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GENERATING DIRICHLET RANDOM VECTORS USING A REJECTION PROPERTY

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In this paper a generalization of Jöhnk and Loukas results reported in [4] and [5] is given and a rejection algorithm for computer generation of Dirichlet random vectors is proposed. Comparisons with other suitable algorithms are also discussed.

1. INTRODUCTION

Let a_j, $1 \leqq j \leqq n + 1$, be $n + 1$ positive real numbers.

The random vector $(X_1, X_2, \ldots, X_n)$ has a Dirichlet $D(n; a_1, a_2, \ldots, a_n, a_{n+1})$ distribution if its probability density function (p.d.f.) is given by (cf. Wilks [14]):

(1) $$f(x_1, x_2, \ldots, x_n) = K x_1^{a_1 - 1} x_2^{a_2 - 1} \ldots x_n^{a_n - 1}(1 - x_1 - x_2 - \ldots - x_n)^{a_{n+1} - 1}$$

for every $(x_1, x_2, \ldots, x_n) \in S_n$, where

$$S_n = \{(x_1, x_2, \ldots, x_n) \mid x_i > 0, 1 \leqq i \leqq n; x_1 + x_2 + \ldots + x_n < 1\}$$

$$K = \Gamma(a_1 + a_2 + \ldots + a_n + a_{n+1})/(\Gamma(a_1)\,\Gamma(a_2) \ldots \Gamma(a_n)\,\Gamma(a_{n+1}))$$

with

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, \mathrm{d}t \,; \quad a > 0 \,.$$

To the best of our knowledge the computer generation of the Dirichlet distribution has not been enough studied in the literature (cf. [3], [6]–[10]).

In this context we can mention a lot of procedures:

- The gamma method based on the relation between Dirichlet and gamma distributions (Wilks [14], p. 179);
- Different rejection techniques (Ştefănescu [12]) or the use of the classical inverse method (considering the marginal distributions, Ştefănescu [12]);
- A one-to-one transform between S_n and $(0,1)^n$ (Ştefănescu [11]);
- The use of a transformation of a uniformly random vector over a bounded domain $D \subset \mathbb{R}^{n+1}$ (Văduva [13]).

The performance of these algorithms were compared in [12]. We can conclude that the gamma method is the fastest; a similar result (for a particular $n, n = 2$) was also obtained by Loukas [5].

In what follows, using a rejection property suggested by Loukas [5], we shall give a new algorithm for generating Dirichlet $D(n; a_1, a_2, \ldots, a_n, a_{n+1})$ random vectors.

2. THE MAIN RESULT

Theorem 1. Let $U_1, U_2, \ldots, U_n, U_{n+1}$ be $n + 1$ independent random variables uniformly distributed over the interval $(0, 1)$ and

$$X_i = U_i^{1/a_i} / (U_1^{1/a_1} + U_2^{1/a_2} + \ldots + U_n^{1/a_n} + U_{n+1}^{1/a_{n+1}}) \tag{2}$$

$1 \leqq i \leqq n$. Then the random vector $(X_1, X_2, \ldots, X_n)$ conditioned on

$$U_1^{1/a_1} + U_2^{1/a_2} + \ldots + U_n^{1/a_n} + U_{n+1}^{1/a_{n+1}} < 1 \tag{3}$$

has a Dirichlet distribution $D(n; a_1, a_2, \ldots, a_n, a_{n+1})$.

Proof. Let introduce a one-to-one transformation $T: (0, 1)^{n+1} \to D$, $D \subset \mathbb{R}^{n+1}$, $T(y_1, \ldots, y_n, y_{n+1}) = (x_1, \ldots, x_n, x_{n+1})$ defined by the relations

$$T: \begin{cases} x_i = y_i/(y_1 + y_2 + \ldots + y_n + y_{n+1}) \, ; & 1 \leqq i \leqq n \\ x_{n+1} = y_1 + y_2 + \ldots + y_n \end{cases} \tag{4}$$

The inverse T^{-1} of the transform T, $T^{-1}: D \to (0, 1)^{n+1}$, $T^{-1}(x_1, x_2, \ldots$ $\ldots, x_n, x_{n+1}) = (y_1, y_2, \ldots, y_n, y_{n+1})$ is given by

$$T^{-1}: \begin{cases} y_i = x_{n+1} x_i/(x_1 + x_2 + \ldots + x_n) \, ; & 1 \leqq i \leqq n \\ y_{n+1} = x_{n+1}(1 - x_1 - x_2 - \ldots - x_n)/(x_1 + x_2 + \ldots + x_n) \end{cases} \tag{5}$$

Let J be the Jacobian of T^{-1},

$$J = D(y_1, y_2, \ldots, y_n, y_{n+1})/D(x_1, x_2, \ldots, x_n, x_{n+1}) = \det(A) \tag{6}$$

where $\det(A)$ is the determinant of the matrix $A = (a_{ij})_{1 \leqq i,j \leqq n+1}$, with $a_{ij} = \partial y_i/\partial x_j$, $1 \leqq i, j \leqq n + 1$.

Adding to the row 1 of the matrix A all remaining rows and using afterwards the last relation (4) we get

$$J = (-1)^{n+2} \, . \det(B) \tag{7}$$

where $B = (b'_{ij})_{1 \leqq i,j \leqq n}$, with $b_{ij} = \partial y_{i+1}/\partial x_j$, $1 \leqq i, j \leqq n$.

From (5) we obtain the derivatives $\partial y_{i+1}/\partial x_j$, $1 \leqq i, j \leqq n$, and therefore

$$\det(B) = (-x_{n+1}/(x_1 + x_2 + \ldots + x_n)^2)^n \det(C) \tag{8}$$

where $C = (c_{ij})_{1 \leqq i,j \leqq n}$, with

$$c_{nj} = 1 \, ; \quad 1 \leqq j \leqq n$$
$$c_{ij} = x_{i+1} \, ; \quad 1 \leqq i \leqq n - 1 \, , \quad 1 \leqq j \leqq n \, , \quad j \neq i + 1$$
$$c_{i,i+1} = x_{i+1} - (x_1 + x_2 + \ldots + x_n) \, ; \quad 1 \leqq i \leqq n - 1$$

Subtracting column 1 of the matrix C from the other columns of C and then moving the last row on the first place we obtain a lower triangular matrix; hence

$$\det(C) = (x_1 + x_2 + \ldots + x_n)^{n-1} \tag{9}$$

From (6)–(9) we conclude that

$$J = x_{n+1}^n/(x_1 + x_2 + \ldots + x_n)^{n+1} \tag{10}$$

Let $U_1, U_2, \ldots, U_n, U_{n+1}$ be $n + 1$ independent random variables uniformly distributed over the interval $(0, 1)$. If $Y_i = U_i^{1/a_i}$, $1 \leqq i \leqq n + 1$, then the p.d.f. g_1 of the random vector $(Y_1, Y_2, \ldots, Y_n, Y_{n+1})$ takes the form

$$g_1(y_1, y_2, \ldots, y_n, y_{n+1}) = \prod_{j=1}^{n+1} a_j y_j^{a_j - 1}\,; \quad 0 < y_j < 1, \ 1 \leqq j \leqq n + 1 \tag{11}$$

Denoting

$$D_1 = \{(y_1, \ldots, y_n, y_{n+1}) \mid 0 < y_j < 1, 1 \leqq j \leqq n + 1, y_1 + \ldots + y_{n+1} < 1\} \tag{12}$$

and using the following identity $(a > 0, b > 0, c > 0)$

$$\textstyle\int_0^a bt^{b-1}(a - t)^c \, dt = a^{b+c}\,\Gamma(b + 1)\,\Gamma(c + 1)/\Gamma(b + c + 1)$$

we obtain the value K_1, that is

$$\begin{aligned} K_1 &= \textstyle\int_{D_1} g_1(y_1, y_2, \ldots, y_n, y_{n+1})\, dy_1\, dy_2 \ldots dy_n\, dy_{n+1} = \\ &= \textstyle\int_0^1 dy_1 \int_0^{1-y_1} dy_2 \ldots \int_0^{1-y_1-\ldots-y_n} g_1(y_1, y_2, \ldots, y_n, y_{n+1})\, dy_{n+1} = \\ &= \Big(\prod_{j=1}^{n+1} \Gamma(a_j + 1)\Big)/\Gamma(a_1 + a_2 + \ldots + a_n + a_{n+1} + 1) \end{aligned} \tag{13}$$

Therefore the p.d.f. g_2 of the random vector $(Y_1, Y_2, \ldots, Y_n, Y_{n+1})$ conditioned on the inequality

$$Y_1 + Y_2 + \ldots + Y_n + Y_{n+1} < 1 \tag{14}$$

has the form

$$g_2(y_1, y_2, \ldots, y_n, y_{n+1}) = g_1(y_1, y_2, \ldots, y_n, y_{n+1})/K_1 \tag{15}$$

for every $(y_1, y_2, \ldots, y_n, y_{n+1}) \in D_1$.

Let $(X_1, X_2, \ldots, X_n, X_{n+1})$ be the random vector obtained from the vector $(Y_1, Y_2, \ldots, Y_n, Y_{n+1})$ by applying the transform T (formula (4)). Then the p.d.f. f_2 of $(X_1, X_2, \ldots, X_n)$ is given by

$$f_2(x_1, x_2, \ldots, x_n, x_{n+1}) = g_2(y_1, y_2, \ldots, y_n, y_{n+1})\,|J| \tag{16}$$

From the relations (5), (10), (11), (13), (15), (16) and using the equality $\Gamma(a + 1) = a\,\Gamma(a)$ we finally obtain

$$f_2(x_1, \ldots, x_n, x_{n+1}) = f(x_1, x_2, \ldots, x_n)\, f_3(x_1, x_2, \ldots, x_n, x_{n+1}) \tag{17}$$

where the function f is given by (1) and f_3 takes the form

$$f_3(x_1, \ldots, x_{n+1}) = (a_1 + \ldots + a_{n+1})\, x_{n+1}^{a_1 + \ldots + a_{n+1} - 1}/(x_1 + \ldots + x_n)^{a_1 + \ldots + a_{n+1}} \tag{18}$$

From (5) we get

$$y_1 + y_2 + \ldots + y_n + y_{n+1} = x_{n+1}/(x_1 + x_2 + \ldots + x_n)$$

and then the inequality (3) is equivalent to $y_1 + \ldots + y_n + y_{n+1} < 1$, that is

$$(19) \qquad 0 < x_{n+1} < x_1 + x_2 + \ldots + x_n$$

Hence the p.d.f. of $(X_1, X_2, \ldots, X_n)$ is given by

$$(20) \qquad f_1(x_1, x_2, \ldots, x_n) = \int_0^{x_1+x_2+\ldots+x_n} f_2(x_1, x_2, \ldots, x_n, x_{n+1})\, dx_{n+1} =$$
$$= f(x_1, x_2, \ldots, x_n) \int_0^{x_1+x_2+\ldots+x_n} f_3(x_1, x_2, \ldots, x_n, x_{n+1})\, dx_{n+1} =$$
$$= f(x_1, x_2, \ldots, x_n)$$

which was required to prove. □

Remark 1. The Beta(a, b) distribution is a unidimensional Dirichlet distribution (Beta$(a, b) \equiv D(1; a, b)$),.

Remark 2. If U is uniformly distributed over $(0, 1)$ then $Y = U^{1/a}$ has a Beta$(a, 1)$ distribution ($Y \sim$ Beta$(a, 1)$).

Considering $n = 1$, respectively $n = 2$, from Theorem 1 it results

Corollary 1 (Jöhnk [4]). If $Y_i \sim$ Beta$(a_i, 1)$, $i = 1, 2$, are independent random variables so that $Y_1 + Y_2 < 1$ then $X = Y_1/(Y_1 + Y_2) \sim$ Beta(a, b).

Corollary 2 (Loukas [5]). If $Y_1 \sim$ Beta$(a_1, 1)$, $Y_2 \sim$ Beta$(a_2, 1)$, $Y_3 \sim$ Beta$(a_3, 1)$ are independent random variables and $Y_1 + Y_2 + Y_3 < 1$ then $(X_1, X_2) = (Y_1/(Y_1 + Y_2 + Y_3), Y_2/(Y_1 + Y_2 + Y_3)) \sim D(2; a_1, a_2, a_3)$.

3. THE REJECTION PROCEDURE

From Theorem 1 we get the following algorithm for generating a random vector $(X_1, X_2, \ldots, X_n)$ having a Dirichlet $D(n; a_1, a_2, \ldots, a_n, a_{n+1})$ distribution.

Algorithm AGDR (Algorithm for **G**enerating a **D**irichlet distribution using a **R**ejection property).

STEP 0. Inputs: $a_1, a_2, \ldots, a_n, a_{n+1}, n \quad (a_j > 0, 1 \leqq j \leqq n + 1)$.

STEP 1. Generate $n + 1$ independent random variables $U_1, U_2, \ldots, U_{n+1}$ uniformly distributed over the interval $(0, 1)$

$$X_j \leftarrow U_j^{1/a_j}, \quad 1 \leqq j \leqq n.$$

STEP 2. $S \leftarrow X_1 + X_2 + \ldots + X_{n-1} + X_n + U_{n+1}^{1/a_{n+1}}$.

STEP 3. If $S \geqq 1$ then go to Step 1.

STEP 4. $X_j \leftarrow X_j/S$, $1 \leqq j \leqq n$.

STEP 5. Print $(X_1, X_2, \ldots, X_{n-1}, X_n)$. STOP.

The "acceptance probability" $\mathsf{P}_{ac}(n; a_1, a_2, \ldots, a_n, a_{n+1})$ (Devroye [3]) in Step 3 of the algorithm AGDR is given by

$$(21)\quad \mathsf{P}_{ac}(n; a_1, \ldots, a_n, a_{n+1}) = \mathsf{P}(U_1^{1/a_1} + U_2^{1/a_2} + \ldots + U_n^{1/a_n} + U_{n+1}^{1/a_{n+1}} < 1) = \\ = \int_{D_2} \mathrm{d}u_1\, \mathrm{d}u_2 \ldots \mathrm{d}u_n\, \mathrm{d}u_{n+1}$$

where

$$(22)\quad D_2 = \{(u_1, \ldots, u_{n+1}) \mid 0 < u_j < 1,\ 1 \leqq j \leqq n + 1;\ u_1^{1/a_1} + \ldots + u_{n+1}^{1/a_{n+1}} < 1\}.$$

Considering the new variables $y_j = u_j^{1/a_j}$, $1 \leqq j \leqq n + 1$, we have

$$(23)\quad \mathsf{P}_{ac}(n; a_1, \ldots, a_{n+1}) = \int_{D_1} a_1 a_2 \ldots a_{n+1}\, y_1^{a_1 - 1} \ldots y_{n+1}^{a_{n+1}-1}\, \mathrm{d}y_1 \ldots \mathrm{d}y_{n+1}$$

where the domain D_1 is defined by (12). Using (11) and (13) we find

$$(24)\quad \mathsf{P}_{ac}(n; a_1, \ldots, a_{n+1}) = K_1 = \Gamma(1 + a_1) \ldots \Gamma(1 + a_n)\, \Gamma(1 + a_{n+1}) \,/ \\ / \,\Gamma(1 + a_1 + \ldots + a_n + a_{n+1}).$$

4. PERFORMANCES

Proposition 1. If $0 < a_j \leqq b_j$, $1 \leqq j \leqq n + 1$, then

$$(25)\quad \mathsf{P}_{ac}(n; a_1, a_2, \ldots, a_n, a_{n+1}) \geqq \mathsf{P}_{ac}(n; b_1, b_2, \ldots, b_n, b_{n+1})$$

Proof. If $0 < a \leqq b$, $c > 0$, $0 < t < 1$ then $t^{a-1} \geqq t^{b-1}$ and hence

$$(26)\quad \int_0^1 t^{a-1}(1 - t)^{c-1}\, \mathrm{d}t \geqq \int_0^1 t^{b-1}(1 - t)^{c-1}\, \mathrm{d}t$$

From (26) it results

$$(27)\quad \Gamma(a)/\Gamma(a + c) \geqq \Gamma(b)/\Gamma(b + c)\,, \quad 0 < a \leqq b\,, \quad c > 0$$

Applying the inequality (27) it obtains successively

$$(28)\quad \mathsf{P}_{ac}(n; a_1, a_2, \ldots, a_n, a_{n+1}) \geqq \mathsf{P}_{ac}(n; b_1, a_2, \ldots, a_n, a_{n+1}) \geqq \ldots \\ \ldots \geqq \mathsf{P}_{ac}(n; b_1, b_2, \ldots, b_n, a_{n+1}) \geqq \mathsf{P}_{ac}(n; b_1, b_2, \ldots, b_n, b_{n+1})$$

the proposition being proved. □

Tables 1 and 4 contain the values of the "acceptance probability" $\mathsf{P}_{ac}(n; a, a \ldots, a)$ obtained by using formula (24) (Table 1) or applying a Monte Carlo procedure (Table 4; 10 000 simulation steps); it considers $a \in \{1{\cdot}0; 0{\cdot}5; 0{\cdot}25; 0{\cdot}2; 0{\cdot}125; 0{\cdot}1; 0{\cdot}05\}$, $n = 1, 2$.

The values from Table 1 and Table 2 are very close.

Table 1. The "acceptance probability" $\mathsf{P}_{ac}(n; a, a, \ldots, a, a)$ given by (24).

a	$a = 1{\cdot}0$	$a = 0{\cdot}5$	$a = 0{\cdot}25$	$a = 0{\cdot}2$	$a = 0{\cdot}125$	$a = 0{\cdot}1$	$a = 0{\cdot}05$
$n = 1$	0·5000	0·7854	0·9270	0·9502	0·9785	0·9857	0·9962
$n = 2$	0·1667	0·5236	0·8103	0·8663	0·9396	—	—

Remark 3. Table 2 and Table 3 give the "acceptance probability" $\mathsf{P}_{ac}(n; a, \ldots, a)$ (formula (24)) for different values of n. Using Proposition 1 and Tables 2, 3 it can be found a good approximation of the "acceptance probability" $\mathsf{P}_{ac}(n; a_1, a_2, \ldots \ldots, a_n, a_{n+1})$ when we haven't $a_1 = a_2 = \ldots = a_n = a_{n+1}$.

Table 2. The "acceptance probability" $\mathsf{P}_{ac}(n; a, a, \ldots, a, a)$ obtained from (24).

a	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
1·0	0·5000	0·1667	0·0417	0·0083	0·0014	0·0002	0·0000
0·9	0·5517	0·2133	0·0639	0·0157	0·0033	0·0006	0·0001
0·8	0·6068	0·2710	0·0970	0·0292	0·0076	0·0018	0·0004
0·7	0·6647	0·3414	0·1452	0·0532	0·0173	0·0050	0·0013
0·6	0·7246	0·4255	0·2138	0·0949	0·0380	0·0140	0·0047
0·5	0·7854	0·5236	0·3084	0·1645	0·0807	0·0369	0·0159
0·4	0·8452	0·6339	0·4335	0·2749	0·1637	0·0922	0·0495
0·3	0·9014	0·7516	0·5888	0·4380	0·3117	0·2134	0·1412
0·2	0·9502	0·8663	0·7631	0·6525	0·5438	0·4429	0·3533
0·1	0·9857	0·9594	0·9232	0·8793	0·8297	0·7762	0·7204

Table 3. The "acceptance probability" $\mathsf{P}_{ac}(n; a, a, \ldots, a, a)$ obtained from (24) for large n and small values of a.

n	$a = 0·1$	$a = 0·05$	$a = 0·01$	$a = 0·005$	$a = 0·001$
10	0·5521	0·8373	0·9915	0·9978	0·9999
20	0·1597	0·5566	0·9692	0·9918	0·9997
30	0·0313	0·3157	0·9356	0·9824	0·9992
40	0·0046	0·1587	0·8931	0·9698	0·9987
50	0·0006	0·0724	0·8438	0·9543	0·9979
60	0·0000	0·0304	0·7900	0·9363	0·9971
70	0·0000	0·0119	0·7332	0·9161	0·9960
80	0·0000	0·0044	0·6753	0·8939	0·9949
90	0·0000	0·0015	0·6173	0·8700	0·9936
100	0·0000	0·0005	0·5605	0·8447	0·9921

Remark 4. From formula (24) it follows

$$(29)\qquad \begin{aligned} &\lim_{a_1 \to 0} \lim_{a_2 \to 0} \ldots \lim_{a_{n+1} \to 0} \mathsf{P}_{ac}(n; a_1, a_2, \ldots, a_n, a_{n+1}) = 1 \\ &\lim_{a_1 \to \infty} \lim_{a_2 \to \infty} \ldots \lim_{a_{n+1} \to \infty} \mathsf{P}_{ac}(n; a_1, a_2, \ldots, a_n, a_{n+1}) = 0 \end{aligned}$$

which proves that the AGDR algorithm is very fast for small values of the parameters $a_1, a_2, \ldots, a_n, a_{n+1}$ (see also Table 3).

We will denote by AGDW (Wilks' algorithm) the procedure for generating Dirichlet $D(n; a_1, a_2, \dots, a_n, a_{n+1})$ random vectors based on the following proposition

Proposition 2 (Wilks [14], p. 179). If Y_j, $1 \leqq j \leqq n+1$, are $n+1$ independent random variables having a gamma distribution with parameter a_j, and

$$X_i = Y_i/(Y_1 + Y_2 + \dots + Y_n + Y_{n+1}); \quad 1 \leqq i \leqq n \tag{30}$$

then the random vector $(X_1, X_2, \dots, X_n)$ has a Dirichlet distribution.

Table 4. The values of "acceptance probability" $\mathsf{P}_{ac}(n; a, \dots, a)$ obtained using a Monte Carlo procedure (100 000 simulations).

a	$a = 1{\cdot}0$	$a = 0{\cdot}5$	$a = 0{\cdot}25$	$a = 0{\cdot}2$	$a = 0{\cdot}125$	$a = 0{\cdot}1$	$a = 0{\cdot}05$
$n = 1$	0·4978	0·7824	0·9248	0·9509	0·9804	0·9879	0·9968
$n = 2$	0·1736	0·5211	0·8086	0·8782	0·9421	—	—

Analysing a lot of procedures for generating Dirichlet deviates we concluded that the algorithm AGDW is the fastest (cf. Ştefănescu [12]; for $n = 2$ Loukas [5] obtained the same result).

Remark 5. The rejection procedure AGDR is based on simple operations; in addition its "acceptance probability" is very large for small values of a_j, $1 \leqq j \leqq n+1$ (see Remark 4).

Table 5. The values of the threshold q for which AGDR algorithm is the fastest (case $a_1 = a_2 = \dots = a_n = a$).

a	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
1·0	1·26	0·02	—	—	—	—	—	—	—	—
0·9	1·40	0·11	—	—	—	—	—	—	—	—
0·8	1·60	0·25	—	—	—	—	—	—	—	—
0·7	2·05	0·38	—	—	—	—	—	—	—	—
0·6	2·37	0·55	—	—	—	—	—	—	—	—
0·5	2·80	0·80	0·08	—	—	—	—	—	—	—
0·4	4·29	1·00	0·34	—	—	—	—	—	—	—
0·3	6·75	1·66	0·66	0·20	—	—	—	—	—	—
0·2	13·43	3·01	1·32	0·75	0·25	0·05	—	—	—	—
0·1	100·00	10·2	4·39	2·07	1·34	0·96	0·66	0·44	0·24	0·11

From Proposition 1 it follows that for $0 < q < a_{n+1}$ the run time of the algorithm AGDR$(n; a_1, a_2, \dots, a_{n+1})$ is less than for the AGDW$(n; a_1, a_2, \dots, a_n, q)$ algorithm. Therefore, for fixed values of $n, a_1, a_2, \dots, a_n$ it can be established a threshold q, $q > 0$, so that for any $0 < a_{n+1} < q$, the algorithm AGDR$(n; a_1, a_2, \dots, a_n, a_{n+1})$ is faster then the AGDW$(n; a_1, a_2, \dots, a_n, a_{n+1})$ algorithm. The value of q depends on the computer, on the specific implementations of these algorithms or on the particular procedure for generating gamma random variates (see Proposition 2).

From the literature [3], [6]–[10] it can be selected the following algorithms for generating random values having a gamma distribution with parameter a, $a < 1$:

- the GT and GBH Cheng-Feast's procedures [2] (in the case of $a > 0{\cdot}5$, respectively $a > 0{\cdot}25$);
- the GS Ahrens-Dieter's procedure [1].

The GS algorithm is the fastest when it generates only one random value.

Table 6. The value of q so that AGDR (2; a_1, a_2, a_3) algorithm is the fastest ($0 < a_3 < q$).

$a_2 =$	1·0	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
$a_1 = 1{\cdot}0$	0·02									
$a_1 = 0{\cdot}9$	0·06	0·11								
$a_1 = 0{\cdot}8$	0·12	0·18	0·25							
$a_1 = 0{\cdot}7$	0·17	0·24	0·32	0·38						
$a_1 = 0{\cdot}6$	0·23	0·30	0·39	0·47	0·55					
$a_1 = 0{\cdot}5$	0·34	0·42	0·49	0·59	0·65	0·80				
$a_1 = 0{\cdot}4$	0·41	0·50	0·59	0·68	0·78	0·90	1·00			
$a_1 = 0{\cdot}3$	0·53	0·62	0·72	0·79	0·94	1·06	1·22	1·66		
$a_1 = 0{\cdot}2$	0·65	0·78	0·84	0·99	1·11	1·29	1·64	2·29	3·01	
$a_1 = 0{\cdot}1$	0·77	0·96	1·12	1·18	1·46	1·73	2·14	2·96	5·08	10·20

Tables 5 and 6 give the q threshold values for different n, a_1, a_2, ..., a_n considering a FORTRAN implementation of the AGDR and AGDW algorithms run on a Romanian FELIX C-256 computer and using the GS Ahrens-Dieter's procedure [1] for gamma random variates.

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