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MEDIUM DISTANCES OF PROBABILITY FUZZY-POINTS AND AN APPLICATION TO LINEAR PROGRAMMING

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In the paper the notions of probability-fuzzy points and medium distances in metric space are defined. These concepts are then applied for solving degenerated linear programs.

1. PROBABILITY-FUZZY POINTS

It is well known that a fuzzy point a in a space X is defined by a characteristic function

$$f_a : X \to [0, 1] .$$

Here $f_a(x)$ presents the probability that $x = a$. If we accept the addition operation of such possibilities then it may happen that the possibility of a is greater than 1 for some $x \in X$. This fact somehow contradicts our normal 'probabilistic' thinking. Therefore we propose now a new concept, namely probability-fuzzy points, which eliminates the mentioned discrepancy.

In what follows we suppose that a σ-algebra $\mathscr{X}$ of subsets in the space X is given. A probability-fuzzy point a in X is characterized by a probability measure μ_a on $(X, \mathscr{X})$ (it means, among others, $\mu_a(X) = 1$). Now for all $A \in \mathscr{X}$ we have

$$0 \leqq \mu_a(A) \leqq 1$$

what expresses the probability that $a \in A$.

A deterministic point $a \in X$ has its characteristic probability measure μ_a of special form

$$\mu_a(A) = \begin{cases} 1 & \text{if} \quad a \in A \\ 0 & \text{if} \quad a \notin A \end{cases}$$

The space X is thus embedded into the space of all probability-fuzzy points, denoted by X_p.

Now we suppose that a σ-finite Haar measure μ is defined on $(X, \mathscr{X})$, where X

is equipped with a group operation '+'. If there exist density functions f_a and f_b of the probability measures μ_a and μ_b with respect to μ, we can easily define the sum $a + b$ by means of convolutions (see [2, 3]). Indeed, the density function corresponding to the sum $a + b$ is naturally defined as follows

$$f_{a+b}(x) = \int_X f_a(z) \cdot f_b(x - z) \,\mathrm{d}\mu(z)$$
$$= \int_X f_a(x - y) \cdot f_b(y) \,\mathrm{d}\mu(y)$$

and

$$\mu_{a+b}(A) = \int_A f_{a+b}(x) \,\mathrm{d}\mu(x) .$$

In virtue of Fubini's Theorem on iterated integrals (cf. [1]) we have

$$\begin{aligned}\mu_{a+b}(X) &= \int_X f_{a+b}(x) \,\mathrm{d}\mu(x)\\ &= \int_X(\int_X f_a(z) \cdot f_b(-z + x) \,\mathrm{d}\mu(z)) \,\mathrm{d}\mu(x)\\ &= \int_X(\int_X f_a(z) \cdot f_b(-z + x) \,\mathrm{d}\mu(x)) \,\mathrm{d}\mu(z)\\ &= \int_X f_a(z) \cdot (\int_X f_b(-z + x) \,\mathrm{d}\mu(x)) \,\mathrm{d}\mu(z)\\ &= \int_X f_a(z) \,\mathrm{d}\mu(z) = 1 .\end{aligned}$$

So μ_{a+b} is in fact a probability measure where all integrals are allowable because the function $f_{a+b}(x)$ is integrable and $\mathscr{X}$-measurable (see [3], Statement 2.1). It should be stressed that by means of probability-fuzzy points one eliminates the trouble appearing with the 'min' operation in the definition of sums of fuzzy quantities as it is done in [2].

2. MEDIUM DISTANCES

Now we suppose that the space X is provided with a metric $\varrho(x, y)$ defined for deterministic points in X. We shall extend the metric $\varrho(x, y)$ to the space X_p of probability-fuzzy points by means of the notion of medium distance proposed as follows.

Let a and b be probability-fuzzy points with characteristic measures μ_a and μ_b respectively. Let $\mu_a \otimes \mu_b$ denote the product measure of μ_a and μ_b on the Cartesian product $(X, \mathscr{X}) \times (X, \mathscr{X})$, where $\mathscr{X}$ contains all open subsets of X. Then the function $\varrho(x, y)$ is integrable and the Lebesgue integral

$$\int_{X \times X} \varrho(x, y) \,\mathrm{d}\mu_a \otimes \mu_b$$

will be called the medium distance of points a and b and denoted by $d_m(a, b)$.

It is easily seen that if a and b are deterministic points then $d_m(a, b)$ coincides with $\varrho(a, b)$. The symmetry property

$$d_m(a, b) = d_m(b, a) \quad \forall a, b \in X_p$$

immediately follows from the definition. As for the triangle inequality it needs a detailed proof.

Let a, b and c be arbitrary probability-fuzzy points. We have

$$\begin{aligned} d_m(a, b) &= \int_{X \times X} \varrho(x, y) \, \mathrm{d}\mu_a \otimes \mu_b(x, y) \\ &= \int_{X \times X} \varrho(x, y) \, \mathrm{d}\mu_a \otimes \mu_b(x, y) \, . \int_X \mathrm{d}\mu_c(z) \\ &= \int_{X \times X \times X} \varrho(x, y) \, \mathrm{d}\mu_a \otimes \mu_b \otimes \mu_c(x, y, z) \\ &\leqq \int_{X \times X \times X} [\varrho(x, z) + \varrho(z, y)] \, \mathrm{d}\mu_a \otimes \mu_b \otimes \mu_c(x, y, z) \\ &= \int_{X \times X \times X} \varrho(x, z) \, \mathrm{d}\mu_a \otimes \mu_b \otimes \mu_c(x, y, z) + \\ &+ \int_{X \times X \times X} \varrho(y, z) \, \mathrm{d}\mu_a \otimes \mu_b \otimes \mu_c(x, y, z) \\ &= \int_{X \times X} \varrho(x, z) \, \mathrm{d}\mu_a \otimes \mu_c(x, z) \, . \int_X \mathrm{d}\mu_b(y) + \\ &+ \int_{X \times X} \varrho(y, z) \, \mathrm{d}\mu_b \otimes \mu_c(y, z) \, . \int_X \mathrm{d}\mu_a(x) \\ &= d_m(a, c) + d_m(c, b) \, . \end{aligned}$$

Now, let $a \in X_p$ and $B \subset X$. The medium distance from a to B is similarly defined as

$$d_m(a, B) = \int_X \varrho(x, B) \, \mathrm{d}\mu_a(x) \, ,$$

where

$$\varrho(x, B) = \inf \{\varrho(x, b) \colon b \in B\}$$

is the usual distance from x to the set B.

Remark. The medium distance does not fulfill the property

$$d_m(a, b) = 0 \Leftrightarrow a = b$$

and it prevents $d_m(a, b)$ to be a metric in the traditional sense. However, if we consider probability-fuzzy points as random points varying with respect to time, then the fact that the distance between a point and itself is positive is acceptable. Here it is in place to remind the famous statement of Heracleitos: 'One cannot enter twice into the same river'.

3. EXAMPLE

Let $X = \mathbb{R}^n$ and

$$S_a = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_i \leqq c \ \forall i \text{ and } \sum_{i=1}^{n} x_i = a\}$$

where c is a fixed positive constant and a is a parameter satisfying

$$(3.1) \qquad |a| \leqq c \, . \, n \, .$$

Since S_a are parallel affine spaces of dimension $n - 1$ there exists the same positive atomless measure v, on the σ-algebra of measurable sets, for all of them, similarly as in $\mathbb{R}^{n-1}$.

Further, we define p_a as a probability-fuzzy point with uniform probability distribution on S_a, i.e. p_a has the density function f_a on S_a such that

$$f_a(x) = 1/v(S_a) \quad \forall x \in S_a \, .$$

The characteristic measure μ_a of p_a is then defined by

$$\mu_a(S) = \nu(S \cap S_a)/\nu(S_a) \quad \forall S \subset \mathbb{R}^n .$$

Let us denote

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon x_i \geqq 0 \ \forall i = 1, \ldots, n\} .$$

We shall consider the medium distance from p_a to $\mathbb{R}^n_+$ and prove that, as a function of parameter a, $d_m(p_a, \mathbb{R}^n_+)$ is decreasingly monotone.

Putting

$$C = \underbrace{(c, \ldots, c)}_{n-\text{times}} \in \mathbb{R}^n$$

then the following lemmas are easily verifiable.

Lemma 3.1. For all $a \in [0, nc]$ we have

(3.2) $$S_a - C = \lambda_a(S_0 - C) ,$$

where

(3.3) $$\lambda_a = (nc - a)/nc .$$

Lemma 3.2. For all $a \in [0, nc]$ we have

$$\nu(S_a) = \nu(S_a - C) =$$
$$= \lambda_a^{n-1}\, \nu(S_0 - C) = \lambda_a^{n-1}\, \nu(S_0)$$

(see Fig. 1).

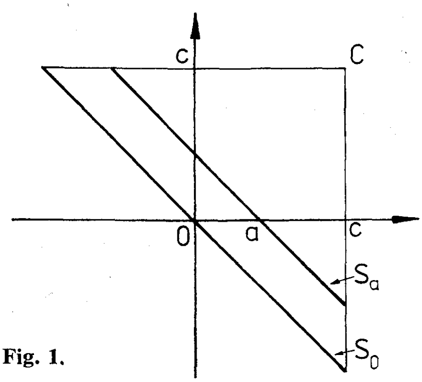

Fig. 1.

Statement 3.1. If $nc \geqq a > b \geqq 0$, then

$$d_m(p_a, \mathbb{R}^n_+) < d_m(p_b, \mathbb{R}^n_+) .$$

Proof. We have

$$d_m(p_a, \mathbb{R}^n_+) = \frac{1}{\nu(S_a)} \int_{S_a} [\sum_{x_i<0} x_i^2]^{1/2}\, d\nu(x) = \frac{1}{\lambda_a^{n-1}\, \nu(S_0)} \int_{S_a - C} [\sum_{y_i<-c} (y_i + c)^2]^{1/2}\, d\nu(y) =$$

(by Lemma 3.2 and substitution $y = x - C$)

$$= \frac{1}{\lambda_a^{n-1}\, \nu(S_0)} \int_{S_0 - C} [\sum_{z_i<-c/\lambda_a} (\lambda_a z_i + c)^2]^{1/2}\, \lambda_a^{n-1}\, d\nu(z) =$$

(by substitution $y = \lambda_a . z$)

$$= \frac{1}{\nu(S_0)} \int_{S_0 - C} [\sum_{z_i<-c/\lambda_a} (\lambda_a z_i + c)^2]^{1/2}\, d\nu(z) .$$

Analogically,

$$d_m(p_b, \mathbb{R}^n_+) = \frac{1}{v(S_0)} \int_{S_0 - C} [\sum_{z_i < -c/\lambda_b} (\lambda_b z_i + c)^2]^{1/2} \, dv(z) .$$

Since $a > b$ one has $\lambda_a < \lambda_b$ (see (3.3)). Then

$$-c/\lambda_a < -c/\lambda_b$$

and hence

$$(\lambda_a . z_i + c)^2 < (\lambda_b . z_i + c)^2 \quad \text{for such } z_i \text{ that} \quad z_i < -c/\lambda_a .$$

Consequently we obtain

$$d_m(p_a, \mathbb{R}^n_+) < d_m(p_b, \mathbb{R}^n_+)$$

and the proof is complete. □

Now consider the case $-nc \leqq a \leqq 0$. Obviously

Lemma 3.3. For all $a \in [-nc, 0]$ we have

(3.4) $$S_a + C = \beta_a(S_0 + C)$$

where

(3.5) $$\beta_a = (nc + a)/nc .$$

Lemma 3.4. For all $a \in [-nc, 0]$ we have

$$v(S_a) = v(S_a + C) = \beta_a^{n-1} v(S_0 + C) = \beta_a^{n-1} v(S_0)$$

(see Fig. 2).

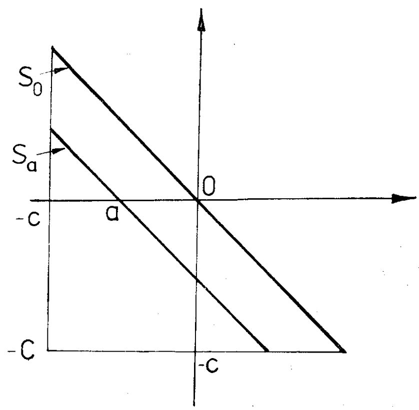

Fig. 2.

Statement 3.2. If $0 \geqq a > b \geqq -nc$, then

$$d_m(p_a, \mathbb{R}^n_+) < d_m(p_b, \mathbb{R}^n_+) .$$

Proof. We have

$$d_m(p_a, \mathbb{R}^n_+) = \frac{1}{v(S_a)} \int_{S_a} [\sum_{x_i < 0} x_i^2]^{1/2} \, dv(x) =$$

$$= \frac{1}{\beta_a^{n-1} v(S_0)} \int_{S_a + C} [\sum_{y_i < c} (y_i - c)^2]^{1/2} \, dv(x) =$$

(by Lemma 3.4 and substitution $y = x + C$)

$$= \frac{1}{\beta_a^{n-1}\, \nu(S_0)} \int_{S_0+C} \Big[\sum_{z_i < c/\beta_a} (\beta_a\, z_i - c)^2 \Big]^{1/2} \beta_a^{n-1}\, d\nu(z) =$$

(by substitution $y = \beta_a \, . \, z$)

$$= \frac{1}{\nu(S_0)} \int_{S_0+C} \Big[\sum_{z_i < c/\beta_a} (\beta_a\, z_i - c)^2 \Big]^{1/2} d\nu(z) \, .$$

Analogously

$$d_m(p_b, \mathbb{R}^n_+) = \frac{1}{\nu(S_0)} \int_{S_0+C} \Big[\sum_{z_i < c/\beta_b} (\beta_b\, z_i - c)^2 \Big]^{1/2} d\nu(z) \, .$$

Since $a > b$ one has $\beta_a > \beta_b$ (see 3.5). Then

$$c/\beta_a < c/\beta_b$$

and hence

$$(\beta_a \, . \, z_i - c)^2 < (\beta_b \, . \, z_i - c)^2 \quad \text{for such } z_i \text{ that} \quad z_i < c/\beta_a \, .$$

Consequently, we obtain

$$d_m(p_a, \mathbb{R}^n_+) < d_m(p_b, \mathbb{R}^n_+)$$

and the proof is complete. □

4. AN APPLICATION TO LINEAR PROGRAMMING

First, we briefly recall the simplex method for solving linear programs. Consider the problem

(4.1) $$f = c \, . \, x \to \max$$

s.t.

(4.2) $$A \, . \, x = b$$

(4.3) $$x \geqq 0$$

where $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $b = (b_1, \ldots, b_m)' \in \mathbb{R}^m$, $A = (a_{ij})_{m \times n}$ and $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$.

Suppose that $m < n$ and rank $(A) = m$. A vector $x \in \mathbb{R}^n$ is a feasible solution if is fulfills (4.2) and (4.3); an optimal solution if it fulfils (4.1), (4.2) and (4.3); a basic solution if it is a feasible solution and the columns a_j of A corresponding to positive components $x_j > 0$ are linearly independent. A basis of a basic solution x is an arbitrary system of m linearly independent columns of A including all those corresponding to positive components of x. A basic solution is degenerated if it has less than m positive components. Problems (4.1)–(4.3) is degenerated if it has degenerated solutions.

Let us have a basic solution $x = (x_1, \ldots, x_n)$ of problems (4.1)–(4.3) with the basis

$$A_\sigma = \{a_j \colon a_j \text{ is the } j\text{th column of } A \text{ and } j \in \sigma\}$$

the basic variables

$$x_\sigma = \{x_j \colon j \in \sigma\}$$

and the nonbasic variables

$$x_\omega = \{x_j : j \in \omega = \{1, \ldots, n\} \setminus \sigma\}$$

where σ denotes the set of indexes of basic variables.

We have then the corresponding simplex tableau

Tableau 1.

x_σ	$-x_\omega$	
	1	$\ldots -x_j \ldots$
f	β_{00}	$\ldots\ \beta_{0j} \ldots$
$\vdots$	$\vdots$	$\vdots$
x_i	β_{i0}	$\ldots\ \beta_{ij} \ldots$
$\vdots$	$\vdots$	$\vdots$

where

(4.4) $$B = (\beta_{ij}) = A_\sigma^{-1} . A_\omega , \quad \beta = (\beta_{i0}) = A_\sigma^{-1} . b , \quad \Delta = (\beta_{0j}) = c_\sigma . B - c_\omega ,$$
$$\beta_{00} = c_\sigma . \beta , \quad c_\sigma = \{c_j : j \in \sigma\} \quad \text{and} \quad c_\omega = \{c_j : j \in \omega\} .$$

The simplex iteration procedure consists of the following steps.

Step 1. Testing of optimality:

(4.5) $$\Delta = (\beta_{0j})_{j\in\omega} \geqq 0$$

If (4.5) holds then x is an optimal solution. Stop. Otherwise go to Step 2.

Step 2. Testing of boundedness:

(4.6) $$\exists j \in \omega : \beta_{0j} < 0 \quad \text{and} \quad \beta_{ij} \leqq 0 \quad \forall i \in \sigma .$$

If (4.6) holds then $c . x$ is not bounded from above. Stop. Otherwise go to Step 3.

Step 3. Constructing of an improved solution:

Choose a $\beta_{0s} < 0$, usually the smallest one, and determine

(4.7) $$\theta = \min \{\beta_{i0}/\beta_{is} : i \in \sigma \text{ and } \beta_{is} > 0\} .$$

Let $\beta_{t0}/\beta_{ts} = \theta$. Perform the Jordan-Gauss elimination procedure:

(4.8) $$\beta'_{ij} = \beta_{ij} - \frac{\beta_{is}\beta_{ts}}{\beta_{ts}} \quad \begin{array}{l} \forall i \in (\sigma' \setminus \{s\}) \cup \{0\} \\ \forall_j \in (\sigma' \setminus \{t\}) \cup \{0\} \end{array}$$
$$\beta'_{sj} = \beta_{tj}/\beta_{ts} \qquad \forall j \in (\omega' \setminus \{t\}) \cup \{0\}$$
$$\beta'_{st} = 1/\beta_{ts}$$
$$\beta'_{it} = -\beta_{is}/\beta_{ts} \qquad \forall i \in (\sigma' \setminus \{t\}) \cup \{0\}$$

where

$$\sigma' = (\sigma \setminus \{t\}) \cup \{s\} , \quad \omega' = (\omega \setminus \{s\}) \cup \{t\} .$$

We obtain a new simplex tableau, Tableau 2, and a new basic solution x' with improved values

(4.9) $$\beta'_{00} = \beta_{00} - \theta \,.\, \beta_{0s} \,.$$

Tableau 2.

$x_{\sigma'}$	$x_{\omega'}$	
	1	$\ldots -x_j \ldots$
f	β'_{00}	$\ldots\ \beta'_{0j} \ldots$
$\vdots$	$\vdots$	$\vdots$
x_i	β_{i0}	$\ldots\ \beta_{ij} \ldots$
$\vdots$	$\vdots$	$\vdots$

Go to Step 1.

If the problem (4.1)–(4.3) is degenerated, there may exist several basic variables x_t with $\beta_{t0}/\beta_{ts} = \theta$ and one faces the question which of them to choose. In practice, the variable with the smallest index is usually chosen. In order to avoid cycling one can use the lexicographical rule.

However these rules do not take into consideration the optimality criterion and the iteration procedure for finding an optimum solution is in many cases unnecessarily prolonged.

Let us consider β'_{0j}, $j \in \omega'$. We know that

$$\beta'_{0t} = -\beta_{0s}/\beta_{ts} > 0 \,.$$

Hence the optimality criterion (4.5) in this iteration reduces to

(4.10) $$\Delta_t = (\beta'_{0j})_{j \in \omega' \setminus \{t\}} \geqq 0$$

which is equivalent to

(4.11) $$d(\Delta_t, \mathbb{R}_+^{n-m-1}) = 0$$

where $d(\cdot, \cdot)$ denotes the normal distance in $\mathbb{R}^{n-m-1}$.

If we know the exact value of Δ_t, it is not a problem to verify condition (4.11). However, if the information about Δ_t is not complete, Δ_t becomes a non-exactly known vector and one naturally turns to 'fuzzy theory' for help. In what follows we shall present one such possible help.

Suppose we know only the sum of all components Δ_t

(4.12) $$\delta_t = \sum_{j \in \omega' \setminus \{t\}} \beta'_{0j}$$

and the data are random enough and do not exceed some limit $c > 0$. Then we can assume that Δ_t is uniformly distributed on the set

$$S_{\delta_t} = \{z = (z_1, \ldots, z_{n-m-1}) \colon \sum_{i=1}^{n-m-1} z_i = \delta_t \text{ and } |z_i| \leqq c \ \forall i = 1, \ldots, n-m-1\} \,.$$

Now Δ_t becomes a probability-fuzzy point in $\mathbb{R}^{n-m-1}$ and one disposes of the medium distance $d_m(\Delta_t, \mathbb{R}^{n-m-1})$ considered in the preceding section. Put

$$M = \{k: k \in \sigma \text{ and } \beta_{k0}/\beta_{ks} = \theta\}.$$

Then the optimality condition (4.11) is equivalent to

$$d(\Delta_t, \mathbb{R}_+^{n-m-1}) = \min \{d(\Delta_k, \mathbb{R}_+^{n-m-1}): k \in M\} = 0. \tag{4.13}$$

So it natural to choose $t \in M$ that such

$$d(\Delta_t, \mathbb{R}_+^{n-m-1}) = \min_{k \in M} d(\Delta_k, \mathbb{R}_+^{n-m-1}). \tag{4.14}$$

However the Δ_k are not exactly known. Therefore we logically use the medium distance and (4.14) becomes

$$d_m(\Delta_t, \mathbb{R}_+^{n-m-1}) = \min_{k \in M} d_m(\Delta_k, \mathbb{R}_+^{n-m-1}). \tag{4.15}$$

It follows from the results of Section 3 that (4.15) is equivalent to

$$\delta_t = \max \{\delta_k: k \in M\}. \tag{4.16}$$

By virtue of (4.8) and (4.12) we have

$$\delta_k = \sum_{j \in \omega \setminus \{s\}} \beta_{0j} - \beta_{0s} \Big(\sum_{j \in \omega \setminus \{s\}} \beta_{kj} \Big) / \beta_{ks}. \tag{4.17}$$

Since $\sum_{j \in \omega \setminus \{s\}} \beta_{0j}$ is invariable and $-\beta_{0s} > 0$, condition (4.16) is equivalent to the following condition

$$\Big(\sum_{j \in \omega \setminus \{s\}} \beta_{tj} \Big) / \beta_{ts} = \max_{k \in M} \Big(\sum_{j \in \omega \setminus \{s\}} \beta_{kj} \Big) / \beta_{ks} \tag{4.18}$$

which is called the Maximum-Sum Rule.

The following examples show the advantage of the Maximum-Sum Rule as compared with the lexicographical rule.

Example 1. Consider the program

$$f = -2x_1 - 3x_2 - 3x_3 - 8x_4 - 4x_5 \to \max$$

s.t.

$$\begin{aligned} 2x_2 \phantom{{}+x_3} + x_4 + 7x_5 &= 14 \\ 5x_2 + x_3 \phantom{{}+x_4} + 2x_5 &= 4 \\ x_1 + x_2 \phantom{{}+x_3+x_4} + 2x_5 &= 8 \\ x_j \geqq 0 \quad \forall j & \end{aligned}$$

Applying the lexicographical rule we get an optimal solution after three simplex tableaux, see Tableau 3, 4 and 5, and the comment attached to them.

In the passage from Tableau 3 to Tableau 4 $\frac{14}{7} = \frac{4}{2} = 2$ is the minimum in (4.7), hence according to the lexicographical rule x_4 is excluded from the basis.

Tableau 3.

	1	$-x_2$	$-x_5$
f	-140	-30	-62
x_4	14	2	7
x_3	4	5	2
x_1	8	1	2

Tableau 4.

	1	$-x_2$	$-x_4$
f	-16	$-86/7$	$62/7$
x_5	2	$2/7$	$1/7$
x_3	0	$31/7$	$-2/7$
x_1	4	$3/7$	$-2/7$

Tableau 5.

	1	$-x_3$	$-x_4$
f	-16	$86/31$	$250/31$
x_5	2		
x_2	0		
x_1	4		

However the Maximum-Sum Rule enables to obtain the optimal solution only after two simplex tableaux as it is shown by Tableau 6 and 7 below. Here $\frac{5}{2} > \frac{2}{7}$,

Tableau 6.

	1	$-x_2$	$-x_5$
f	-140	-30	-62
x_4	14	2	7
x_3	4	5	2
x_1	8	1	2

Tableau 7.

	1	$-x_2$	$-x_3$
f	-16	125	31
x_4	0		
x_5	2		
x_1	4		

hence $\frac{5}{2}$ is the maximum in (4.18). So according to the Maximum-Sum Rule x_3 is excluded from the basis.

Example 2. Let us consider the well known Beale's example (see Gass [4], Chapter 7)

$$\frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4 \to \max$$

s.t.

$$\begin{aligned}
\tfrac{1}{4}x_1 - 60x_2 - \tfrac{1}{25}x_3 + 9x_4 + x_5 &= 0\\
\tfrac{1}{2}x_1 - 90x_2 - \tfrac{1}{50}x_3 + 3x_4 + x_6 &= 0\\
x_3 + x_7 &= 1
\end{aligned}$$

$$x_j \geqq 0 \qquad \forall j$$

This example is degenerated. If one choses among the variables realizing the minimum in (4.7) the one with the smallest index, cycling will appear. In order to avoid cycling we can use the lexicographical rule and obtain the optimal solution after six simplex tableaux, see Gass [4], Chapter 7.

However, applying our Maximum-Sum Rule one gets the optimal solution much faster, after three simplex tableaux, as it is shown by Tableau 8, 9 and 10 below.

Tableau 8.

	1	$-x_1$	$-x_2$	$-x_3$	$-x_4$
f	0	$-3/4$	150	$-1/50$	6
x_5	0	1/4	-60	$-1/25$	9
x_6	0	1/2	-90	$-1/50$	3
x_7	1	0	0	1	0

Tableau 9.

	1	$-x_6$	$-x_2$	$-x_3$	$-x_4$
f	0	3/2	15	$-1/20$	21/2
x_5	0	$-1/2$	-15	$-3/100$	15/2
x_1	0	2	-180	$-1/25$	6
x_7	1	0	0	1	0

Tableau 10.

	1	$-x_6$	$-x_2$	$-x_7$	$-x_4$
f	1/20	3/2	15	1/20	21/2
x_5	3/1000				
x_1	1/25				
x_3	1				

In the passage from Tableau 8 to Tableau 9 we have $(-60 - \frac{1}{25} + 9) : \frac{1}{4} = -204 \cdot 16$; $(-90 - \frac{1}{50} + 3) : \frac{1}{2} = -174{\cdot}04$. So according to the Maximum-Sum Rule x_6 is excluded from the basis.

(Received January 25, 1988.)

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