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CONTROLLABILITY OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

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Sufficient conditions are established for nonlinear Volterra integrodifferential systems with implicit derivative. The result is obtained using the measure of noncompactness of a set and the Darbo fixed point theorem.

1. INTRODUCTION

Controllability of nonlinear systems and nonlinear delay systems has been studied by several authors by means of fixed point principles [4]. The Darbo fixed point theorem is used first by Dacka [5] to study the controllability of nonlinear systems with implicit derivative and subsequently by Balachandran [1 – 3] to study the same problem for various nonlinear delay systems. In this paper we shall extend the method of Dacka and Balachandran to nonlinear Volterra integrodifferential systems. For mathematical preliminaries relating to measure of noncompactness of a set and Darbo's theorem the reader can refer to [1, 2, 5].

Consider the following nonlinear Volterra integrodifferential system

$$\dot{x}(t) = A(t)\,x(t) + \int_{t_0}^{t} H(t,s)\,x(s)\,\mathrm{d}s + B(t)\,u(t) + f(t, x(t), \dot{x}(t)) \tag{1}$$

where the state $x(t)$ is an n-vector and the control $u(t)$ is an m-vector. The matrix functions

$$A: [t_0, t_1] \to \mathbb{R}^{n^2} \quad \text{and} \quad B: [t_0, t_1] \to \mathbb{R}^{nm}$$

are assumed to be continuous. The matrix function $H: \Delta \to \mathbb{R}^{n^2}$, $\Delta = \{(t, s): t_0 \leqq s \leqq t \leqq t_1\}$ is continuous and f is an n-vector function.

Define the norm of a continuous $n \times m$ matrix valued function $D: I \to \mathbb{R}^{nm}$ by

$$\|D(t)\| = \max_i \sum_{j=1}^{m} \max_{t \in I} |d_{ij}(t)|$$

where d_{ij} are the elements of D and I will be either $[t_0, t_1]$ or Δ in what follows.

Assume that, for $t \in [t_0, t_1]$ and $(t, s) \in \Delta$ we have

(2) $$\|A(t)\| \leqq \alpha\,, \quad \|H(t, s\| \leqq \beta\,, \quad \|B(t)\| \leqq \gamma$$

and that f is continuous and, for $x, y, \bar{y} \in \mathbb{R}^n$ and $t \in [t_0, t_1]$ satisfies the conditions

(3) $$|f(t, x, y)| \leqq \delta$$

(4) $$|f(t, x, y) - f(t, x, \bar{y})| \leqq k|y - \bar{y}|$$

where α, β, γ, δ and k are positive constants with $k < 1$.

The solution of the system (1) is given by

(5) $$x(t) = R(t, t_0)\, x_0 + \int_{t_0}^t R(t, s)\, B(s)\, u(s)\, \mathrm{d}s + \int_{t_0}^t R(t, s) f(s, x(s), \dot{x}(s))\, \mathrm{d}s\,.$$

where

(6) $$\frac{\partial R(t, s)}{\partial s} + R(t, s)\, A(s) + \int_s^t R(t, \eta)\, H(\eta, s)\, \mathrm{d}\eta = 0$$

$$R(t, t) = \text{identity} \quad \text{for} \quad t_0 \leqq s \leqq t \leqq t_1$$

In fact, $R(t, s)$ is continuous on Δ and from (2) and (6) we have by Gronwall's inequality

(7) $$\|R(t, s)\| \leqq \exp\left[\alpha(t - s) + (t - s)^2\, \beta\right]$$

We say that system (1) is completely controllable if for any $x_0, x_1 \in \mathbb{R}^n$ there exists continuous control function $u(t)$, defined on $[t_0, t_1]$ such that a solution x of (1) satisfies $x(t_1) = x_1$.

Define the controllability matrix W by

(8) $$W(t_0, t) = \int_{t_0}^t R(t, s)\, B(s)\, B^*(s)\, R^*(t, s)\, \mathrm{d}s$$

where the star denotes the matrix transpose.

2. MAIN RESULT

Theorem. Given the system (1) with conditions (2)–(4) and suppose that the matrix $W(t_0, t_1)$ is nonsingular for some $t_1 > t_0$. Then the system is completely controllable on $[t_0, t_1]$.

Proof. Take $x_1 \in \mathbb{R}^n$ arbitrarily and define the nonlinear operator T on the Banach space $C_n^1[t_0, t_1] = C^1([t_0, t_1]: \mathbb{R}^n)$ (see [1]) by the formula

(9) $$T(x)(t) = R(t, t_0)\, x_0 + \int_{t_0}^t R(t, s)\, B(s)\, u(s)\, \mathrm{d}s + \int_{t_0}^t R(t, s) f(s, x(s), \dot{x}(s))\, \mathrm{d}s$$

where the control $u(t)$ is given by

(10) $$u(t) = B^*(t)\, R^*(t_1, t)\, W^{-1}(t_0, t_1)\left[x_1 - R(t_1, t_0)\, x_0 - \int_{t_0}^{t_1} R(t_1, s) f(s, x(s), \dot{x}(s))\, \mathrm{d}s\right]$$

Substituting equation (10) into (9), we obtain

$$(11) \qquad T(x)(t) = R(t, t_0)\, x_0 + \int_{t_0}^{t} R(t, s)\, B(s)\, B^*(s)\, R^*(t_1, s)\, W^{-1}(t_0, t_1)\, .$$
$$.\, [x_1 - R(t_1, t_0)\, x_0 - \int_{t_0}^{t_1} R(t_1, s) f(s, x(s), \dot{x}(s))\, ds]\, ds +$$
$$+ \int_{t_0}^{t} R(t, s) f(s, x(s), \dot{x}(s))\, ds$$

Since all the functions involved in the definition of the operator T are continuous, the mapping T is continuous and transforms the space $C_n^1[t_0, t_1]$ into itself. Define the closed convex subset Q by

$$(12) \qquad Q = \{x: x \in C_n^1[t_0, t_1],\ \|x\| \leqq N_1\,,\ \|\dot{x}\| \leqq N_2\}$$

where the positive constants N_1 and N_2 are given by

$$N_1 = \exp\,[\alpha(t_1 - t_0) + (t_1 - t_0)^2\, \beta]\,[|x_0| + (t_1 - t_0)\, \gamma N_3 + (t_1 - t_0)\, \delta]$$
$$N_2 = \alpha N_1 + (t_1 - t_0)\, \beta N_1 + \gamma N_3 + \delta$$
$$N_3 = \|W^{-1}(t_0, t_1)\|\, [|x_1| + (\exp \alpha(t_1 - t_0) + (t_1 - t_0)^2\, \beta)\, .$$
$$.\, (|x_0| + (t_1 - t_0)\, \delta)]$$

The operator T maps Q into itself. As easily seen, all the functions $T(x)(t)$ are equicontinuous, since they have uniformly bounded derivatives. Now we shall find an estimate of the modulus of continuity of the functions $D\, T(x)(t)$ for $t, s \in [t_0, t_1]$

$$(13) \qquad |D\, T(x)(t) - D\, T(x)(s)| \leqq |A(t)\, T(x)(t) - A(s)\, T(x)(s)| +$$
$$+ \left|\int_{t_0}^{t} H(t, \eta)\, x(\eta)\, d\eta - \int_{t_0}^{s} H(s, \eta)\, x(\eta)\, d\eta\right| + |B(t)\, u(t) - B(s)\, u(s)| +$$
$$+ |f(t, x(t), \dot{x}(t)) - f(s, x(s), \dot{x}(s))|$$

The first three terms of the right hand side of inequality (13) can be estimated as $\beta_0(|t - s|)$ with $\beta(h) = O(h)$. In the same manner the last term of (13) can be estimated as $\beta_1(|t - s|) + k|\dot{x}(t) - \dot{x}(s)|$. Letting $\beta^* = \beta_0 + \beta_1$, we finally obtain

$$|D\, T(x)(t) - D\, T(x)(s)| \leqq k|\dot{x}(t) - \dot{x}(s)| + \beta^*(|t - s|)$$

hence we conclude that for any set $E \subset Q$

$$\mu(TE) \leqq k\, \mu(E)\,,$$

where μ stands for the measure of noncompactness. Consequently, by the Darbo fixed point theorem the operator T has at least one fixed point, therefore there exists a function $x \in C_n^1[t_0, t_1]$ such that

$$(14) \qquad x(t) = T(x(t))$$

Differentiating with respect to t, we easily verify that $x(t)$ given by (14) is a solution to the system (1) for the control $u(t)$ given by (10) and satisfies $x(t_0) = x_0$ and $x(t_1) = x_1$. Thus the control $u(t)$ steers the system (1) from x_0 to x_1, and hence the system (1) is completely controllable on $[t_0, t_1]$. □

Remark. If we assume that the function f satisfies also the Lipschitz condition with respect to x, then we can obtain the unique response by any control.

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