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BOUNDS ON THE THROUGHPUT OF AN UNSLOTTED ALOHA CHANNEL IN THE CASE OF A HETEROGENEOUS USERS' POPULATION

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In this paper we give a lower and an upper bound on the intensity of successful packet transmissions and the throughput of an unslotted ALOHA channel in the case of a heterogeneous users' population. These bounds have been derived under the only assumptions of stationarity and independence of traffic streams generated by the users. They are obtained by means of the results from the theory of random pulse streams [9] and depend only on the mean packet transmission times and the intensities of packet generation.

1. INTRODUCTION

For the last ten years there has been a significant interest in performance evaluation of random access protocols (see [1] and references therein). Most of the studies in this area have been devoted to protocols with slotted time e.g. the slotted ALOHA. The main reason is that slotted protocols can be conveniently described by Markov chains. Unfortunately, the use of Markov chains to unslotted protocols is not straightforward (see e.g. [2]). In consequence, there are only few results on unslotted protocols in the literature.

Most of these results has been obtained for the unslotted ALOHA channel. The first approach to that channel has been made under the assumption that the overall packet traffic on the multiaccess channel is Poissonian [3]. It has been recognized that this assumption is not particulary valid for small users' populations and heavy loads. Another approach has been presented in [4] where it has been assumed that the starting points of packet transmissions at each transmitter form a renewal process. Next, this approach has been generalized in [5]. In all above mentioned papers it has been assumed that transmitted packets have the same constant length. In the case of variable length packets the throughput analysis is more laborious. The research in this case was begun in [6] and [7] under the assumption of exponentially distributed "think" time of each user. Another approach has been presented in [8] where it has been assumed that the packet transmissions form collectively a Poisson point process.

In this paper we are concerned with an unslotted-ALOHA-type channel and a heterogeneous population of users. We present bounds on the intensity of successful packet transmissions and throughput under the most relaxed assumptions one can find in the literature, namely of stationarity and independence of traffic streams generated by the users. Our analysis is based on the results from the general theory of random pulse streams [9].

2. THE MODEL

Let us assume that the considered system consists of M users which collectively share a broadcast channel for communication with the common receiver. Each user sends occasionally his information in the form of variable length packets. Then, the signal emitted by each user can be viewed as a sequence of alternating pulses and idle periods of different lengths.

Let

$$A_m(t) = \begin{cases} 1 & \text{if user } m \text{ is active at time } t \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad m = 1, 2, \ldots, M$$

Thus, $\{A_m(t), t \in (-\infty. +\infty)\}$, $m = 1, 2, \ldots, M$, are stochastic processes and let us assume that

1) all these processes are stationary, therefore

$$\mathsf{P}[A_m(t) = 1] = \mathsf{E}[A_m(t)] = g_m\,, \quad m = 1, 2, \ldots, M\,,$$

2) all these processes are mutually independent, therefore

$$\mathsf{P}[A_m(t) = 1 \quad \text{and} \quad A_n(t) = 1] = g_m g_n\,, \quad m, n = 1, 2, \ldots, M, m \neq n\,.$$

We shall denote the average packet transmission time and the average idle period of user m ($m = 1, 2, \ldots, M$) by τ_m and α_m, respectively. Thus, the intensity of packet generation at user m is $\lambda_m = 1/(\alpha_m + \tau_m)$ whereas $g_m = \lambda_m \tau_m$.

Next, we assume that the only source of errors is overlapping of packets in the multiaccess channel.

3. BOUNDS ON THE INTENSITY OF SUCCESSFULLY TRANSMITTED PACKETS

The stream of pulses generated by user m can be conveniently represented as $PS(m) = (\ldots, \overline{x_m^{(i-1)} y_m^{(i-1)}}, \overline{x_m^{(i)} y_m^{(i)}}, \overline{x_m^{(i+1)} y_m^{(i+1)}}, \ldots)$ where $\overline{x_m^{(i)} y_m^{(i)}}$ is pulse i of user m that is the time interval from $x_m^{(i)}$ to $y_m^{(i)}$ when user m remains active. Next, let us denote by $PS(\Sigma)$ the set of all pulses generated by all users: $PS(\Sigma) = (\overline{x_m^{(i)} y_m^{(i)}}, m = 1, 2, \ldots, M, i = 0, \mp 1, \mp 2, \ldots)$.

Now let us focus our attention on a family of pulse streams that can be derived from $PS(1), PS(2), \ldots, PS(M)$. Let t be a randomly chosen observation point on the time axis. Then, one may define the pulse stream $PS[\min(k, M)]$, $k = 1, 2, \ldots, M$, as follows

$$t \in PS[\min(k, M)] \quad \text{if and only if } t \in \text{at least } k \text{ elements of } PS(\Sigma)$$

Note that any point t may belong to at most M elements of $PS(\Sigma)$ or does not belong to $PS(\Sigma)$ at all. Figure 1 presents a sample of pulse streams $PS(m)$ and $PS[\min(k, M)]$ for $M = 3$.

In this paper we first of all focus ourselves on the pulse stream denoted by $PS(+)$ which is formed from the conflict-free pulses that is non-overlapping in time (see Fig. 1). The intensity of $PS(+)$ will be denoted by λ_+. We shall relate it with the

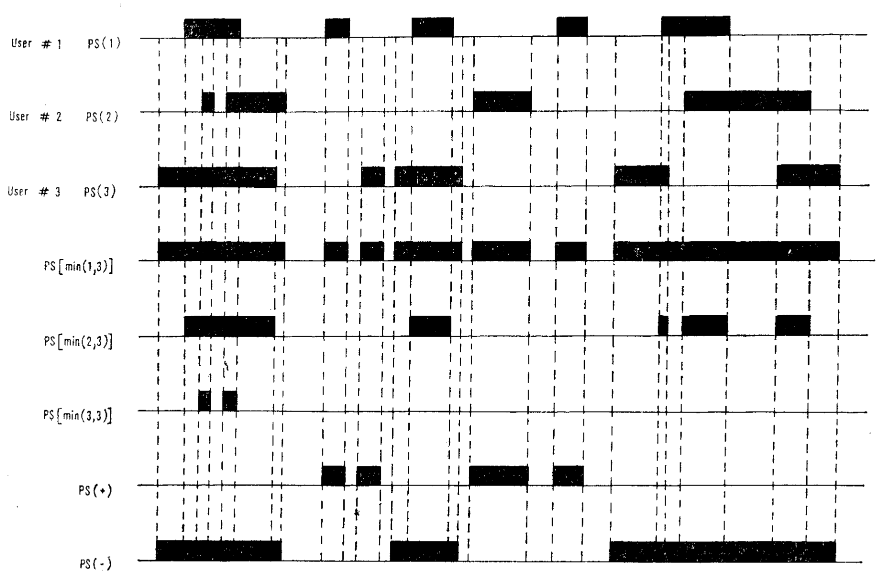


Fig. 1. The considered pulse streams for a three user population ($M = 3$).

intensities $\lambda_{\min(k,M)}$ and, in turn, with λ_Σ that is the intensities of $PS[\min(k, M)]$ and $PS(\Sigma)$, respectively.

The intensity $\lambda_{\min(k,M)}$ has been derived in [9] under the assumptions mentioned in Section 2 in the following form

$$(1) \qquad \lambda_{\min(k,M)} = \sum_{l=k}^{M} \sum_{n=1}^{\binom{M}{l}} \left\{ \prod_{m=1}^{l} g_m \prod_{\substack{j=1 \\ j \neq m}}^{M-l} (1 - g_j) \left[\sum_{m=1}^{l} \frac{1}{\tau_m} - \sum_{\substack{j=1 \\ j \neq m}}^{M-l} \frac{\lambda_j}{1 - g_j} \right] \right\}_{(n)}$$

where λ_m is the intensity of $PS(m)$, τ_m is the average length of pulses generated by user m, M is the number of users, $g_m = \lambda_m \tau_m$ and (n) denotes that we have the nth combination of l indices m in the brackets.

In order to explain the notation used in (1), we present below (1) for $k = 2$ and $M = 3$.

$$\lambda_{\min(2,3)} = g_1 g_2 (1 - g_3)\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{1}{1 - g_3}\right) +$$

$$+ g_1 g_3 (1 - g_2)\left(\frac{1}{\tau_1} + \frac{1}{\tau_3} - \frac{1}{1 - g_2}\right) +$$

$$+ g_2 g_3 (1 - g_1)\left(\frac{1}{\tau_2} + \frac{1}{\tau_3} - \frac{1}{1 - g_1}\right) +$$

$$+ g_1 g_2 g_3 \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3}\right)$$

It is to be observed that the intensity of conflict-free packets λ_+ must be less than $\lambda_{\min(1,M)}$ since $PS[\min(1, M)]$ also comprises pulses formed from the overlapping ones. Thus, using (1) we get that

(2) $$\lambda_+ < \lambda_{\min(1,M)} = \sum_{m=1}^{M} \lambda_m \prod_{\substack{j=1 \\ j \neq m}}^{M-1} (1 - g_j) = \lambda_U$$

The right-hand side of the above expression denoted by λ_U is, in fact, the upper bound on intensity λ_+.

Let us turn now to deriving for a lower bound, λ_L. We note that a pulse stream $PS[\min(1, M)]$ may be splitted into two streams, that is the stream of conflict-free pulses $PS(+)$ of intensity λ_+ and the stream of pulses $PS(-)$ of intensity λ_- which is formed from the interfering ones. Thus, we have that

(3) $$\lambda_- = \lambda_{\min(1,M)} - \lambda_+$$

On the other hand, we are able to observe that

(4) $$\lambda_- < \lambda_{\min(2,M)}$$

since every pulse of $PS(-)$ meets with at least one pulse of $PS[\min(2, M)]$. Substituting (3) into (4), we get that

(5) $$\lambda_+ > \lambda_{\min(1,M)} - \lambda_{\min(2,M)}$$

Using (1) we have that

(6) $$\lambda_+ > \sum_{m=1}^{M} g_m \prod_{\substack{j=1 \\ j \neq m}}^{M-1} (1 - g_j)\left[\frac{1}{\tau_m} - \sum_{\substack{j=1 \\ j \neq m}}^{M-1} \frac{\lambda_j}{1 - g_j}\right] = \lambda_L^*$$

It is to be noted that the result $\lambda_{\min(1,M)} - \lambda_{\min(2,M)}$ may be less than zero for the sufficiently large values of $\lambda_1, \lambda_2, \ldots, \lambda_M$. Therefore, in order to determine a lower bound, we take the maximum of λ_L^* and zero:

(7) $$\lambda_L = \max(0, \lambda_L^*)$$

From the above considerations we can finally conclude that the intensity λ_+ in a multiaccess channel has belong to the interval (λ_L, λ_U), that is

(8) $$\lambda_L < \lambda_+ < \lambda_U$$

where λ_L and λ_U are given by (7) and (2), respectively.

Let us consider the special case of a homogeneous population of users in the sense that $\lambda_1 = \lambda_2 = \ldots = \lambda_M = \lambda$ and $\tau_1 = \tau_2 = \ldots = \tau_M = \tau$, then (1) reduces to the following

(9) $$\lambda_{\min(k,M)} = \lambda \sum_{l=k}^{M} \binom{M}{l} g^{l-1} (1 - g)^{M-l-1} (l - Mg)$$

where $g = \lambda\tau$.

Using the above result we get the lower and the upper bound as follows

(10) $$\lambda_L = \max [0, M\lambda(1 - g)^{M-2} (1 - Mg)]$$

and

(11) $$\lambda_U = M\lambda(1 - g)^{M-1}$$

4. BOUNDS ON THE THROUGHPUT

In many cases we are interested in the throughput of a multiaccess channel instead of the intensity λ_+. Classically, the throughput S is defined as the average number of successfully transmitted bits (i.e. without errors) per time unit. However, in the theory of multiaccess channels we are often faced with another definition that can be expressed as follows

(12) $$S = \lambda_+ \tau_+$$

where τ_+ is the average length of pulses of the stream $PS(+)$. This definition does not take into account the non-demaged bits in the overlapped packets. It is assumed that packets involved in collisions will be retransmitted after elapsing the proper timeout interval. In the following we focus ourselves on the throughput defined by (12).

In order to find an upper bound on the throughput we define the pulse stream $PS(1, M)$:

$t \in PS(1, M)$ if and only if $t \in$ exactly one element of $PS(\Sigma)$

Let $\lambda_{(1,M)}$ and $\tau_{(1,M)}$ be the intensity and the average pulse length of $PS(1, M)$. Then, $g_{(1,M)} = \lambda_{(1,M)}\tau_{(1,M)}$ and it is given by the following formula [9]:

(13) $$g_{(1,M)} = \sum_{m=1}^{M} g_m \prod_{\substack{j=1 \\ j \neq m}}^{M-1} (1 - g_j)$$

where $g_m = \lambda_m \tau_m$.

The quantity $g_{(1,M)}$ represents the probability that at a randomly chosen point on the time axis exactly one user is active.

We observe that a pulse stream $PS(1, M)$ comprises all successfully transmitted packets plus non-overlapped parts of packets involved in collisions. Thus, $g_{(1,M)}$ is the unconditional channel throughput in terms of error-free bits transmitted. The above remark simply leads us to an upper bound on the throughput defined by (12):

$$S < g_{(1,M)} = S_U \tag{14}$$

where $g_{(1,M)}$ is given by (13).

In the special case of a homogeneous system in the sense that $g_1 = g_2 = \ldots = g_M = g$ we get that

$$S < G\left(1 - \frac{G}{M}\right)^{M-1} = S_U \tag{15}$$

where G is the total traffic in the channel $(G = Mg)$.

The elementary differential calculus shows that the right-hand side of (15) is maximized at $G_0 = 1$ and then it is equal to

$$C = \left(1 - \frac{1}{M}\right)^{M-1} \tag{16}$$

The above result is in fact the channel capacity since the average number of successfully transmitted packets per time unit is always less than C. The same result has been obtained in the other way in [5] but under the assumption of constant length packets. It is worth to note that for very large M we have $C \approx 1/e$ as for the slotted ALOHA channel. Next, it is to be observed also that (14), (15), and (16) do not depend on the type of distribution of traffic generated by users but only on g_m, $m = 1, 2, \ldots, M$.

Unfortunately, it is impossible to derive a lower bound under such general assumptions using the pulse streams theory since we cannot boun dτ_+. The only exception is that all users generate identical fixed-length packets. Then $\tau_1 = \tau_2 = \ldots = \tau_M = \tau$ and $\tau_+ = \tau$, and hence

$$S > \lambda_L \tau = S_L = \max\left(0, \sum_{n=1}^{M} \left\{g_m \prod_{\substack{j=1 \\ j \neq m}}^{M-1} (1 - g_j)\left[1 - \sum_{\substack{j=1 \\ j \neq m}}^{M-1} \frac{g_j}{1 - g_j}\right]\right\}_{(n)}\right) \tag{17}$$

where λ_L is given by (7).

Let us consider the symmetric case, i.e. $\lambda_1 \tau = \lambda_2 \tau = \ldots = \lambda_M \tau = G/M$. Then, one may simply find that

$$S_L = \max\left[0, G(1 - G)\left(1 - \frac{G}{M}\right)^{M-2}\right] \tag{18}$$

In the limit as $M \to \infty$, we get that

$$S_L = \max\left[0, G(1 - G)^{-G}\right] \tag{19}$$

One can prove that S_L given by (18) has the maximum at

$$G_1 = \frac{3M - 1 - \sqrt{(5M^2 - 6M + 1)}}{2M} \tag{20}$$

equal to

$$S_{L,\max} = \frac{-4M^2 + 5M - 1 + (2M - 1)\sqrt{(5M^2 - 6M + 1)}}{2M^2} \cdot \left(1 - \frac{3M - 1 - \sqrt{(5M^2 - 6M + 1)}}{2M^2}\right) \tag{21}$$

Finally, we conclude that the maximum throughput, $S_{\max}$, can be bounded as given below

$$S_{L,\max} < S_{\max} < \left(1 - \frac{1}{M}\right)^{M-1} \tag{22}$$

and in the limit case as $M \to \infty$, we get that

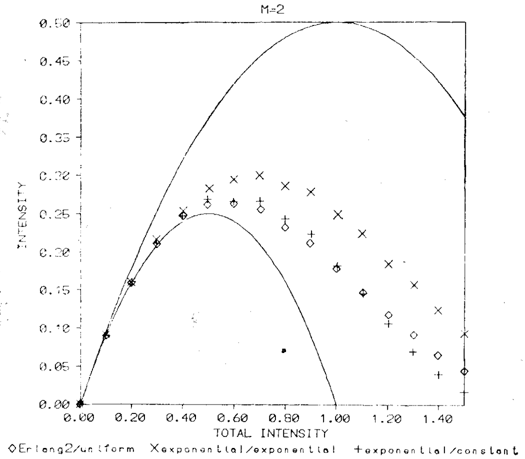
$$(\sqrt{5} - 2)\, \mathrm{e}^{(\sqrt{5}-3)/2} < S_{\max} < 1/\mathrm{e} \tag{23}$$

At this moment we recall the well-known Abramson's [3] result on the maximum throughput in the case of infinite population of users and constant length packets that is equal to $1/(2\,\mathrm{e})$ and, of course, satisfies (23).

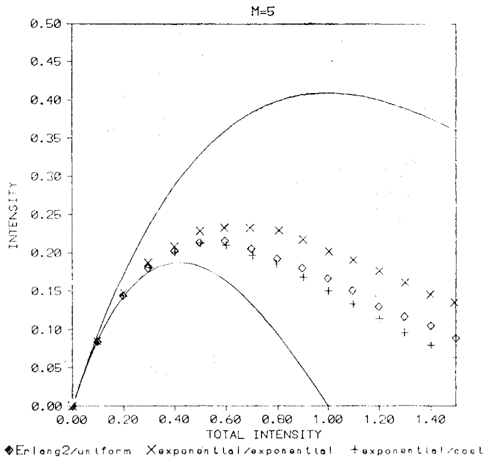
5. NUMERICAL EXAMPLES

In the following we shall present numerical examples in order to demonstrate the behaviour of the derived bounds.

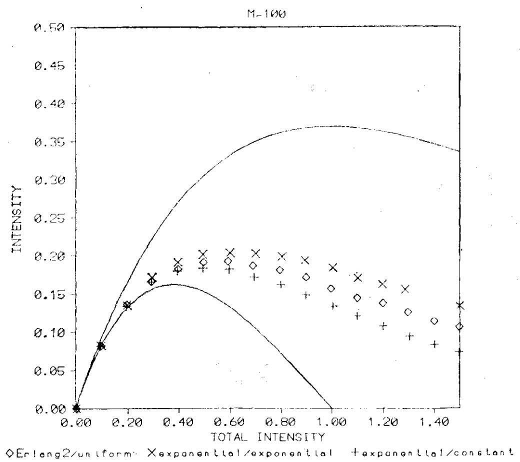
To begin with, we compare the bounds on the intensity of successfully transmitted packets given by (9) and (10) for a homogeneous population of users with the simulation results. This is displayed in Figs. 2a, b, and c for different number of users, namely 2, 5, and 100, respectively. The simulation results have been gathered for different types of idle periods/packet transmission times distributions, namely Erlang2/uniform, exponential/exponential, and exponential/constant ones. The parameters of packet transmission time distributions have been chosen in such a way that the average transmission time equals one ($\tau = 1$) in all cases. We observe that all results obtained via simulation lie between the bounds. In the limit cases when $G = Mg = 0$ or ∞ both bounds are equal to zero, thus for very light and heavy loads they are very close to real values of the throughput. Furthermore, one may note that for the load within the range $[0, G_1]$, where G_1 maximizes the lower bound and is given by (20), in the case of $\tau = 1$, the gap between the bounds is little. Therefore, we may approximate the intensity of successfully transmitted packets by $\lambda_+ \approx (\lambda_L + \lambda_U)/2$ in the considered range.



a)



b)

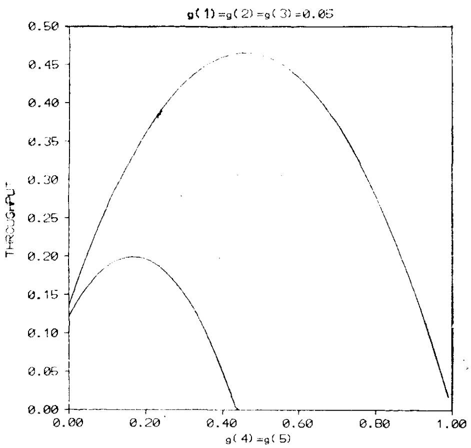


c)

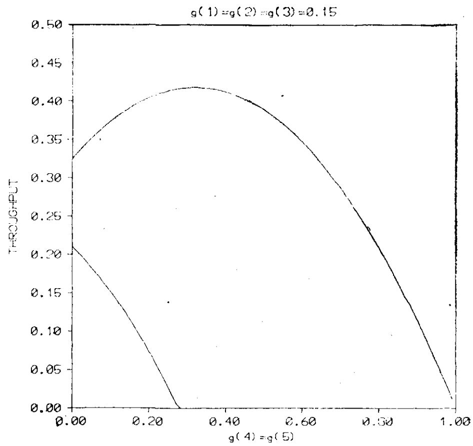
Fig. 2. The bounds on the intensity of successfully transmitted packets vs. simulation results for a homogeneous population of users, different number of users (M), and types of idle periods/packet transmission times distributions.

We note that out of this range λ_+ is considerably sensitive to the type of idle period and packet transmission time distributions.

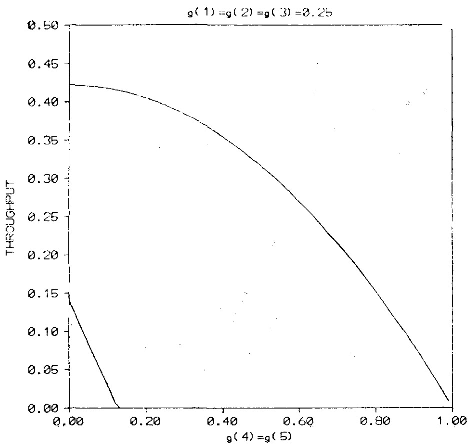
Let us turn now to the case of a heterogeneous population of users. Figures 3a, b, c, d and e show the bounds for five user population ($M = 5$) in the case of fixed traffic generated by three users ($g_1 = g_2 = g_3$) and variable traffic from two users (but $g_4 = g_5$). We may easy recognize that the behaviour of the bounds is very similar to that presented in Figure 2. The upper bound is valid for any type of idle period and packet transmission time distributions. The greater than zero part of the lower bound is valid only for identical and constant packet transmission times of all users. In the case when $\tau = 1$ we have that $\lambda_L \equiv S_L$ and $\lambda_U \equiv S_U$. When the total traffic offered to the channel is sufficiently large then the lower bound is equal to zero what can be observed in Figures 3d and e.



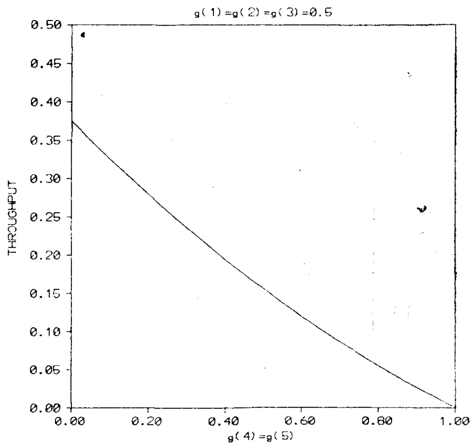
a)



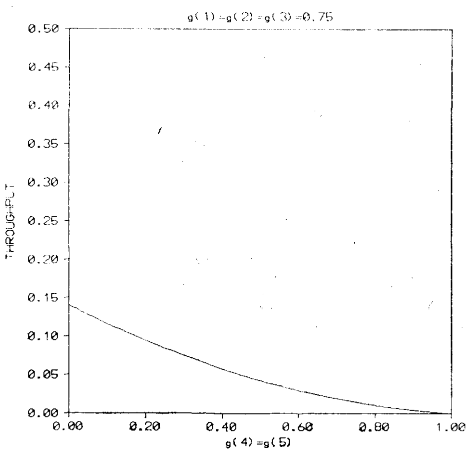
b)



c)



d)



e)

Fig 3. The bounds on the throughput for a heterogeneous population of users ($M = 5$) in the case of fixed $g_1 = g_2 = g_3$ and variable $g_4 = g_5$.

6. CONCLUSIONS

The lower and upper bounds on the intensity of successfully transmitted packets and throughput of an unslotted-ALOHA-type channel were given in the case of arbitrary packet lengths and "think" time distributions. According to the best authors' knowledge they are the first such presented in the open literature. These bounds were obtained under the only assumptions of stationarity and mutual independence of traffic streams generated by the users. The presented bounds are rather applicable to ALOHA systems (i.e. with feedback and retransmissions of erroneous packets) if users' buffers are emptied according to a random selection discipline or to the case of packets identical in length. It is to be mentioned that the assumed model of an ALOHA channel is appropriate to the system with an entry controlling discipline and non-empty users' buffers.

It is worth to note that the presented upper bounds are identical with the throughput in the case of open multiaccess channel where these quantities are measured in terms of successfully transfered bits (even if they are contained in collided packets) per time unit.

The derived bounds are very easy to obtain because they only depend on the average packet transmission times and intensities of packet streams transmitted by users. These parameters are in most cases easy to predict or estimate.

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