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STATIONARY DISTRIBUTION OF SOME NONLINEAR AR(1) PROCESSES

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Let e_t be a sequence of independent identically distributed random variables such that $\mathsf{P}(e_t = 0) = p$, $\mathsf{P}(e_t = c) = 1 - p$, where $c > 0$ and $p \in (0, 1)$ are given numbers. Let F be a stationary distribution function of the nonlinear AR(1) process $X_t = aX_{t-1}^{1/h} + e_t$, where $a > 0$, $h > 1$. A method for calculating F and its moments is introduced in the paper. The results are demonstrated on some numerical examples.

1. INTRODUCTION

Let e_t be a strict white noise, i.e. a sequence of independent random variables with the same distribution function $H(x) = \mathsf{P}(e_t < x)$. Let X_0 be a random variable independent of $\{e_1, e_2, \ldots\}$. Define

$$(1.1) \qquad X_t = a\lambda(X_{t-1}) + e_t \quad (t = 1, 2, \ldots)$$

where λ is a given function and a is a parameter. The process X_t given by (1.1) is called the nonlinear AR(1) process. Jones [2], [3] and [4] investigated conditions for stationarity, proposed some numerical methods for finding a stationary distribution and dealt with the problem of estimating the parameter a. Recently, Loges [5] proved that under very general conditions the least squares estimator of a is strictly consistent. This result was derived without any assumption about stationarity of X_t.

Assume that $e_t \geqq 0$ and $a > 0$. Let λ be a nonnegative increasing function on $[0, \infty)$.

Theorem 1.1. A distribution function F corresponds to a stationary distribution of the process X_t if and only if it satisfies the equation

$$(1.2) \qquad F(x) = \int_0^x F\left[\lambda^{-1}\left(\frac{x - t}{a}\right)\right] \mathrm{d}H(t) \qquad (x \geqq 0)\,.$$

Proof. If X_{t-1} has a distribution function F, then the distribution function of

$Y = a\lambda(X_{t-1})$ is

$$G(y) = F\left[\lambda^{-1}\left(\frac{y}{a}\right)\right].$$

Using (1.1) we can see that the distribution function F^* of X_t is

$$F^*(x) = \int_0^x G(x - t)\,\mathrm{d}H(t)\,.$$

The distribution function F is stationary if and only if $F^* = F$. From here we get (1.2). □

Generally, it is very difficult to solve the equation (1.2) when H is given. Anděl [1] suggested a method for calculating F in the special case when

$$X_t = X_{t-1}^{1/2} + e_t\,, \quad \mathsf{P}(e_t = 0) = \mathsf{P}(e_t = 1) = \tfrac{1}{2}\,.$$

In our paper we generalize this procedure to the model

$$X_t = aX_{t-1}^{1/h} + e_t\,, \quad \mathsf{P}(e_t = 0) = p\,, \quad \mathsf{P}(e_t = c) = 1 - p\,, \tag{1.3}$$

where $a > 0$, $h > 1$, $p \in (0, 1)$, $c > 0$. If the model (1.3) is valid, then we have from (1.2) that

$$F(x) = pF\left[\left(\frac{x}{a}\right)^h\right] \quad \text{for} \quad x \in (0, c]\,, \tag{1.4}$$

$$F(x) = pF\left[\left(\frac{x}{a}\right)^h\right] + (1 - p)\,F\left[\left(\frac{x - c}{a}\right)^h\right] \quad \text{for} \quad x > c\,. \tag{1.5}$$

The results are slightly different for $a \geqq 1$ and for $a \in (0, 1)$.

2. CASE $a \geqq 1$

In this section we assume that $a \geqq 1$. Define $m = \min(a, c)$. Let z be the largest real root of the equation (6.1).

Theorem 2.1. If $x \leqq m$, then $F(x) = 0$.

Proof. Assume that there exists $x \in (0, m)$ such that $F(x) > 0$. Then (1.4) gives

$$F(x) = pF\left[\left(\frac{x}{a}\right)^h\right] < F\left[\left(\frac{x}{a}\right)^h\right]. \tag{2.1}$$

However from $0 < x/a \leqq x/m < 1$ we obtain $(x/a)^h < x/a \leqq x$ and thus $F[(x/a)^h] \leqq$ $\leqq F(x)$. This is a contradiction to (2.1). Therefore, $F(x) = 0$ for $x < m$. Since F is left-continuous, we get also $F(m) = 0$. □

Theorem 2.2. If $z \geqq c + ac^{1/h}$, then

$$F(x) = pF\left[\left(\frac{x}{a}\right)^h\right] \quad \text{for} \quad x \in (m, c + ac^{1/h}]\,, \tag{2.2}$$

(2.3) $$F(x) = pF\left[\left(\frac{x}{a}\right)^h\right] + (1 - p)\,F\left[\left(\frac{x - c}{a}\right)^h\right] \quad \text{for} \quad x \in (c + ac^{1/h}, z]\,,$$

(2.4) $$F(x) = 1 \quad \text{for} \quad x > z\,.$$

Proof. Formula (2.2) follows from (1.4) and from Theorem 2.1. Formula (2.3) is a special case of (1.5). It remains to prove (2.4). Assume that there exists $y > z$ such that $F(y) < 1$. Define

$$w = \sup\{x\colon F(x) \leqq F(y)\}\,.$$

Since F is left-continuous, we have $F(w) = F(y)$. If $x > w$, then $F(x) > F(w)$. But $[(x - c)/a]^h > x$ for $x > z$ (see Remark 6.2) and thus also $(x/a)^h > x$ for $x > z$. From (6.2) we obtain $w > c$ and using (1.5) we derive

$$F(w) = pF\left[\left(\frac{w}{a}\right)^h\right] + (1 - p)\,F\left[\left(\frac{w - c}{a}\right)^h\right] < p\,F(w) + (1 - p)\,F(w) = F(w)\,.$$

Thus $F(x) = 1$ for $x > z$. □

Let us remark that $F(z) = 1$. Indeed, it follows from (2.3) if $x \to z^-$.

Theorem 2.3. Let $0 < \alpha < \beta$ be numbers such that

(2.5) $$a\beta^{1/h} \leqq c + ac^{1/h}\,.$$

If $F(x) = f$ for $x \in (\alpha, \beta]$, then

$$F(x) = pf \quad \text{for} \quad x \in J = (a\alpha^{1/h}, a\beta^{1/h}]\,.$$

Proof. It is clear that $(x/a)^h \in (\alpha, \beta]$ if and only if $x \in J$. In this case $x < a\beta^{1/h} \leqq c + ac^{1/h}$ and thus the assertion follows from (2.2) and (1.4). □

Theorem 2.4. Let $0 < \alpha < \beta$ be numbers such that

(2.6) $$a\alpha^{1/h} \geqq z - 2c\,.$$

If $F(x) = f$ for $x \in (\alpha, \beta]$, then

$$F(x) = p + (1 - p)f \quad \text{for} \quad x \in I = (c + a\alpha^{1/h}, c + a\beta^{1/h}]\,.$$

Proof. If $(x - c)/a \in (\alpha, \beta]$, i.e. if $x \in I$, then $F[((x - c)/a)^h] = f$. Using (1.5) we get

$$F(x) = pF[(x/a)^h] + (1 - p)f \quad \text{for} \quad x \in I\,.$$

If $x \in I$, then $x > c + a\alpha^{1/h}$. From (2.6) we obtain

$$\left(\frac{x}{a}\right)^h > \left(\frac{c + a\alpha^{1/h}}{a}\right)^h \geqq \left(\frac{c + z - 2c}{a}\right)^h = \left(\frac{z - c}{a}\right)^h = z\,,$$

so that $F[(x/a)^h] = 1$. □

Theorem 2.5. Let $\alpha > 0$. If

(2.7) $$a\alpha^{1/h} \geqq (2a^h)^{1/(h-1)} - c\,,$$

then (2.6) is fulfilled.

Proof. We have from (6.2) that

$$z - 2c < \max\left[0, (2a^h)^{1/(h-1)} - c\right].$$

If (2.7) holds, then

$$a\alpha^{1/h} \geqq \max\left[0, (2a^h)^{1/(h-1)} - c\right] > z - 2c. \qquad \square$$

Remark 2.6. In the formulation of Theorem 2.2 we used the assumption that $z \geqq c + ac^{1/h}$. Generally, it can happen that

$$z < c + ac^{1/h}. \tag{2.8}$$

Indeed, we can see from (6.2) that (2.8) holds if

$$\max\left[2c, c + (2a^h)^{1/(h-1)}\right] < c + ac^{1/h}. \tag{2.9}$$

Put $c = a$. Then (2.9) is equivalent to

$$\max\left[a, (2a^h)^{1/(h-1)}\right] < a^{(h+1)/h}$$

which is satisfied for any given $h > 1$, if a is sufficiently large.

3. CASE $a \in (0, 1)$

Theorem 3.1. If $x \geqq z$, then $F(x) = 1$.

Proof is the same as that of (2.4). $\square$

Theorem 3.2. Let $0 < \alpha < \beta$ be numbers such that $a\beta^{1/h} \leqq c$. If $F(x) = f$ for $x \in (\alpha, \beta]$, then

$$F(x) = pf \quad \text{for} \quad x \in J = (a\alpha^{1/h}, a\beta^{1/h}].$$

Proof is similar to that of Theorem 2.3. $\square$

Theorem 3.3. Let $0 < \alpha < \beta$ be numbers such that

$$a\alpha^{1/h} \geqq z - 2c. \tag{3.1}$$

If $F(x) = f$ for $x \in (\alpha, \beta]$, then

$$F(x) = p + (1 - p)f \quad \text{for} \quad x \in I = (c + a\alpha^{1/h}, c + a\beta^{1/h}].$$

The condition (3.1) is fulfilled if (2.7) holds.

Proof is the same as in the case of $a \geqq 1$. $\square$

4. AN EXAMPLE

The results of Sections 2 and 3 enable to calculate the stationary distribution function F. The method is demonstrated on the following example. Consider the model

$$X_t = 2X_{t-1}^{1/2} + e_t, \quad \mathsf{P}(e_t = 0) = 0{\cdot}6, \quad \mathsf{P}(e_t = 3) = 0{\cdot}4.$$

The equation (6.1)

$$\left(\frac{x-3}{2}\right)^2 = x$$

has the roots $x_1 = 1$, $x_2 = 9$. Thus $z = 9$. Theorem 2.1 gives $F(x) = 0$ for $x \leqq 2$ and (2.4) states that $F(x) = 1$ for $x > 9$. The inequality (2.5) holds for $\beta \in (0, \beta_0]$, where $\beta_0 = [(3 + 2.3^{1/2})/2]^2 = 10{\cdot}446152$. Since $F(x) = 1$ for $x \in (z, \beta_0]$, Theorem 2.3 yields

$$F(x) = 0{\cdot}6 \quad \text{for} \quad x \in J = (2z^{1/2}, 2\beta_0^{1/2}] = (6,\ 6{\cdot}4641016].$$

Denote $\alpha = 6$, $\beta = 6{\cdot}4641016$. It can be checked that the assumptions of Theorem 2.3 are satisfied and thus

$$F(x) = 0{\cdot}36 \quad \text{for} \quad x \in (2\alpha^{1/2}, 2\beta^{1/2}] = (4{\cdot}8989795,\ 5{\cdot}0849195]\,.$$

But (2.6) is also satisfied and so applying Theorem 2.4 one gets

$$F(x) = 0{\cdot}84 \quad \text{for} \quad x \in (3 + 2\alpha^{1/2}, 3 + 2\beta^{1/2}] = (7{\cdot}8989795,\ 8{\cdot}0849195]\,.$$

In the same way further values of $F(x)$ can be calculated.

5. A NUMERICAL STUDY

We have proved that under some conditions the distribution function F satisfies $F(x) = 0$ for $x \leqq u$ and $F(x) = 1$ for $x \geqq z$, where $u = \min(a, c)$ if $a \geqq 1$ and $u = 0$ if $a \in (0, 1)$. Moreover, as we have shown in Section 4, it is possible to calculate intervals $(c_1, d_1], \ldots, (c_n, d_n]$ and values $0 < f_1 < f_2 < \ldots < f_n < 1$ such that

$$u < c_1 < d_1 < c_2 < d_2 < \ldots < c_n < d_n < z$$

and that

$$F(x) = f_i \quad \text{for} \quad x \in (c_i, d_i]\,.$$

If $n \to \infty$, then $\max(f_i - f_{i-1}) \to 0$. It implies that the distribution function F can be calculated with any given accuracy. Nevertheless, our calculations must be restricted to a finite n. Let this number n be fixed. Define the distribution functions F_L and F_U in the following way. Let

$$F_U(x) = \begin{cases} 0 & \text{for} \quad x \leqq c_1\,, \\ f_i & \text{for} \quad x \in (c_i, c_{i+1}]\,, \quad i = 1, \ldots, n-1\,, \\ f_n & \text{for} \quad x \in (c_n, z]\,, \\ 1 & \text{for} \quad x > z\,, \end{cases}$$

$$F_L(x) = \begin{cases} 0 & \text{for} \quad x \leqq u\,, \\ f_1 & \text{for} \quad x \in (u, d_1]\,, \\ f_i & \text{for} \quad x \in (d_{i-1}, d_i]\,, \quad i = 2, \ldots, n\,, \\ 1 & \text{for} \quad x > d_n\,. \end{cases}$$

It is clear that

$$F_U(x) \leqq F(x) \leqq F_L(x) . \tag{5.1}$$

For $k = 1, 2, \ldots$ define

$$m_k = \int_0^\infty x^k \, dF(x) , \quad m_k^{(L)} = \int_0^\infty x^k \, dF_L(x) , \quad m_k^{(U)} = \int_0^\infty x^k \, dF_U(x) .$$

Then (5.1) yields

$$m_k^{(L)} \leqq m_k \leqq m_k^{(U)} , \quad k = 1, 2, \ldots$$

The moments $m_k^{(L)}$ and $m_k^{(U)}$ can be calculated using known formulas

$$m_k^{(L)} = k \int_0^\infty x^{k-1}[1 - F_L(x)] \, dx , \quad m_k^{(U)} = k \int_0^\infty x^{k-1}[1 - F_U(x)] \, dx ,$$

respectively. Since both F_L and F_U are step functions, the calculations of $m_k^{(L)}$ and $m_k^{(U)}$ are elementary. For the variance $\sigma^2 = m_2 - m_1^2$ we obtain the bounds

$$m_2^{(L)} - [m_1^{(U)}]^2 \leqq \sigma^2 \leqq m_2^{(U)} - [m_1^{(L)}]^2$$

and similar inequalities can easily be written down also for the central moments of higher order. Then we can derive also the bounds for skewness and curtosis.

Three models were investigated in detail.

$$\begin{aligned}
&\text{I.} \quad X_t = X_{t-1}^{1/2} + e_t , \quad \mathrm{P}(e_t = 0) = 0{\cdot}5 , \quad \mathrm{P}(e_t = 1) = 0{\cdot}5 ; \\
&\text{II.} \quad X_t = 2X_{t-1}^{1/2} + e_t , \quad \mathrm{P}(e_t = 0) = 0{\cdot}6 , \quad \mathrm{P}(e_t = 3) = 0{\cdot}4 ; \\
&\text{III.} \quad X_t = 5X_{t-1}^{1/4} + e_t , \quad \mathrm{P}(e_t = 0) = 0{\cdot}4 , \quad \mathrm{P}(e_t = 6) = 0{\cdot}6 .
\end{aligned}$$

Table 1. Lower bound (l.b.) and upper bound (u.b.) for statistical characteristics of stationary distributions.

Characteristic	Model I		Model II		Model III	
	l.b.	u.b.	l.b.	u.b.	l.b.	u.b.
expectation m_1	1·84199	1·84199	7·08899	7·08899	11·56259	11·56379
variance σ^2	0·29104	0·29105	2·54244	2·54262	8·964	9·011
skewness	−0·007	−0·007	−3·464	−3·450	0·341	0·432
curtosis	−1·506	−1·504	−1·361	−1·341	−2·226	−1·162

In each case $2^{16} = 65\,536$ values f_i were calculated. The corresponding distribution function F is given in Figure 1 for the model I, in Figure 2 for the model II and in Figure 3 for the model III. Numerical results are presented in Table 1. They can serve for comparison in the cases when some approximations for stationary distribution of a nonlinear AR(1) process are proposed (see [4], for example). It can be seen from Table 1 that $n = 2^{16}$ does not give sufficient accuracy for all characteristics. We had to restrict ourselves to this n because for larger n time needed for computations rapidly increased.

Fig. 1.

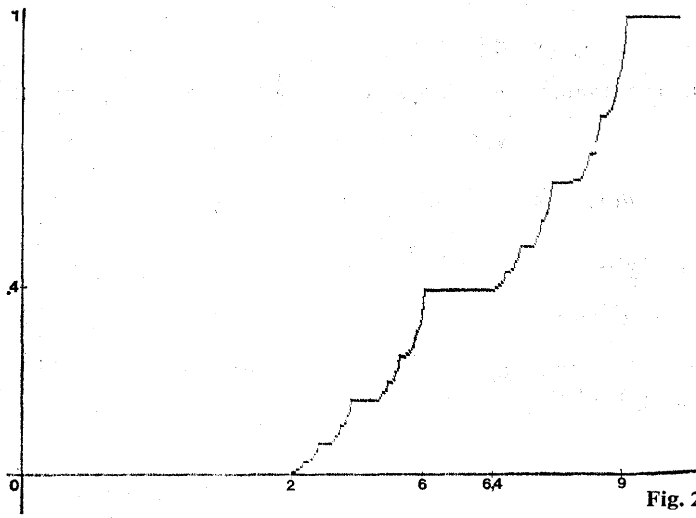

Fig. 2.

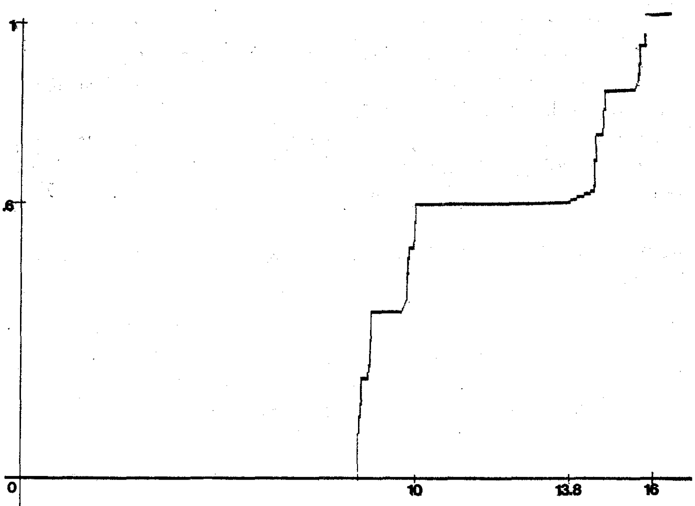

Fig. 3.

6. APPENDIX

Lemma 6.1. Let $h > 1$, $a > 0$, $c > 0$. Then the equation

$$\left(\frac{x - c}{a}\right)^h = x \tag{6.1}$$

has a unique real root z larger than c. This root satisfies

$$c + a(a/h)^{1/(h-1)} < z < \max\left[2c, c + (2a^h)^{1/(h-1)}\right]. \tag{6.2}$$

Proof. Define

$$f(x) = \left(\frac{x - c}{a}\right)^h - x, \quad x_0 = c + a(a/h)^{1/(h-1)}.$$

Then

$$f'(x) = \frac{h}{a}\left(\frac{x - c}{a}\right)^{h-1} - 1,$$

$$f(c) = -c < 0, \quad f'(x) < 0 \quad \text{for} \quad x \in (c, x_0), \quad f'(x) > 0 \quad \text{for} \quad x > x_0.$$

Thus $f(x) = 0$ has a unique root z larger than c. It is clear that $x_0 < z$. Now, let

$$g(x) = x - c - ax^{1/h}.$$

We can write

$$g(x) = x - c - a(x - c)^{1/h}\left(1 + \frac{c}{x - c}\right)^{1/h}.$$

If $x > 2c$, then $c/(x - c) < 1$ and $(1 + c/(x - c))^{1/h} < 2^{1/h}$. Then

$$g(x) > x - c - 2^{1/h}a(x - c)^{1/h} = (x - c)^{1/h}\left[(x - c)^{(h-1)/h} - 2^{1/h}a\right] = g_0(x).$$

If $x > c + (2a^h)^{1/(h-1)}$, then $g_0(x) > 0$ and the function $g(x)$ has no root for $x \geqq \max\left[2c, c + (2a^h)^{1/(h-1)}\right]$. □

Remark 6.2. Define $\psi(x) = ((x - c)/a)^h$. Then we have $\psi(x) > x$ for $x > z$.

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