_n$ are terms and p an n-ary predicate symbol then $p(t_1, \ldots, t_n)$ is a (atomic) formula.

(c) If A and B are formulae then $A \Rightarrow B$ and $(\forall x)\, A$ are formulae.

We introduce the following abbreviations of formulae:

$$\neg A := A \Rightarrow \mathbf{0} \qquad \text{(negation)}$$
$$A \vee B := (A \Rightarrow B) \Rightarrow B \qquad \text{(disjunction)}$$
$$A \wedge B := \neg((A \Rightarrow B) \Rightarrow B) \qquad \text{(conjunction)}$$
$$A \,\&\, B := \neg(A \Rightarrow \neg B) \qquad \text{(bold conjunction)}$$
$$A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A) \qquad \text{(equivalence)}$$
$$(\exists x) A := \neg(\forall x) \neg A \qquad \text{(existential quantifier)}$$
$$A^k := \underbrace{A \,\&\, A \,\&\, \dots \,\&\, A}_{k-\text{times}} \qquad \text{(power)}$$

A set of all the terms of a language J is denoted by M_J and a set of all the formulae by F_J.

Analogously as in classical logic we introduce the notions of *free* and *bound* variables and a *substitutible term*. If t is a term and A a formula then $A_x[t]$ is a formula resulting from A when substituting the term t instead of each free occurrence of x in A.

Two formulae A, B are congruent,

$$A \sim B$$

if there is a formula C and *bound* variables $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n$ such that A or B is a result of replacement of $z_1, \dots, z_n$ in C by the variables $x_1, \dots, x_n$ or $y_1, \dots, y_n$ respectively.

Obviously, $\sim$ is an equivalence and it is a congruence with respect to $\Rightarrow$ (and, thence, to $\vee$, $\wedge$, $\&$ as well). We define

$$\bigwedge_{t \in M_J} |A_x[t]| := |(\forall x) A(x)|$$

where M_J is a set of all the terms without variables and $|\cdot|$ denotes an equivalence class with respect to $\sim$.

We obtain the algebra of formulae

$$\mathscr{F}_J = \langle F_J|_\sim, \vee, \wedge, \&, \Rightarrow, \{\mathbf{a}; a \in L\}, \bigvee, \bigwedge \rangle$$

which is of the same type as $\mathscr{L}$ (enriched by $\bigvee$, $\bigwedge$ and $\{\mathbf{a}; a \in L\}$ being considered as a set of nullary operations on L).

A function

$$C\colon F_J|_\sim \to L$$

is called a *Q-homomorphism* if it has the following properties:

$$C|\mathbf{a}| = a\,, \quad a \in L, \tag{5}$$
$$C|A \Rightarrow B| = C|A| \to C|B| \tag{6}$$

provided that A and B are closed formulae,

$$C(\bigwedge_{t \in M_J} |A_x[t]| = C|(\forall x) A| = \bigwedge_{t \in M_J} C|A_x[t]| \tag{7}$$
$$C|A(x_1, \dots, x_n)| = \bigwedge_{t_1, \dots, t_n \in M_J} C|A_{x_1} \dots_{x_n}[t_1, \dots, t_n]|\,. \tag{8}$$

In general, by Q-homomorphism we call any homomorphism from one algebra with generalised operations to another one which preserves also generalised operations (cf. [10]).

4. SEMANTICS

4.1 Structures and truth valuation

A *structure* for the basic language J of first-order fuzzy logic is

$$\mathscr{D} = \langle D, p_D, \ldots; f_D, \ldots; u, v, \ldots \rangle$$

where D is a set, $p_D \subsetneqq D^n, \ldots$ are n-ary relations adjoined to each n-ary predicate symbol $p, \ldots, f_D$ are n-ary (ordinary) functions defined on D and adjoined to each n-ary functional symbol f, and $u, v, \ldots \in D$ are elements which are assigned to each constant $\mathbf{u}, \mathbf{v}$ of the language J.

We will assume that J contains one constant $\mathbf{d} \in J$ associated with each element $d \in D$ (a name of $\mathbf{d}$). Let $\mathbf{u} \in J$ be a constant. Then its interpretation is an element $\mathscr{D}(\mathbf{u}) \in D$ which was assigned to $\mathbf{u}$ in the structure $\mathscr{D}$. Let f_D be a function assigned to f and $t_1, \ldots, t_n$ be terms without variables. Then

$$\mathscr{D}f(t_1, \ldots, t_n)) = f_D(t_1, \ldots, t_n) .$$

We have introduced ordinary functions since fuzzy functions, being defined in fuzzy set theory (cf. [6]), can be understood to be special fuzzy relations. Introducing them instead of the ordinary functions would lead to greater complexity of the language and the definition of interpretation. Note that functional symbols are introduced only for the sake of completeness and they can be dropped away since they can be replaced by special predicates.

Truth valuation of formulae

Let $\mathscr{D}$ be a structure for the basic language J. A truth valuation of formulae in $\mathscr{D}$ is a function

$$\mathscr{D}: F_J \to L$$

which assigns a truth value to every formula $C \in F_J$ as follows.

(i) $\mathscr{D}(\mathbf{a}) = a\,, \quad a \in L,$

(ii) $\mathscr{D}(p(t_1, \ldots, t_n)) = p_D(\mathscr{D}(t_1), \ldots, \mathscr{D}(t_n))$

where $\mathscr{D}(t_i) \in D$, $i = 1, \ldots, n$ is an interpretation of the term $t_i \in J$ and t_i is a term without variables.

(iii) $\mathscr{D}(A \Rightarrow B) = \mathscr{D}(A) \to \mathscr{D}(B)$

provided that A and B are closed formulae.

(iv) $\mathscr{D}((\forall x)\, A(x)) = \bigwedge_{d\in D} \mathscr{D}(A_x[\mathbf{d}])$

where $\mathbf{d}$ is a name of the element $d \in D$.

(v) $\mathscr{D}(A(x_1, \ldots, x_n)) = \bigwedge_{\substack{d_i\in D\\ i=1,\ldots,n}} \mathscr{D}(A_{x_1\ldots x_n}[\mathbf{d}_1, \ldots, \mathbf{d}_n])$

From the definition of the truth valuation and Lemma 1 we immediately obtain

$$\begin{aligned}
&\mathscr{D}(A \wedge B) = \mathscr{D}(A) \wedge \mathscr{D}(B)\\
&\mathscr{D}(A \vee B) = \mathscr{D}(A) \vee \mathscr{D}(B)\\
&\mathscr{D}(A \,\&\, B) = \mathscr{D}(A) \otimes \mathscr{D}(B)\\
&\mathscr{D}(A^k) \quad\;\, = (\mathscr{D}(A))^k\\
&\mathscr{D}(\neg A) \quad = \neg \mathscr{D}(A) = \mathscr{D}(A) \to \mathbf{0}\\
&\mathscr{D}(A \Leftrightarrow B) = \mathscr{D}(A) \longleftrightarrow \mathscr{D}(B)\\
&\mathscr{D}((\exists x)\, A(x)) = \bigvee_{d\in D} \mathscr{D}(A_x[\mathbf{d}])\,.
\end{aligned}$$

Lemma 2. Let $A, B \in F_J$. If $A \sim B$ then

$$\mathscr{D}(A) = \mathscr{D}(B)$$

holds in any structure $\mathscr{D}$ for the language J.

Proof. If A contains no bound variables then B is A and the equality trivially holds true. Let $A := (\forall x)\, C(x)$ and $B := (\forall y)\, C_x[y]$. Then

$$\mathscr{D}(A) = \mathscr{D}((\forall x)\, C(x)) = \bigwedge_{d\in D} \mathscr{D}(C_x[\mathbf{d}]) = \bigwedge_{d\in D} \mathscr{D}(C_x[y]_y\, [\mathbf{d}]) = \mathscr{D}(B)\,. \qquad \square$$

Lemma 3. Let $\mathscr{D}$ be a structure for J and put

$$T|A| = \mathscr{D}(A)\,, \quad A \in F_J\,.$$

Then T is a Q-homorphism

$$T\colon \mathscr{F}_J \to \mathscr{L}\,.$$

Proof. It follows from Lemma 2 that the value of T does not depend on the choice of a representative from $|A|$. Hence

$$\begin{aligned}
&T|A| = \mathscr{D}(\boldsymbol{a}) = a\\
&T(|A| \Rightarrow |B|) = T(|A \Rightarrow B|) = \mathscr{D}(A \Rightarrow B) = \mathscr{D}(A) \to \mathscr{D}(B) = T|A| \to T|B|\\
&T \bigwedge_{t\in M_J} |A_x[t]| = T|(\forall x)\, A| = \mathscr{D}((\forall x)\, A) = \bigwedge_{d\in D} \mathscr{D}(A_x[\mathbf{d}]) = \bigwedge_{t\in M_J} T|A_x[t]|
\end{aligned}$$

since M_J contains all the names for all the elements from D. $\square$

Canonical structure for the basic language of fuzzy logic

Let

$$T\colon \mathscr{F}_J \to \mathscr{L}$$

be a Q-homomorphism. Put

$$D_0 = M_J\,,$$

and

$$\mathscr{D}_0(t) = t\,, \quad t \in M_J$$

if t is a constant. The functions f_{D_0} are defined as follows. Let f be an n-ary functional symbol and $t_1, \ldots, t_n \in M_J$ terms. Then we put

$$f_{D_0}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)\,.$$

Relations p_{D_0} are defined by

$$p_{D_0}(t_1, \ldots, t_n) = T|p(t_1, \ldots, t_n)|$$

for all the terms $t_1, \ldots, t_r \in M_J$. At the same time we suppose J to contain at least two different constants. This can be done, for example, by adding names of all the terms to J. The structure

$$D_0 = \langle D_0, p_{D_0}, \ldots, f_{D_0}, \ldots, \mathbf{u}, \ldots \rangle$$

is called the *canonical structure for J*.

4.2 The operation of semantic consequence

The operation of semantic consequence can be introduced on the basis of Lemmas 2 and 3. Let $X \underset{\sim}{\subseteq} F_J$ be a fuzzy set of formulae. Then the fuzzy set of semantic consequences of the fuzzy set X is

$$(C^{\text{sem}}X)\,A = \bigwedge\{\mathscr{D}(A);\ \mathscr{D} \text{ is a structure for } J \text{ and } (\forall B \in F_J)\,(X(B) \leqq \mathscr{D}(B))\} \tag{9}$$

($X(B) \in L$ is the grade of membership of B in X).

Lemma 4. C^{sem} is a closure operation on L^{F_J}, i.e. it fulfils the conditions

(a) $X \subseteq C^{\text{sem}}X$

(*b*) $X \subseteq y$ implies $C^{\text{sem}}X \subseteq C^{\text{sem}}y$

(c) $C^{\text{sem}}(C^{\text{sem}}X) = C^{\text{sem}}X$.

Proof. (a) and (b) are obvious. (c) follows from the fact that $(C^{\text{sem}}X)\,A \leqq \mathscr{D}(A)$ for every structure $\mathscr{D}$ from the right hand side of (9). □

A formula $A \in F_J$ is an *a-tautology* if

$$a = (C^{\text{sem}}\,\mathbf{0})\,A$$

and we write $\models_a A$. We write $\models A$ if $a = \mathbf{1}$ and say that A is a tautology.

Lemma 5.

(a) $\models A \Rightarrow B$ iff $\mathscr{D}(A) \leq \mathscr{D}(B)$

(b) $\models A \Leftrightarrow B$ iff $\mathscr{D}(A) = \mathscr{D}(B)$

holds in every structure $\mathscr{D}$.

Proof. Obvious. □

4.3 Tautologies of first-order fuzzy logic

In this section we present the most important schemes of tautologies which later will take the role of logical axioms.

$$\models (\mathbf{a} \Rightarrow \mathbf{b}) \Leftrightarrow \overline{(\mathbf{a} \Rightarrow \mathbf{b})} \tag{T 1}$$

where $\overline{(\mathbf{a} \Rightarrow \mathbf{b})}$ denotes the atomic formula for the truth value $a \rightarrow b$ when a and b are given.

$$\models A \Rightarrow A \tag{T2}$$

$$\models A \Rightarrow \mathbf{1} \tag{T3}$$

$$\models ((A \,\&\, B) \Rightarrow C) \Leftrightarrow (A \Rightarrow (B \Rightarrow C)) \tag{T4}$$

$$\models (A \,\&\, B) \Leftrightarrow (B \,\&\, A) \tag{T5}$$

$$\models (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \tag{T6}$$

$$\models (A \Rightarrow B) \vee (B \Rightarrow A) \tag{T7}$$

$$\models (A \vee B)^n \Rightarrow (A^n \vee B^n), \quad n > 0 \tag{T8}$$

$$\models (\forall x)\, A \Rightarrow A_x[t] \tag{T9}$$

for any term t.

$$\models (\forall x)\,(A \Rightarrow B) \Leftrightarrow (A \Rightarrow (\forall x)\, B) \tag{T10}$$

provided that x is not free in A.

$$\models (\exists y)\,(A_x[y] \Rightarrow (\forall x)\, A)^n \tag{T11}$$

If L is a finite chain then

$$\models (A \Rightarrow \mathbf{a}_k) \vee (\mathbf{a}_{k+1} \Rightarrow A) \tag{TK}$$

for $k < m$.

If $L = \langle 0, 1 \rangle$ then

$$\models ((\mathbf{a} \Rightarrow B)^n \Rightarrow \mathbf{b}) \Rightarrow ((\mathbf{a}' \Rightarrow B)^n \Rightarrow \mathbf{b}') \tag{TI1}$$

for $b < b' < 1, \quad 0 \leqq a' < a, \quad n, a + b \leqq n, a' + b'$

$$\models ((A \Rightarrow \mathbf{a})^n \Rightarrow \mathbf{b}) \Rightarrow ((A \Rightarrow \mathbf{a}')^n \Rightarrow \mathbf{b}') \tag{TI2}$$

for $b < b' < 1,\ a < a', n, a' - b' \leqq na - b$.

All the tautologies can be easily proved using Lemma 5, the definition of a truth valuation and the properties of $\mathscr{L}$. We will demonstrate e.g. (T9): Due to Lemma 5(a), it must hold

$$\mathscr{D}(\forall x)\, A) \leqq \mathscr{D}A_x[t])$$

in every structure $\mathscr{D}$. By the definition and the properties of infimum

$$\bigwedge_{d \in D} \mathscr{D}(A_x[\mathbf{d}]) \leqq \mathscr{D}(A_x[\mathbf{d}])$$

holds for every $d \in D$. If t is a term without variables then there in $\mathbf{d} \in J$ such that $\mathscr{D}(t) = d$. Otherwise the inequality holds trivially by the definition of $\mathscr{D}$.

5. DEDUCTION

5.1 Rules of inference

An *n-ary rule of inference* r is a couple

$$r = \langle r^{\text{syn}}, r^{\text{sem}} \rangle$$

where r^{syn} is its *syntactic* part which is a partial n-ary operation on F_J and r^{sem} is a *semantic* part which is an n-ary operation on L preserving arbitrary non-empty joints in each argument (semicontinuity).

A fuzzy set $V \subseteq F_J$ is *closed* with respect to r if

$$X(r^{\text{syn}}(A_1, \ldots, A_n)) \geqq r^{\text{sem}}(X(A_1), \ldots, X(A_n))$$

holds for all $A_1, \ldots, A_n \in F_J$ for which r^{syn} is defined.

A rule of inference r is *sound* if

$$T|r^{\text{syn}}(A_1, \ldots, A_n)| \geqq r^{\text{sem}}(T|A_1|, \ldots, T|A_n|) \tag{10}$$

holds for every $A_1, \ldots, A_n \in \text{Dom}\, r^{\text{syn}}$ and any Q-homomorphism

$$T: \mathscr{F}_J \to \mathscr{L}.$$

Lemmas 2 and 3 assure us that (10) holds also for every structure $\mathscr{D}$.

The rules of inference are usually written in the form

$$r: \frac{A_1, \ldots, A_n}{r^{\text{syn}}(A_1, \ldots, A_n)} \left(\frac{a_1, \ldots, a_n}{r^{\text{sem}}(a_1, \ldots, a_n)} \right)$$

where $a_i \in L$ are truth valuations of the respective formulae A_i $i = 1, \ldots, n$.

Lemma 6. The following rules of inference are sound:

(a) Modus ponens

$$r_{\text{MP}}: \frac{A, A \Rightarrow B}{B} \left(\frac{a, b}{a \otimes b} \right)$$

(b) a-lifting rule

$$r_{\text{Ra}}: \frac{B}{a \Rightarrow B} \left(\frac{b}{a \to b} \right)$$

(c) Generalisation

$$r_{\text{G}}: \frac{A}{(\forall x)\, A} \left(\frac{a}{a} \right)$$

Proof. The semicontinuity and soundness of the rules (a), (b) was proved in [9]. Semicontinuity of (c) is obvious. Soundness:

$$T|r_{\text{G}}^{\text{syn}}(A)| = T|(\forall x)\, A| = \bigwedge_{t \in M_v} T|A_x[t]| = T|A(x)| = r_{\text{G}}^{\text{sem}}(T|A|) \qquad \square$$

5.2 The operation of syntactic consequence

In fuzzy logic we deal with fuzzy sets of logical and special axioms. Let $A_L \subsetneqq F_J$ be a fuzzy set of logical axioms and R a set of sound rules of inference. Then the couple

$$\langle A_L, R\rangle$$

is a *syntax of fuzzy logic.*

Let $X \subsetneqq F_J$ be a fuzzy set of formulae. Then

$$(C^{\text{syn}}X)\,A = \bigwedge\{U(A)\,;\ \ U \subsetneqq F_J\,,\ \ U \text{ is closed with respect to all } r \in R \text{ and } A_L, X \subseteq U\}$$

defines a fuzzy set of *syntactic consequences* of the fuzzy set X.

A *proof* of a formula A from the fuzzy set X is a sequence

$$w := A_0[a_0; P_0]\,,\ \ A_1[a_1; P_1], \ldots, A_n[a_n; P_n]$$

such that A_n is A and P_i $i \leqq n$ is LA or SA if A_i is a logical or a special axiom respectively, or P_i is r_i if A_i is a formula

$$r^{\text{syn}}(A_{i_1}, \ldots, A_{i_n})\,,\ \ i_1, \ldots, i_n < i$$

and r_i is an n-ary sound rule of inference. The

$$a_i = \text{Val}_x\,(w_{(i)})$$

is the *value* of the proof

$$w_{(i)} := A_0[a_0; P_0], \ldots, A_i[a_i; P_i]$$

defined as follows:

$$\text{Val}_x\,(w_{(i)}) = \begin{cases} A_L(A_i) & \text{if} \ \ P_i = \text{LA} \ \ (\text{i.e. } A_i \text{ is a logical axiom}) \\ X(A_i) & \text{if} \ \ P_i = \text{SA} \ \ (\text{i.e. } A_i \text{ is a special axiom}) \\ r_i^{\text{sem}}(\text{Val}_x\,(w_{(i_1)}), \ldots, \text{Val}_x\,(w_{(i_n)})) & \text{if} \ \ A_i = r_i^{\text{syn}}(A_{i_1}, \ldots, A_{i_n}) \end{cases}$$

Note that the above definition of a proof is a generalisation of the classical one. If we confine ourselves to $\{0, 1\}$ then $\text{Val}_x\,(w_{(i)}) = 1$ expresses the existence of a proof $w_{(i)}$ of the formula A_i in the classical sense.

Theorem 1.

$$(C^{\text{syn}}X)\,A = \bigvee\{\text{Val}_x\,(w);\ w \text{ is a proof of } A \text{ from } X \subsetneqq F_J\}$$

Proof. It is a verbatim repetition of the proof of Theorem 16 from [9]-I. □

It follows from this theorem that finding a proof, say w, of a formula A ensures only, that the degree in which it is a theorem is greater than or equal to $\text{Val}_x\,(w)$. If $\text{Val}_x\,(w) \neq 1$ then it is difficult to assure ourselves that we cannot find a proof with a greater value.

The syntax is *sound* if

$$A_L \subsetneq C^{\text{sem}}\, \emptyset$$

and each rule $r \in R$ is sound.

The syntax of first-order fuzzy logic consists of:

(a) The fuzzy set A_L of logical axioms defined as follows:

$$A_L\boldsymbol{a} = a\,, \quad a \in L,$$

$$A_L(\boldsymbol{a} \Rightarrow \boldsymbol{b}) = a \rightarrow b, \quad a, b \in L,$$

$A_L(A) = \mathbf{1}$ if A is any of the formulae of the form (T1)–(T11) and either (TI1), (TI2) if $L = \langle 0, 1\rangle$ or (TK) if L is a finite chain,

$A_L(A) = \mathbf{0}$ otherwise.

In the case when $A := B \Leftrightarrow C$ we understand that both $A_L(B \Rightarrow C) = \mathbf{1}$ as well as $A_L(C \Rightarrow B) = \mathbf{1}$.

(b) The set of rules of inference is

$$R = \{r_{\text{MP}}, r_{\text{G}}, \{r_{\text{R}a};\, a \in L\}\}\,.$$

6. THEORIES IN FIRST-ORDER FUZZY LOGIC

6.1 Properties of fuzzy theories

A *theory* T in the language J of first-order fuzzy logic (a *fuzzy theory*) is a triple

$$T = \langle A_L, A_S, R\rangle$$

where $\langle A_L, R\rangle$ is the above defined syntax of fuzzy logic and $A_S \subsetneq F_J$ is a fuzzy set of special axioms. By $J(T)$ we denote the language of fuzzy theory. *Fuzzy predicate calculus* is the fuzzy theory with $A_S = \emptyset$.

Let $\mathscr{D}$ be a structure for $J(T)$. Then $\mathscr{D}$ is a *model* of the theory T, $\mathscr{D} \models T$, if

$$A_S(A) \leqq \mathscr{D}(A)$$

holds for every A $F_{J(T)}$. It follows from the definition of logical axioms that

$$A_L(A) \leqq \mathscr{D}(A)$$

holds in any model $\mathscr{D} \models T$ for every formula $A \in F_{J(T)}$. Then

$$(C^{\text{sem}} A_S)\, A = \bigwedge\{\mathscr{D}(A);\ \mathscr{D} \models T\}\,.$$

If $(C^{\text{sem}} A_S)\, A = a$ then the formula A is *true in the degree a* in the theory T and we write

$$T \models_a A\,.$$

If $(C^{\text{syn}} A_S)\, A = a$ then A is a *theorem in the degree a* of the theory T and we write

$$T \vdash_a A\,.$$

We write $T \vdash A$, $T \models A$ instead of $T \vdash_1 A$, $T \models_1 A$ respectively and say that A is a theorem (true) of the theory T.

It follows from the definition that

$$\vdash_a \boldsymbol{a} \quad \text{and} \quad \models_a \boldsymbol{a}$$

holds in fuzzy predicate calculus for every $a \in L$. If w is a proof in T then we write $\mathrm{Val}_T(w)$ for its value. If T is predicate calculus then we omit the subscript T.

Theorem 2. The following schemes formulae are theorems of fuzzy predicate calculus:

$$\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \tag{D1}$$

$$\vdash (A \wedge B) \Rightarrow A \tag{D2}$$

$$\vdash (A \wedge B) \Leftrightarrow (B \wedge A) \tag{D3}$$

$$\vdash (A \vee B) \Leftrightarrow (B \vee A) \tag{D4}$$

$$\vdash (C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B))) \tag{D5}$$

$$\vdash A \Rightarrow (A \vee B) \tag{D6}$$

$$\vdash (A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C)) \tag{D7}$$

$$\vdash A \Rightarrow (B \Rightarrow (A \mathbin{\&} B)) \tag{D8}$$

$$\vdash ((A \Rightarrow B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)) \tag{D9}$$

$$\vdash \neg\neg A \Leftrightarrow A \tag{D10}$$

$$\vdash \neg(A \Rightarrow B) \Leftrightarrow (A \mathbin{\&} \neg B) \tag{D11}$$

$$\vdash (A \wedge B) \Leftrightarrow \neg(\neg A \vee \neg B) \tag{D12}$$

$$\vdash (A \Rightarrow B) \Leftrightarrow \neg(A \mathbin{\&} \neg B) \tag{D13}$$

$$\vdash (A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A) \tag{D14}$$

$$\vdash (A \mathbin{\&} B) \Rightarrow A \tag{D15}$$

$$\vdash A\, x[t] \Rightarrow (\exists x)\, A \tag{D16}$$

$$\vdash (\forall x)\, A \Leftrightarrow \neg(\exists x)\, \neg A \tag{D17}$$

$$\vdash (\forall x)\,(A \Rightarrow B) \Rightarrow ((\forall x)\, A \Rightarrow (\forall x)\, B) \tag{D18}$$

provided that x is not free in B.

$$\vdash ((\exists x)\, A \Rightarrow B) \Leftrightarrow (\forall x)\,(A \Rightarrow B) \tag{D20}$$

provided that x is not free in B

$$\vdash (A \mathbin{\&} \neg A) \Rightarrow B \tag{D21}$$

$$\vdash A \Rightarrow (B \Rightarrow C) \Leftrightarrow (B \Rightarrow (A \Rightarrow C)) \tag{D22}$$

$$\vdash A \Rightarrow (B \Rightarrow A) \tag{D23}$$

$$\vdash (A^m \Rightarrow (B \Rightarrow C)) \Rightarrow ((A^n \Rightarrow B) \Rightarrow (A^{m+n} \Rightarrow C)), \quad m, n > 0. \tag{D24}$$

Proof. We use logical axioms and the above defined rules of inference. It is advantageous to prove these theorems in a certain order and to use previously proved ones as intermediate results for proofs of the next ones. We will demonstrate e.g. (D2). Let us assume we have already proved (D1), (D10), (D14) and (D23). Note that (D2) is a short for a formula

$$\neg((A \Rightarrow B) \Rightarrow \neg A) \Rightarrow A . \tag{11}$$

Let us denote $C := (A \Rightarrow B) \Rightarrow \neg A$. Let w_1 be a proof of $\neg A \Rightarrow C$ (formula (D23)), w_2 a proof of $\neg\neg A \Rightarrow A$, w_3 a proof of $\neg A \Rightarrow C) \Rightarrow (\neg C \Rightarrow \neg\neg A)$ formula ((D14)) and w_4 a proof of $(\neg\neg A \Rightarrow A) \Rightarrow ((\neg C \Rightarrow \neg\neg A) \Rightarrow (\neg C \Rightarrow A))$ (formula (D1)) where

$$\mathrm{Val}(w_1) = \mathrm{Val}(w_2) = \mathrm{Val}(w_3) = \mathrm{Val}(w_4) = \mathbf{1}.$$

Then

$$w := w_1[\mathbf{1}], w_3[\mathbf{1}], \neg C \Rightarrow \neg\neg A[\mathbf{1}; r_{\mathrm{MP}}], w_2[\mathbf{1}], w_4[\mathbf{1}],$$
$$(\neg C \Rightarrow \neg\neg A) \Rightarrow (\neg C \Rightarrow A)\, [\mathbf{1}; r_{\mathrm{MP}}], \neg C \Rightarrow A[\mathbf{1}; r_{\mathrm{MP}}]$$

is a proof of the formula (11). □

Note that to each of all the above formulae there exists a proof with the value equal to **1**.

Lemma 7.

$$T \vdash A \Leftrightarrow B \quad \text{iff} \quad T \vdash A \Rightarrow B \quad \text{and} \quad T \vdash B \Rightarrow B .$$

Proof. Let $T \vdash A \Leftrightarrow B$ and let w_1 be a proof of $A \Leftrightarrow B$, $\mathrm{Val}_T(w_1) = a$ and w_2 a proof of a theorem $(A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$ (D2), $\mathrm{Val}(w_2) = \mathbf{1}$. Then

$$w := w_1[a]\,, \quad w_2[\mathbf{1}]\,, \quad A \Rightarrow B[a; r_{\mathrm{MP}}]$$

is a proof of $A \Rightarrow B$, $\mathrm{Val}_T(w) = a$. But $\bigvee\{\mathrm{Val}_T(w); w_1\} = \mathbf{1}$ due to the assumption which gives $T \vdash A \Rightarrow B$. Analogously, using theorem (D3) we obtain $T \vdash B \Rightarrow A$.

Conversely, let w_1 be a proof of $A \Rightarrow B$, $\mathrm{Val}_T(w_1) = a$ and w_2 a proof of $B \Rightarrow A$. $\mathrm{Val}_T(w_2) = b$ and w_3 a proof of

$$(\mathbf{1} \Rightarrow (A \Rightarrow B)) \Rightarrow ((\mathbf{1} \Rightarrow (B \Rightarrow A)) \Rightarrow (\mathbf{1} \Rightarrow (A \Leftrightarrow B)))$$

(D5), $\mathrm{Val}(w_3) = \mathbf{1}$. Then

$$w := w_1[a], \mathit{1} \Rightarrow (A \Rightarrow B)\, [\mathbf{1} \to a = a; r_{\mathrm{R1}}], w_2[b], \mathit{1} \Rightarrow (B \Rightarrow A)\, [b; r_{\mathrm{R1}}],$$
$$w_3[\mathbf{1}], (\mathit{1} \Rightarrow (B \Rightarrow A)) \Rightarrow (\mathit{1} \Rightarrow (A \Leftrightarrow B))\, [a; r_{\mathrm{MP}}], \mathit{1} \Rightarrow (A \Leftrightarrow B)$$
$$[a \otimes b; r_{\mathrm{MP}}], \mathit{1}[\mathbf{1}; \mathrm{LA}], A \Leftrightarrow B[\mathbf{1} \otimes a \otimes b; r_{\mathrm{MP}}]$$

is a proof of $A \Leftrightarrow B$. But $\bigvee\{\mathrm{Val}_T(w); w_1, w_2\} = \mathbf{1}$ by the assumption which gives $T \vdash A \Leftrightarrow B$. □

Lemma 8. Let T, T' be theories and A, A' formulae. If for any $a, b \in L$

$$T \vdash_a A \quad \text{and} \quad T' \vdash_b A' \quad \text{implies} \quad a \leq b$$

and at the same time for any $c, d \in L$

$$T' \vdash_c A' \quad \text{and} \quad T \vdash_d A \quad \text{implies} \quad c \leqq d$$

then

$$T \vdash_a A \quad \text{iff} \quad T' \vdash_a A' .$$

Proof. Obvious. □

Lemma 9.

(a) $C^{\text{syn}} A_S \subseteq C^{\text{sem}} A_S$.

(b) If $T \vdash_a A$, $\mathscr{D} \models T$ and $\mathscr{D}(A) = b$ then $a \leqq b$.

Proof. Due to Lemmas 2 and 3 $\mathscr{D}$ is a Q-homomorphism $\mathscr{D}$: $F_{J(r)} \to L$. Since every Q-homomorphism is closed with respect to the rules of inference we obtain (a) from the definition of C^{syn} and C^{sem}. (b) is a consequence of (a). □

Theorem 3 (validity theorem).

$$\text{If} \quad T \vdash_a A, \quad T \models_b A \quad \text{then} \quad a \leqq b .$$

Proof. This is a corollary of Lemma 9. □

This theorem demonstrates that the balance between syntax and semantics is sound. Saying freely, if we derive formally some result then its semantic interpretation is true at least in the same degree as the value of its formal derivation.

Lemma 10.

(a) Let $T \vdash_a A \Rightarrow B$ and $T \vdash B \Leftrightarrow B'$. Then $T \vdash_a A \Rightarrow B'$.

(b) Let $T \vdash A \Leftrightarrow A'$. Then

$$T \vdash (A \Rightarrow B) \Leftrightarrow (A' \Rightarrow B) .$$

Proof. (a) Let w_1 be a proof of $A \Rightarrow B$, $\text{Val}_T(w_1) = a'$, w_2 a proof of $B \Leftrightarrow B'$, $\text{Val}_T(w_2) = b'$ and w_3 a proof of $(B \Leftrightarrow B') \Rightarrow (B \Rightarrow B')$ (D2), $\text{Val}(w_3) = \mathbf{1}$. Then

$$\begin{aligned} w := {} & w_1[a'], w_2[b'], w_3[\mathbf{1}], B \Rightarrow B'[b'; r_{\text{MP}}], \\ & (A \Rightarrow B) \Rightarrow ((B \Rightarrow B') \Rightarrow (A \Rightarrow B'))\,[\mathbf{1}; LA_{\text{T6}}], \\ & (B \Rightarrow B') \Rightarrow (A \Rightarrow B')\,[a'; r_{\text{MP}}], \quad A \Rightarrow B'[a' \otimes b'; r_{\text{MP}}] \end{aligned}$$

is a proof of $A \Rightarrow B'$. Due to the assumption we obtain

$$\bigvee\{\text{Val}_T(w); w_1, w_2\} = a \otimes \mathbf{1} = a .$$

thence $T \vdash_c A \Rightarrow B'$, $a \leqq c$. Analogously, from $T \vdash B' \Rightarrow B$ we obtain $T \vdash_d A \Rightarrow B$, $c \leqq d$ and from the assumption $a \leqq c \leqq d = a$.

(b) Let w_1 be a proof of $A \Leftrightarrow A'$, $\text{Val}_T(w_1) = a'$ and w_2 a proof of $(A \Leftrightarrow A') \Rightarrow$ $\Rightarrow (A \Rightarrow A')$, $\text{Val}(w_2) = \mathbf{1}$. Then we can write down a proof

$$\begin{aligned} w := {} & w_1[a'], w_2[\mathbf{1}], A \Rightarrow A'[a'; r_{\text{MP}}], (A \Rightarrow A') \Rightarrow ((A' \Rightarrow B) \Rightarrow \\ & \Rightarrow (A \Rightarrow B))\,[\mathbf{1}; LA_{\text{T6}}], (A' \Rightarrow B) \Rightarrow (A \Rightarrow B)\,[a'; r_{\text{MP}}] \end{aligned}$$

and from the assumption

$$T \vdash (A' \Rightarrow B) \Rightarrow (A \Rightarrow B).$$

Analogously we obtain

$$T \vdash (A' \Rightarrow B) \Rightarrow (A \Rightarrow B)$$

which follows

$$T \vdash (A' \Rightarrow B) \Leftrightarrow (A \Rightarrow B)$$

by Lemma 7. □

A theory T is *contradictory* if there is a formula A and proofs w_1 and w_2 of A and $\neg A$, respectively such that

$$\mathrm{Val}_T(w_1) \otimes \mathrm{Val}_T(w_2) > \mathbf{0}.$$

In the opposite case it is *consistent*. Obviously, if T is contradictory then $T \vdash_a A$, $T \vdash_b \neg A$ and $a \otimes b > \mathbf{0}$.

Lemma 11. Let T be a consistent theory.

(a) If $T \vdash_a A$ and $T \vdash_b B$ then

$$T \vdash_c A \Rightarrow B \quad \text{implies} \quad c \leqq a \to b.$$

(b) If $T \vdash_a (\forall x) A$ then $a \leqq \bigwedge\{b; T \vdash_b A_x[t], t \in M_J\}$.

Proof. (a) The case $a \leqq b$ is trivial. Let $b < a$ and $c > a \to b$ and w_1 be a proof of A, $\mathrm{Val}_T(w_1) = a'$ and w_2 a proof of $A \Rightarrow B$, $\mathrm{Val}_T(w_2) = c'$. Then

$$w := w_1[a'], w_2[c'], B[a' \otimes c', r_{\mathrm{MP}}]$$

is a proof of B and

$$\bigvee\{\mathrm{Val}_T(w); w_1, w_2\} = a \otimes c$$

whence

$$T \vdash_d B, \quad a \otimes c \leqq d.$$

But $a \to b = \bigvee\{e; a \otimes e \leqq b\} < c$ and so $b < a \otimes c \leqq d$ – a contradiction.

(b) Let $a > c = \bigwedge\{b; T \vdash_b A_x[t], t \in M_J\}$. Then there is $b' \in L$ and a term $t' \in M_J$ such that $T \vdash_b A_x[t']$ and $b' > a$. Let w_1 be a proof of $(\forall x) A$, $\mathrm{Val}_T(w_1) = a'$. We write down a proof

$$w := w_1[a'], (\forall x) A \Rightarrow A_x[t'] [\mathbf{1}; \mathrm{LA}_{\mathrm{T9}}], A_x[t'] [a'; r_{\mathrm{MP}}]$$

whence $\bigvee\{\mathrm{Val}_T(w); w_1\} = a$, i.e. $T \vdash_a A_x[t']$ and $b' < a \leqq d$ – a contradiction. □

Theorem 4. A theory T is contradictory iff $T \vdash A$ holds for every formula $F_{J(T)}$.

Proof. Let w_1, w_2 be proofs of A and $\neg A$ respectively, $\mathrm{Val}_T(w_1) = a$ and $\mathrm{Val}_T(w_2) = b$, $\mathbf{0} < a \otimes b < \mathbf{1}$ (the case $a \otimes b = \mathbf{1}$ is trivial). Let w_3 be a proof of $A \Rightarrow (\neg A \Rightarrow (A \& \neg A))$ (D8), $\mathrm{Val}(w_3) = \mathbf{1}$ and w_4 a proof of $A \& \neg A \Rightarrow \mathbf{0}$

(D2), Val $(w_4) = \mathbf{1}$. We write down a proof

$$w := w_1[a], w_2[b], w_3[\mathbf{1}], \neg A \Rightarrow (A \& \neg A)[a; r_{MP}],$$
$$A \& \neg A[a \otimes b; r_{MP}], w_4[\mathbf{1}], \mathit{0}[a \otimes b; r_{MP}], \overline{a \otimes b} \Rightarrow \mathit{0}$$
$$[\mathbf{1}; r_{Ra\otimes b}; r_{MP}], (\overline{a \otimes b} \to \mathbf{0}) \to \neg(\overline{a \otimes b})[\mathbf{1}; LA_{T1}], \neg(\overline{a \otimes b})[\mathbf{1}; r_{MP}],$$

i.e. $T \vdash \boldsymbol{c}$. for some truth value $c < \mathbf{1}$. Since c is nilpotent with respect to $\otimes$, let us take n such that $c^n = \mathbf{0}$. Then using theorem (D8) we obtain $T \vdash \boldsymbol{c}^n$. Let w_5 be a proof of $\boldsymbol{c}^n$, $\mathrm{Val}_T(w_5) = d$ and $B \in F_{J(T)}$ be an arbitrary formula. Consider the proof

$$w := w_5[d], \boldsymbol{c}^n \Rightarrow \mathit{0}[c^n \to \mathbf{0} = \mathbf{1}; r_{Rc^n}], \mathit{0}[d; r_{MP}],$$
$$B[e; SA], \mathit{0} \Rightarrow B[\mathbf{0} \to e = \mathbf{1}; r_{RO}], B[d; r_{MP}].$$

Then $\bigvee\{\mathrm{Val}_T(w); w_5\} = \mathbf{1}$, i.e. $T \vdash B$. The converse implication is obvious. □

Corollary. (a) T is contradictory iff there is a formula A and a proof w of $A \& \neg A$ such that $\mathrm{Val}_T(w) > \mathbf{0}$.

(b) T is contradictory iff there are $a < \mathbf{1}$ and a proof w of it such that $\mathrm{Val}_T(w) > a$.

Proof. (a) The proof proceeds analogously as that of Theorem 4 using theorem (D21).

(b) Let w be such a proof. Then there are proofs

$$w := \mathit{1} \Rightarrow \mathit{0}[\mathbf{1} \to \mathbf{0}; LA], \boldsymbol{a} \Rightarrow (\mathit{1} \Rightarrow \mathit{0})[a \to \mathbf{0}; r_{Ra}],$$
$$w[b], \mathit{1} \Rightarrow \mathit{0}[b \otimes \neg a; r_{MP}]$$
$$w' := \mathit{1}[\mathbf{1}; LA]$$

with $\mathrm{Val}_T(w) \otimes \mathrm{Val}_T(w') = \mathbf{1} \otimes b' \otimes \neg a > \mathbf{0}$, i.e. T is contradictory. The converse is obvious. □

This theorem is a surprizing result stating that if we find a proof of $A \& \neg A$ in a non-zero degree then necessarily all the formulae of the given theory are theorems in the highest degree $\mathbf{1}$. Such a theory is, of course, useless just in the same sense as in classical logic. Thus, we cannot think of some general degrees of contradictoriness. However, interesting could be the analysis of the weaker case considering only $A \wedge \neg A$. This is not done in the present paper.

Lemma 12. Let T be a consistent theory, $T \vdash_a A$ and $T \vdash A \Leftrightarrow B$. Then $T \vdash_a B$.

Proof. Let $T \vdash_b B$. Then by Lemmas 7 and 11(a) we obtain

$$\mathbf{1} \leqq a \to b, \quad \text{i.e.} \quad a \leqq b$$

and at the same time

$$\mathbf{1} \leqq b \to a, \quad \text{i.e.} \quad b \leqq a$$

whence $b = a$. □

The following lemma is simple but important

Lemma 13. Let T has a model $\mathscr{D}$. Then T is consistent.

Proof. Let $\mathscr{D}(A) = a$. Then $\mathscr{D}(\neg A) = \neg a$ and let $T \vdash_b A$ $T \vdash_c \neg A$. Due to Lemma 9(b) $b \leqq a$ and $c \leqq \neg a$ which follows $b \otimes c \leqq a \otimes \neg a = \mathbf{0}$. Hence, for any formula A and any proofs w of A and w' of $\neg A$

$$\mathrm{Val}_T(w) \otimes \mathrm{Val}_T(w') = \mathbf{0}. \qquad \square$$

Lemma 14. Let $A \in F_{J(T)}$ and $t_1, \ldots, t_n$ be terms substitutible into A for the variables $x_1, \ldots, t_n$. Then $T \vdash_a A$ and $T \vdash_b A_{x_1 \ldots x_n}[t_1, \ldots, t_n]$ implies $a \leqq b$.

Proof. Let w_1 be a proof of A, $\mathrm{Val}_T(w) = a'$. Write down a proof

$$w := w_1[a'], (\forall x_1) A[a'; r_G], (\forall x_1) A \Rightarrow A_{x_1}[t_1]$$
$$[\mathbf{1}; \mathrm{LA}_{T9}], A_{x_1}[t_1][a'; r_{MP}] \ldots$$
$$\ldots (\forall x_n) A_{x_1 \ldots x_{n-1}}[t_1, \ldots, t_{n-1}] \Rightarrow A_{x_1 \ldots x_n}[t_1, \ldots, t_1]$$
$$[\mathbf{1}; LA_{T9}], A_{x_1 \ldots y_n}[t_1, \ldots, t_n].$$

Then $\bigvee\{\mathrm{Val}_T(w); w_1\} = a$ whence $T \vdash_b A_{x_1 \ldots x_n}[t_1, \ldots, t_n]$, $a \leqq b$. $\square$

Corollary. Let $y_1, \ldots, y_n$ be variables which do not occur in a formula A. Then

$$T \vdash_a A \quad \text{iff} \quad T \vdash_a A_{x_1 \ldots x_n}[y_1, \ldots, y_n].$$

Proof. It follows from Lemma 14 that

$$T \vdash_b A_{x_1 \ldots x_n}[y_1, \ldots, y_n], \quad a \leqq b.$$

But, again from Lemma 14.

$$T \vdash_c A_{x_1 \ldots x_n}[y_1, \ldots, y_n]_{y_1 \ldots y_n}[x_1, \ldots, x_n],$$

$b \leqq c$ since $x_1, \ldots, x_n$ are substitutible into $A_{x_1 \ldots x_n}[y_1, \ldots, y_n]$ for $y_1, \ldots, y_n$. But the resulting formula is A, thence

$$a \leqq b \leqq c = a. \qquad \square$$

The following two theorems demonstrate that fuzzy logic behaves in a way analogous to the classical one.

Theorem 5 (closure theorem). Let $A \in F_{J(T)}$ and A' be its closure. Then

$$T \vdash_a A \quad \text{iff} \quad T \vdash_a A'.$$

Proof. Let w be a proof of A. Using the rule r_G we obtain a proof w' of A' such that $\mathrm{Val}_T(w) = \mathrm{Val}_T(w')$. Hence

$$T \vdash_b A', \quad a \leqq b.$$

Conversely, let $T \vdash_c A'$ and w be a proof of A', $\mathrm{Val}_T(w) = c'$. Write down a proof

$$w := w'[c'], A' \Rightarrow (\forall x_2) \ldots (\forall x_n) A_{x_1}[y_1][\mathbf{1}; \mathrm{LA}_{T9}],$$

$$(\forall x_2) \ldots (\forall x_n)\, A_{x_1}[y_1]\,[c'; r_{MP}], \ldots, A_{x_1 \cdots x_n}[y_1, \ldots, y_n]\,[c'; r_{MP}]$$

where $y_1, \ldots, y_n$ are variables which do not occur in A'. Then

$$\bigvee\{\mathrm{Val}_T(w); w'\} = c$$

whence

$$T \vdash A_{x_1 \cdots x_n}[y_1, \ldots, y_n], \quad c \leqq d.$$

Using the corollary of Lemma 14 we obtain $T \vdash_d A$. The proposition then follows from Lemma 8. □

Corollary.

$$T \vdash_a A \quad \text{iff} \quad T \vdash_a (\forall x)\, A.$$

Theorem 6 (equivalence theorem). Let A be a formula, $B_1, \ldots, B_n$ some of its subformulae and $T \vdash B_i \Leftrightarrow B_i'$, $i = 1, \ldots, n$. Then

$$T \vdash A \Leftrightarrow A'$$

where A' is a result of replacing the subformulae $B_1, \ldots, B_n$ in A by $B_1', \ldots, B_n'$ respectively.

Proof. By the complexity of A: If $n = 1$ and $A := B_1$ then the proposition follows trivially from the assumption.

If A is atomic formula then the only its subformula is A and so the proposition follows again from the assumption.

Let $A := B \Rightarrow C$, $T \vdash B \Leftrightarrow B'$ and $T \vdash C \Leftrightarrow C'$. Then

$$T \vdash (B \Rightarrow C) \Leftrightarrow (B' \Rightarrow C)$$

due to Lemma 10(b). Furthermore, using theorem (D1) we immediately obtain

$$T \vdash (B' \Rightarrow C) \Leftrightarrow (B' \Rightarrow C').$$

Then

$$T \vdash (B \Rightarrow C) \Leftrightarrow (B' \Rightarrow C')$$

using Lemmas 10(a) and 7.

Let $A := (\forall x)\, B$ and $T \vdash B \Leftrightarrow B'$ and let w' be a proof of $A \Rightarrow A'$, $\mathrm{Val}_T(w') = a$ and w'' a proof of $(\forall x)(A \Rightarrow A') \Rightarrow ((\forall x)\, A \Rightarrow (\forall x)\, A')$ (D18), $\mathrm{Val}_T(w'') = \mathbf{1}$. Write down the proof

$$w := w'[a], (\forall x)(A \Rightarrow A')\,[a; r_G],\; w''[\mathbf{1}], (\forall x)\, A \Rightarrow (\forall x)\, A'\,[a \otimes \mathbf{1}; r_{MP}].$$

Due to Lemma 7 and Theorem 2

$$\bigvee\{\mathrm{Val}_T(w); w'\} = \mathbf{1},$$

i.e. $T \vdash (\forall x)\, A \Rightarrow (\forall x)\, A'$. Analogously we obtain

$$T \vdash (\forall x)\, A' \Rightarrow (\forall x)\, A$$

which follows

$$T \vdash (\forall x)\, A \Leftrightarrow (\forall x)\, A'$$

due to Lemma 7. □

6.2 Completeness of fuzzy theories

In this section, we introduce the generalisation of the classical notion of a complete theory. Such a theory inherits the properties of the lattice $\mathscr{L}$ of truth values.

A theory T is *complete* if it is consistent and

$$T \vdash_a A \quad \text{implies} \quad T \vdash A \Rightarrow \boldsymbol{a}$$

for every closed formula A and every $\boldsymbol{a}$. Using the rule r_{Ra} we easily prove that

$$T \vdash_a A \quad \text{implies} \quad T \vdash A \Leftrightarrow \boldsymbol{a}$$

holds in a complete theory.

Lemma 15. Let T be a complete theory and $T \vdash_a A$. Then

$$T \vdash A \Rightarrow \boldsymbol{b}$$

for every $b \geqq a$.

Proof. This can be easily demonstrated using theorem (D1) when realizing that $A_L(\boldsymbol{a} \Rightarrow \boldsymbol{b}) = \mathbf{1}$. □

Lemma 16. Let T be a complete theory. Then

$$T \vdash_a A \quad \text{implies} \quad T \vdash_{\neg a} \neg A$$

for every closed formula A and every a.

Proof. Let $T \vdash_a A$ and w' be a proof of $A \Rightarrow \boldsymbol{a}$, $\mathrm{Val}_T(w') = b$. Write down a proof

$$w := \boldsymbol{0}[0; \mathrm{LA}], \boldsymbol{a} \Rightarrow \boldsymbol{0}[a \to 0; r_{\mathrm{Ra}}], w'[b], (A \Rightarrow \boldsymbol{a}) \Rightarrow ((\boldsymbol{a} \Rightarrow \boldsymbol{0}) \Rightarrow$$
$$\Rightarrow (A \Rightarrow \boldsymbol{0}))\,[1; \mathrm{LA}_{\mathrm{T6}}], (\boldsymbol{a} \Rightarrow \boldsymbol{0}) \Rightarrow (A \Rightarrow \boldsymbol{0})\,[b; r_{\mathrm{MP}}], A \Rightarrow$$
$$\Rightarrow \boldsymbol{0}[\neg a \otimes b; r_{\mathrm{MP}}].$$

Then

$$\bigvee \{\mathrm{Val}_T(w); w'\} = \neg a$$

whence $T \vdash_c \neg A$, $\neg a \leqq c$. Since T is consistent it follows from Lemma 11(a) that

$$c \leqq a \to \mathbf{0}$$

i.e. $c = \neg a$. □

It is obvious that if T is a complete classical theory then $T \vdash A$ implies $T \vdash A \Rightarrow \boldsymbol{1}$ and $T \vdash \neg A$ is the same as $T \vdash A \Rightarrow \boldsymbol{0}$.

Lemma 17. Let $T \vdash_a A$ and $T \vdash_{\neg a} \neg A$. Then

$$\mathscr{D}(A) = a\,, \quad \mathscr{D}(\neg A) = \neg a\,, \quad \mathscr{D}(A \Rightarrow \boldsymbol{a}) = \mathbf{1} \quad \text{and} \quad \mathscr{D}(\neg(A \Rightarrow \boldsymbol{a})) = \mathbf{0}$$

holds in every model $\mathscr{D}$ of the theory T.

Proof. Let $\mathscr{D}$ be a model of T. Then we have

$$\mathscr{D}(A) \geqq a \quad \text{as well as} \quad \mathscr{D}(\neg A) = \neg \mathscr{D}(A) \geqq \neg a$$

due to validity theorem. It follows that

$$\neg\neg\mathscr{D}(A) \leqq \neg\neg a\,,$$

i.e.

$$a \leqq \mathscr{D}(A) \leqq a\,.$$

The rest is obvious. □

Theorem 7. Let T be a complete theory and $T \vdash_a A$ and $T \vdash_b B$. Then

$$T \vdash_c A \Rightarrow B \quad \text{iff} \quad c = a \rightarrow b\,.$$

Proof. It follows immediately from Theorem 6 and Lemma 12 if we realize that in a consistent theory

$$T \vdash_c \boldsymbol{a} \Rightarrow \boldsymbol{b}\,, \quad c = a \rightarrow b\,.$$ □

The existence of complete theories will be proved in [8].

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