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EQUIVALENCE, INVARIANCE AND DYNAMICAL SYSTEM CANONICAL MODELLING

Part II. Invariant Properties of Reachable Models and Associated Transformations

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The second part of this paper considers two different classes of models for linear reachable multivariable systems: state-space models and polynomial input — partial state — output models. Equivalence relations that do not affect the input-output behavior of the considered models are then introduced, as well as associated sets of canonical forms directly parametrized by the image of all the models belonging to the same equivalence class in a complete set of independent invariants for the considered equivalence relations. Results regarding systems that are both completely reachable and completely observable are then considered.

1. INTRODUCTION

From a conceptual point of view the content of many sections of this second part of the paper is perfectly dual to the content of some sections of the first part. From an algebraic point of view, however, only the results regarding completely reachable state-space models can be easily derived from those regarding completely observable ones; it has thus been considered as preferable to avoid the use of duality considerations. The paper consists of five sections with the following contents.

Section 2. This section deals with a well-known set of canonical forms for the given equivalence relation on Σ_c and Σ_m and shows how these canonical models are parametrized by the image in a complete set of independent invariants of every element beloging to the same equivalence class.

Section 3. This section defines a set of canonical forms for the considered equivalence relation on S_c and S_{co} and shows that their parametrization is the image in a complete set of independent invariants of every element belonging to the same equivalence class.

Section 4. The canonical forms that have been independently defined on both Σ_c and S_c are compared and the elementary algebraic links between these formally different representations deduced.

Sections 5. General results regarding minimal systems and some concluding remarks are reported in this section.

References to the contents of parts I and II are made according to the following rules: Definitions, theorems, lemmas, corollaries, properties, remarks, figures and algorithms: (p, n) where p refers to the considered part and n is a progressive number. Equations, relations, formulae and examples: (p, s, n) where p refers to parts, s to sections and n is a progressive number.

2. CANONICAL FORMS ON Σ_c AND Σ_m

Let (F, G, H) be an element of Σ_c with $\dim(F) = n$ and

$$H = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \qquad G = [g_1 \dots g_r]\,. \tag{2.2.1}$$

Consider then the r sequences of vectors given by

$$\begin{matrix} g_1 \; Fg_1 \dots F^{(n-r+1)}g_1 \\ \vdots \\ g_r \; Fg_r \; \dots F^{(n-r+1)}g_r \end{matrix} \tag{2.2.2}$$

Now order vectors (2.2.2) as follows

$$g_1 \dots g_r \; Fg_1 \dots Fg_r \dots F^{(n-r+1)}g_r \tag{2.2.3}$$

and select, in sequence (2.2.3), the vectors linearly independent of the preceding ones. Let $F^{v_i{}^c}g_i$ be the first vector, belonging to the ith row of (2.2.2), linearly dependent on the preceding ones in (2.2.3), i.e. such that

$$F^{v_i{}^c}g_i = \sum_{j=1}^{r} \sum_{k=1}^{v_{ji}{}^c} \alpha_{ijk}^c F^{(k-1)}g_j \tag{2.2.4}$$

holds where, because of the order of the vectors in sequence (2.2.3), the integers v_{ij}^c are given by

$$\begin{aligned} v_{ij}^c &= v_i^c && \text{for} \quad i = j \\ v_{ij}^c &= \min(v_j^c + 1, v_i^c) && \text{for} \quad j > i \\ v_{ij}^c &= \min(v_j^c, v_i^c) && \text{for} \quad j < i \end{aligned} \tag{2.2.5}$$

The *total* number of scalars α_{ijk}^c thus defined is therefore given by

$$\eta = \sum_{i=1}^{r} \sum_{j=1}^{r} v_{ij}^c\,. \tag{2.2.6}$$

As is well known, dependence relation (2.2.4) also implies the linear dependence of all subsequent vectors belonging to the ith row of (2.2.2) (i.e. of the type $F^{(v_i{}^c+k)}g_i$, $k \geq 1$) on their antecedents in sequence (2.2.3).

The linearly independent vectors selected in sequence (2.2.3) are called *regular vectors* [1].

Remark 2.1. Since rank $(G) = r$, all the integers ν_i^c are greater than zero.

Remark 2.2. Because of the complete observability of all the elements of Σ_c, the regular vectors constitute a basis for $\mathscr{X}$, i.e. $\nu_1^c + \dots + \nu_r^c = n$.

Now denote

$$c_{ijk}^c = \langle h_i^T, F^{(k-1)} g_j \rangle \tag{2.2.7}$$

the scalar products of the columns of H^T with the regular vectors.

Definition 2.1. The integers ν_i^c $(i = 1, \dots, r)$ obtained by means of the outlined procedure are called in the literature *Kronecker invariants* of the pair (F, G) since they are coincident (modulo ordering) with Kronecker's minimal column indices for the singular matrix pencil $[zI - F \mid G]$ [2], [3]. These indices will be called in the following *structural invariants* of (F, G) or *control invariants* of (F, G, H).

The scalars α_{ijk}^c which appear in (2.2.4) will be called *characteristic parameters* of the pair (F, G), and the scalars c_{ijk}^c which appear in (2.2.7) will be called *output parameters* of (F, G, H).

A set of scalars $(\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$ has been associated to every element (F, G, H) of Σ_c. A function

$$f_c = (f_i^{c\nu}, f_{ijk}^{c\alpha}, f_{ijk}^{cc}) \colon \Sigma_c \to N^r \times \mathscr{F}^\eta \times \mathscr{F}^{(n\times m)}$$

has thus been implicitly defined. Here, and in the following, N denotes the set of natural numbers. It is now possible to prove the following theorem.

Theorem 2.1. The function $f_c = (f_i^{c\nu}, f_{ijk}^{c\alpha}, f_{ijk}^{cc})$ constitutes a complete set of independent invariants for equivalence relation (1.2.6) on Σ_c.

Proof.

Invariance of f_c

Let (F, G, H) and (F', G', H') be two elements of Σ_c with $(F, G, H)\ E\ (F', G', H')$. It will be proved that $f_c(F, G, H) = f_c(F', G', H')$. Because of the given definition for E there exists a nonsingular matrix $T \in \mathscr{F}^{(n\times n)}$ such that $F' = TFT^{-1}$, $G' = TG$ and $H' = HT^{-1}$. Sequence (2.2.3) for (F', G', H') is given by

$$Tg_1 \dots Tg_r \dots TF^{(n-r+1)} g_r \,. \tag{2.2.8}$$

Because of the nonsingularity of T the linear dependence relationships among vectors (2.2.8) are obviously the same as among vectors (2.2.3). It follows, therefore, that $f_i^{c\nu}(F, G, H) = f_i^{c\nu}(F', G', H')$ and $f_{ijk}^{c\alpha}(F, G, H) = f_{ijk}^{c\alpha}(F', G', H')$. Now denote with R the basis of $\mathscr{X}$ given by the regular vectors ordered as follows

$$R = [g_1 \dots F^{(\nu_1{}^c - 1)} g_1 | \dots | \, g_r \dots F^{(\nu_1{}^c - 1)} g_r] \,. \tag{2.2.9}$$

Because of the given definition (2.2.7) the scalars c^{c}_{ijk} are the entries of the matrix HR. When the triple (F', G', H') is considered, because of (2.2.8) it follows immediately that the scalars c'^{c}_{ijk} are the entries of the matrix $H'R' = HT^{-1}TR = HR$. Therefore $f^{cc}_{ijk}(F, G, H) = f^{cc}_{ijk}(F', G', H')$ and, consequently, $f_c(F, G, H) = f_c(F', G', H')$.

Completeness of f_c

Let (F, G, H) and (F', G', H') be two elements of Σ_c such that $f_c(F, G, H) = f_c(F', G', H') = (\nu^c_i, \alpha^c_{ijk}, c^c_{ijk})$. It will be proved that $(F, G, H)\, E\, (F', G', H')$. Since $\nu^c_i = \nu'^c_i$ it follows that the regular vectors associated to (F', G', H') are generated exactly in the same way as vectors (2.2.9), i.e.

$$R' = [g'_1 \dots F'^{(\nu_1{}^c - 1)} g'_1 | \dots | g'_r \dots F'^{(\nu_r{}^c - 1)} g'_r]\,. \tag{2.2.10}$$

Now define the nonsingular matrix

$$T = R'R^{-1} \tag{2.2.11}$$

so that

$$R' = TR \tag{2.2.12}$$

$$F'^{(k-1)} g'_i = TF^{(k-1)} g_i \quad (i = 1, \dots, r;\ k = 1, \dots, \nu^c_i)\,. \tag{2.2.13}$$

Relation (2.2.13) for $i = 1, \dots, r$ and $k = 1$ implies $G' = TG$. Moreover, since $\alpha^c_{ijk} = \alpha'^c_{ijk}$ it also holds that

$$F'^{\nu_i{}^c} g'_i = TF^{\nu_i{}^c} g_i \quad (i = 1, \dots, r)\,. \tag{2.2.14}$$

From (2.2.13) and (2.2.14) it is possible to write

$$F'R' = TFR$$

and, consequently,

$$F' = TFRR'^{-1} = TFRR^{-1}T^{-1}$$

$$F' = TFT^{-1}\,.$$

From condition $c^c_{ijk} = c'^c_{ijk}$ it follows that

$$H'R' = HR$$

or, also,

$$H' = HRR^{-1}T^{-1} = HT^{-1}\,.$$

It has thus been proved that $(F, G, H)\, E\, (F', G', H')$ and, therefore, that set $(f^{c\nu}_i, f^{c\alpha}_{ijk}, f^{cc}_{ijk})$ constitutes a complete invariant for E.

Independence of f_c

Let $(\nu^c_1, \dots, \nu^c_r)$ be an arbitrary element of N^r with $n = \nu^c_1 + \dots + \nu^c_r$, $\nu^c_i \neq 0$, (α^c_{ijk}) an arbitrary element of $\mathscr{F}^n$ and (c^c_{ijk}) and arbitrary element of $\mathscr{F}^{(n \times m)}$. It will be proved that there exists an element $(F, G, H) \in \Sigma_c$ such that $f_c(F, G, H) =$

$= (\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$ i.e. that f_c is surjective with respect to $N^r \times \mathscr{F}^n \times \mathscr{F}^{(n \times m)}$. This will ensure the independence of the considered set of functions.

Choose an arbitrary basis, R, of $\mathscr{X}$ and denote its vectors as follows.

$$R = \left[e_{11} \dots e_{1\nu_1{}^c} \mid e_{21} \dots e_{2\nu_2{}^c} \mid \dots \mid e_{r1} \dots e_{r\nu_r{}^c}\right]. \tag{2.2.15}$$

Now define the columns of the $(n \times r)$ matrix G as

$$g_i = e_{i1} \quad (i = 1, \dots, r) \tag{2.2.16}$$

while the columns of the $(n \times n)$ matrix FR are defined by means of the following relations

$$Fe_{ij} = e_{i(j+1)} \tag{2.2.17a}$$

$$Fe_{i\nu_i{}^c} = \sum_{j=1}^{r} \sum_{k=1}^{\nu_j{}^c} \alpha_{jik}^c e_{jk}. \tag{2.2.17b}$$

Since R is nonsingular, the n relations (2.2.17) univocally define F. Similarly, the rows of HR (and, consequently, of H) are defined by means of the relations

$$h_i R = \left[c_{i11} \dots c_{i1\nu_1{}^c} \mid \dots \mid c_{ir1} \dots c_{ir\nu_r{}^c}\right]. \tag{2.2.18}$$

It is now necessary to verify that the image in f_c of the triple (F, G, H) defined by relations (2.2.16), (2.2.17) and (2.2.18) is $(\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$. From (2.2.16), (2.2.17a) and (2.2.17b) it follows that

$$e_{ij} = Fe_{i(j-1)} = \dots = F^{(j-1)}e_{i1} = F^{(j-1)}g_i. \tag{2.2.19}$$

Substitution of (2.2.19) in (2.2.17b), in (2.2.15) and, consequently, in (2.2.18) directly leads to relations (2.2.4) and (2.2.7). It is thus proved that $f_{ijk}^{c\alpha}(F, G, H) = (\alpha_{ijk}^c)$, $f_{ijk}^{cc}(F, G, H) = (c_{ijk}^c)$. Now let $\hat{\nu}_i^c = f_i^{c\nu}(F, G, H)$; from the substitution of (2.2.19) in (2.2.17b) it follows that $\hat{\nu}_i^c \leqq \nu_i^c$ but the substitution of (2.2.19) in (2.2.15) leads to relation $\hat{\nu}_1^c + \dots + \hat{\nu}_r^c = n$ so that $\hat{\nu}_i^c = \nu_i^c$ and $f_i^{c\nu}(F, G, H) = (\nu_i^c)$. □

The following corollary directly follows from Property 1.1.

Corollary 2.1. Let $g\colon N^r \times \mathscr{F}^n \times \mathscr{F}^{(n \times m)} \to N^r \times \mathscr{F}^n \times \mathscr{F}^{(n \times m)}$ be a bijection. The function $g \cdot f_c$ is a complete set of independent invariants for E on Σ_c.

In [1] it is proved (with a weaker definition of independence) that $f_c' = (f_i^{c\nu}, f_{ijk}^{c\alpha})$ constitutes a complete set of independent invariants for equivalence relation (1.2.6) on the set of pairs (F, G). The image of f_c', however, does not allow to parametrize the quotient set Σ_c/E.

Canonical Forms on Σ_c

$f_c = (f_i^{c\nu}, f_{ijk}^{c\alpha}, f_{ijk}^{cc})$ is a complete set of independent invariants for E on Σ_c. The image of f_c, $(\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$ can therefore be used to parametrize Σ_c/E i.e. to construct a set of canonical forms for E on Σ_c.

Definition of the Set of Canonical Forms C_c

Very useful canonical forms are the multicompanion ones that can be directly obtained from the set of scalars $(\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$. This canonical subset of Σ_c will be denoted with C_c. The elements of C_c can be constructed by means of relations (2.2.15) to (2.2.18), choosing the natural basis for $\mathscr{X}$. From $R = I_n$, in fact, it follows that

$$\tilde{G} = \begin{array}{c}\left[\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right]\end{array}\begin{array}{l} \leftarrow 1 \\ \\ \\ \leftarrow (\nu_1^c + 1) \\ \\ \\ \leftarrow (\nu_1^c + \dots + \nu_{r-1}^c + 1) \\ \\ \\ \\ \end{array} \tag{2.2.20}$$

$$\tilde{F} = [\tilde{F}_{ij}] \qquad (i, j = 1, \dots, r) \tag{2.2.21a}$$

$$\underset{(\nu_i^c \times \nu_i^c)}{\tilde{F}_{ii}} = \begin{bmatrix} 0 & 0 & \dots & 0 & \alpha_{ii1}^c \\ 1 & 0 & \dots & 0 & \alpha_{ii2}^c \\ 0 & 1 & \dots & 0 & \alpha_{ii3}^c \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & \alpha_{ii\nu_i^c}^c \end{bmatrix} \tag{2.2.21b}$$

$$\underset{(\nu_i^c \times \nu_j^c)}{\tilde{F}_{ij}} = \begin{bmatrix} 0 & \dots & 0 & \alpha_{ij1}^c \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \alpha_{ij\nu_{ij}^c}^c \\ 0 & \dots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{2.2.21c}$$

$$\tilde{H} = [\tilde{H}_1 \dots \tilde{H}_r] \qquad \underset{(m \times \nu_i^c)}{\tilde{H}_i} = [\tilde{h}_{i1} \dots \tilde{h}_{i\nu_i^c}] = \begin{bmatrix} c_{1i1}^c & \dots & c_{1i\nu_i^c}^c \\ \vdots & & \vdots \\ c_{mi1}^c & \dots & c_{mi\nu_i^c}^c \end{bmatrix} \tag{2.2.22}$$

It is well known how the canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ is algebraically linked to a generic triple (F, G, H) equivalent under E. In fact, $\tilde{F} = TFT^{-1}$, $\tilde{G} = TG$, $\tilde{H} = HT^{-1}$ where T is the matrix of regular vectors (2.2.9).

Other canonical forms for E on Σ_c can be parametrized by means of sets of scalars bijectively obtained from $(\nu_i^c, \alpha_{ijk}^c, c_{ijk}^c)$ [4], [5].

Example 2.2.1.

Let us consider the triple $(F, G, H) \in \Sigma_c$ given by

$$F = \begin{bmatrix} -0{\cdot}5 & 1 & 0 & 1{\cdot}5 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0{\cdot}5 & 0 & 1 & -1{\cdot}5 & -1 \\ 0{\cdot}5 & 0 & 0 & -0{\cdot}5 & 1 \end{bmatrix} \tag{2.2.23}$$

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{2.2.24}$$

$$H = \begin{bmatrix} 0 & 0{\cdot}5 & 0 & 0{\cdot}5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2.25}$$

The sequence of vectors (2.2.3) is given by:

$$\begin{array}{cc|cc|cc|cc} 0 & 1 & 1 & 2 & 1 & -4 & -3 & 8 \\ 1 & 0 & 0 & 0 & -1 & -2 & 1 & 10 \\ 1 & 0 & 0 & 0 & 1 & 2 & 1 & -6 \;\ldots \\ 0 & 1 & 1 & -2 & -1 & 4 & 3 & -8 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & -4 \\ \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet \end{array} \tag{2.2.26}$$

where the vectors linearly independent on their antecedents have been denoted with the abstract symbol $\circ$, the linearly dependent ones with the symbol $\bullet$. The scalars ν_1^c and ν_2^c are therefore given by $\nu_1^c = 3$ and $\nu_2^c = 2$.

The scalars α_{ijk}^c can be obtained by computing the dependence coefficients of the first dependent vectors in (2.2.26), i.e. $F^2 g_2$ and $F^3 g_1$ from their antecedents. The obtained values are

$$\begin{aligned} \alpha_{121}^c &= 0 & \alpha_{221}^c &= 2 \\ \alpha_{122}^c &= -2 & \alpha_{222}^c &= -3 \\ \alpha_{123}^c &= 2 \\ \alpha_{111}^c &= 1 & \alpha_{211}^c &= 1 \\ \alpha_{112}^c &= -1 & \alpha_{212}^c &= -1{\cdot}5 \\ \alpha_{113}^c &= 0 \end{aligned}$$

The scalars c_{ijk}^c can then be determined as scalar products of the transposed rows of H with the regular vectors in sequence (2.2.26). The obtained values are

$$\begin{aligned} c_{111}^c &= 0{\cdot}5 & c_{121}^c &= 0{\cdot}5 \\ c_{112}^c &= 0{\cdot}5 & c_{122}^c &= -1 \\ c_{113}^c &= -1 \end{aligned}$$

$$c^c_{211} = 0 \qquad c^c_{221} = 1$$
$$c^c_{212} = 0 \qquad c^c_{222} = 0$$
$$c^c_{213} = 0$$

The scalars computed in this way are the image $f_c(F, G, H)$. The canonical form (2.2.20)–(2.2.22) directly parametrized by this image is thus given by the following triple.

$$\tilde{F} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1{\cdot}5 & 1 & -3 \end{array}\right] \tag{2.2.27}$$

$$\tilde{G} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \tag{2.2.28}$$

$$\tilde{H} = \left[\begin{array}{ccc|cc} 0{\cdot}5 & 0{\cdot}5 & -1 & 0{\cdot}5 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{2.2.29}$$

Remark 2.3. The canonical forms (2.2.20)–(2.2.22) that have been considered on Σ_c can obviously be considered also on Σ_m since Σ_m is a subset of Σ_c which is closed with respect to equivalence relation (1.2.6).

3. CANONICAL FORMS ON S_c AND S_{co}

In this section a subset, K_c, of S_c is defined. It is then proved that K_c is a set of canonical forms for equivalence relation (1.2.8) on S_c. The transformation of a generic element of S_c to the corresponding canonical form is then considered and a transformation algorithm is given. The invariance properties of this transformation are then investigated and a numerical example proposed.

Definition of the Set of Canonical Forms K_c

Consider a subset, K_c, of S_c whose elements $(\tilde{R}(z), \tilde{S}(z))$ are characterized by the following conditions:

1) The polynomials on the main diagonal of $\tilde{R}(z)$ are monic;
2) The relations among the degrees of the entries of $\tilde{R}(z)$ are

$$\deg\{\tilde{r}_{jj}(z)\} \geqq \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad j > i \tag{2.3.1a}$$

$$\deg\{\tilde{r}_{jj}(z)\} > \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad j < i \tag{2.3.1b}$$

$$\deg\{\tilde{r}_{ii}(z)\} > \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad i \neq j\,; \tag{2.3.1c}$$

3) The relation between the degrees of the entries of $\tilde{R}(z)$ and $\tilde{S}(z)$ is

$$\deg\{\tilde{r}_{jj}(z)\} > \deg\{\tilde{s}_{ij}(z)\}\,. \tag{2.3.2}$$

The entries of the elements of K_c will be denoted as follows:

$$\tilde{R}(z) = \begin{bmatrix} \tilde{r}_{11}(z) & \dots & \tilde{r}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{r}_{r1}(z) & \dots & \tilde{r}_{rr}(z) \end{bmatrix} \tag{2.3.3}$$

$$\tilde{S}(z) = \begin{bmatrix} \tilde{s}_{11}(z) & \dots & \tilde{s}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{s}_{m1}(z) & \dots & \tilde{s}_{mr}(z) \end{bmatrix} \tag{2.3.4}$$

$$\tilde{r}_{jj}(z) = z^{\nu_j{}^c} - \alpha^c_{jj\nu_j{}^c} z^{(\nu_j{}^c - 1)} - \dots - \alpha^c_{jj2} z - \alpha^c_{jj1} \tag{2.3.5a}$$

$$\tilde{r}_{ij}(z) = -\alpha^c_{ij\nu_{ij}{}^c} z^{(\nu_{ij}{}^c - 1)} - \dots - \alpha^c_{ij2} z - \alpha^c_{ij1} \tag{2.3.5b}$$

$$\tilde{s}_{ij}(z) = \gamma^c_{ij1} z^{(\nu_i{}^c - 1)} + \dots + \gamma^c_{ij(\nu_i{}^c - 1)} z + \gamma^c_{ij\nu_i{}^c}\,. \tag{2.3.6}$$

Remark 2.4. Because of relations (2.3.1) it follows that the column degrees in $\tilde{R}(z)$ are the degrees of $\tilde{r}_{11}(z), \dots, \tilde{r}_{rr}(z)$, i.e. $\nu^c_1, \dots, \nu^c_r$. Moreover it holds that

$$\deg\det\{\tilde{R}(z)\} = \sum_{i=1}^{r} \nu^c_i = n\,. \tag{2.3.7}$$

Remark 2.5. The total number of significant coefficients in the entries of $\tilde{R}(z)$ is given by

$$\eta = \sum_{i=1}^{r} \sum_{j=1}^{r} \nu^c_{ij} \quad (\nu^c_{ii} = \nu^c_i) \tag{2.3.8}$$

while the total number of coefficients in the entries of $\tilde{S}(z)$ is given by

$$\sum_{i=1}^{m} \sum_{j=1}^{r} \nu^c_i = \sum_{i=1}^{m} n = n \times m\,. \tag{2.3.9}$$

Theorem 2.2. K_c constitutes a set of canonical forms for E on S_c.

Proof. The proof will be decomposed into the following steps:

a) *For every element of* S_c $(R(z), S(z))$ *there exists an element of* K_c, $(\tilde{R}(z), \tilde{S}(z))$, *equivalent to* $(R(z), S(z))$.

A constructive proof of the existence of this element is given by Algorithm 2.1.

b) *The element of* K_c *equivalent to a given element of* S_c *is unique.*

Assume that for a given element $(R(z), S(z))$ of S_c there exist two different elements of K_c, $(\tilde{R}'(z), \tilde{S}'(z))$ and $(\tilde{R}''(z), \tilde{S}''(z))$ equivalent to $(R(z), S(z))$. From this assumption it immediately follows that $(\tilde{R}'(z), \tilde{S}'(z))\ E\ (\tilde{R}''(z), \tilde{S}''(z))$, i.e. that there exists a unimodular matrix $M(z)$ such that $\tilde{R}''(z) = R'(z)\,M(z)$ and $\tilde{S}''(z) = \tilde{S}'(z)\,M(z)$. Let us now consider the jth column of $\tilde{R}''(z)$; this is a linear combination of the columns of $\tilde{R}'(z)$ multiplied by the elements of the jth column of $M(z)$. Since $(\tilde{R}''(z), \tilde{S}''(z))$ is an element of K_c, the elements of the jth column of $\tilde{R}''(z)$ must satisfy conditions (2.3.1a) and (2.3.1b). Since, however, the elements of the jth column of $M(z)$ are polynomials in z (and not rational functions), and the elements of $\tilde{R}'(z)$ satisfy conditions (2.3.1a) and (2.3.1b) it follows that, necessarily, $m_{jj}(z) \neq 0$ and that the column degree of this column is $\nu_j^{c''} = \nu_j^{c'} + \deg \{m_{jj}(z)\}$. Since $M(z)$ is unimodular, $\deg \det \{\tilde{R}'(z)\} = \deg \det \{\tilde{R}''(z)\}$ i.e. $\sum_{i=1}^{r} \nu_i^{c'} = \sum_{i=1}^{r} \nu_i^{c''}$ and, therefore, $\deg \{m_{jj}(z)\} = 0$ ($j = 1, \ldots, r$). It has thus been established that $\tilde{R}'(z)$ and $\tilde{R}''(z)$ share the same ordered set of row degrees.

The elements of the jth column of $\tilde{R}''(z)$ must also satisfy row conditions (2.3.1c) with respect to the on-diagonal elements of the subsequent columns; this necessarily leads to the conditions $m_{ij}(z) = 0$ for $i > j$ on the jth column of $M(z)$. Similarly the elements of the jth column of $\tilde{R}''(z)$ must satisfy row condition (2.3.1c) with respect to the on-diagonal elements of the preceding columns, and this leads to the conditions $m_{ij}(z) = 0$ for $i < j$ on the jth column of $M(z)$. It has thus been established that $M(z) = \operatorname{diag} \{m_{jj}(z)\}$ with $\deg \{m_{jj}(z)\} = 0$; $M(z)$ is therefore a diagonal real matrix. Since the polynomials on the main diagonal of $\tilde{R}'(z)$ and $\tilde{R}''(z)$ are monic it follows that $M(z) = I$ and, consequently, $\tilde{R}'(z) = \tilde{R}''(z)$.

c) *Elements of S_c in the same equivalence class (with respect to E) are equivalent to the same element of K_c.*

Let $(R'(z), S'(z))$, $(R''(z), S''(z))$ be two equivalent elements of S_c, $(\tilde{R}'(z), \tilde{S}'(z))$, $(\tilde{R}''(z), S''(z))$ the two corresponding equivalent elements of K_c. Because of the equivalence between $(R'(z), \tilde{S}'(z))$ and $(R''(z), S''(z))$ it also follows that $(\tilde{R}'(z), \tilde{S}'(z))$ $E(\tilde{R}''(z), \tilde{S}''(z))$ and since, because of step b), the equivalence classes with respect to E in K_c have a single element, it follows that $(\tilde{R}'(z), \tilde{S}'(z)) = (\tilde{R}''(z), \tilde{S}''(z))$.

d) *Elements of S_c which do not belong to the same equivalence class are equivalent to distinct elements of K_c.*

Let $(R'(z), S'(z))$ and $(R''(z), S''(z))$ be two elements of S_c belonging to distinct equivalence classes. It follows that $R'(z) \neq R''(z)\, M(z), S'(z) \neq S''(z)\, M(z)$ for every unimodular matrix $M(z)$. If there exists an element of K_c, $(\tilde{R}(z), \tilde{S}(z))$ equivalent to $(R'(z), S'(z))$ and to $(R''(z), S''(z))$ then $\tilde{R}(z) = R'(z)\, M'(z) = R''(z)\, M''(z)$, $\tilde{S}(z) = S'(z)\, M'(z) = S''(z)\, M''(z)$ and, consequently, $R'(z) = R''(z)\, M''(z)\, M'^{-1}(z)$, $S'(z) = S''(z)\, M''(z)\, M'^{-1}(z)$. $(R'(z), S'(z))$ and $(R''(z), S''(z))$ are therefore equivalent to distinct elements of K_c.

According to Definition 1.7 it has thus been proved that K_c is a set of canonical forms for equivalence relation (1.2.8) on S_c. □

Transformation to the Canonical Forms on S_c

Step a) in the proof of Theorem 2.2 will be constructively established by means of the following algorithm which allows, given a generic element $(R(z), S(z))$ of S_c, transformation to the corresponding canonical form $(\tilde{R}(z), \tilde{S}(z))$ of K_c, to be performed.

Algorithm 2.1.

STEP 1. Matrices $R(z)$ and $S(z)$ are postmultiplied for a suitable unimodular matrix $M(z)$ such that $R(z)\,M(z)$ is column-proper. A detailed description of this step can be found in [6].

STEP 2. *Achievement of column condition* (2.3.1a). By means of exchanges of columns, in every column of $R(z)$ polynomials whose degree equals the column degree are moved on the main diagonal. The same column exchanges are performed on $S(z)$. This operation is always possible if $R(z)$ is column-proper (a proof can be found in [7] modulo duality considerations).

STEP 3. *Achievement of column condition* (2.3.1b). The entries $r_{r,r-1}(z)$, $r_{r,r-2}(z), \dots, r_{r,1}(z), r_{r-1,r-2}(z), \dots, r_{r-1,1}(z), \dots, r_{2,1}(z)$ are tested in the given order with respect to column condition (2.3.1b). If $\deg\{r_{ij}(z)\} < \deg\{r_{jj}(z)\}$ no operation is performed. When $\deg\{r_{ij}(z)\} = \deg\{r_{jj}(z)\}$ and $\deg\{r_{ij}(z)\} = \mu_{ij} \geqq \deg\{r_{ii}(z)\} = \mu_{ii}$ the degree of $r_{ij}(z)$ is lowered by subtracting from the jth column of $R(z)$ the ith column multiplied by $\alpha z^{\mu_{ij}-\mu_{ii}}$ where α is the ratio of the maximal degree coefficients in $r_{ij}(z)$ and $r_{ii}(z)$.

If $\deg\{r_{ij}(z)\} = \deg\{r_{jj}(z)\}$ and $\deg\{r_{ij}(z)\} = \mu_{ij} < \deg\{r_{ii}(z)\} = \mu_{ii}$ it is sufficient to exchange the jth column of $R(z)$ with the difference of the ith column and of the jth column multiplied for $\alpha z^{\mu_{ii}-\mu_{ij}}$ where α is the ratio of the maximal degree coefficients in $r_{ii}(z)$ and $r_{ij}(z)$. The same elementary operations performed on $R(z)$ are obviously performed also on $S(z)$. It is important to note that this step does not change *all* conditions obtained in previous steps.

STEP 4. *Achievement of row condition* (2.3.1c) *in the left-lower triangular part of* $R(z)$. The polynomials considered in Step 3 are tested in the same order with respect to row condition (2.3.1c). If $\mu_{ij} < \mu_{ii}$ no operation is performed. When $\mu_{ij} \geqq \mu_{ii}$ the degree of $r_{ij}(z)$ is lowered by subtracting from the jth column of $R(z)$ the ith column multiplied by $\alpha z^{\mu_{ij}-\mu_{ii}}$ where α is the ratio of the maximal degree coefficients in $r_{ij}(z)$ and $r_{ii}(z)$. After the described operation the next polynomial in the given sequence must be tested even if condition (2.3.1c) with respect to $r_{ij}(z)$ has not been achieved. This entire step is repeated until condition (2.3.1c) on the left lower triangular part of $R(z)$ is achieved. The same elementary column operations are performed on $S(z)$. Note that the operations performed at this step to reduce the order of $r_{ij}(z)$ do not change the column conditions obtained at Step 3 or the row condition (2.3.1c) on the polynomials tested before $r_{ij}(z)$.

STEP 5. *Achievement of row condition* (2.3.1c) *in the right upper triangular part of* $R(z)$. The entries $r_{1,2}(z), r_{1,3}(z), \dots, r_{1,r}(z), r_{2,3}(z), \dots, r_{2,r}(z), \dots, r_{r-1,r}(z)$ are tested in the given order with respect to row condition (2.3.1c). If $\mu_{ij} < \mu_{ii}$ no operation is performed. When $\mu_{ij} \geqq \mu_{ii}$ the degree of $r_{ij}(z)$ is lowered by subtracting from the jth column of $R(z)$ the ith column multiplied by $\alpha z^{\mu_{ij}-\mu_{ii}}$ where α is the ratio of the maximal degree coefficients in $r_{ij}(z)$ and $r_{ii}(z)$. After this operation the next polynomial in the given sequence must be tested even if condition (2.3.1c) with respect to $r_{ij}(z)$ has not been achieved. The same elementary column operations are performed on $S(z)$. This entire step is repeated until condition (2.3.1c) is achieved on the right upper triangular part of $R(z)$.

This step does not change all conditions obtained in previous steps.

STEP 6. Adjustment of the coefficients on the main diagonal of $R(z)$. The first, second, ..., rth columns of $R(z)$ and $S(z)$ are divided for the maximal degree coefficients in $r_{11}(z), r_{22}(z), \ldots, r_{rr}(z)$ respectively. After this step the polynomials on the main diagonal of $R(z)$ are monic.

Given a generic element $(R(z), S(z))$ of S_c, Algorithm 2.1 leads (by means of steps equivalent to the postmultiplication of $R(z)$ and $S(z)$ for unimodular matrices) to the equivalent canonical pair $(\tilde{R}(z), \tilde{S}(z))$. The algorithm is based on the fact that every step does not change *all* previously obtained conditions.

By means of Algorithm 2.1 a function $\phi^c = (\phi_i^{c\nu}, \phi_{ijk}^{c\alpha}, \phi_{ijk}^{c\gamma}): S_c \to N^r \times \mathscr{F}^n \times \mathscr{F}^{(n \times m)}$ has been implicitly defined. The image $\phi^c(R(z), S(z)) = (\nu_i^c, \alpha_{ijk}^c, \gamma_{ijk}^c)$ has been used for the parametrization of the elements of K_c, i.e. for the parametrization of the canonical forms on S_c. The following theorem can therefore be established.

Theorem 2.3. $\phi^c = (\phi_i^{c\nu}, \phi_{ijk}^{c\alpha}, \phi_{ijk}^{c\gamma})$ constitutes a complete set of independent invariants for equivalence relation (1.2.8) on S_c.

Proof.

Invariance of ϕ^c

Let $(R'(z), S'(z))$ and $(R''(z), S''(z))$ be two elements of S_c with $(R'(z), S'(z))$ $E(R''(z), S''(z))$. It must be proved that $\phi^c(R'(z), S'(z)) = \phi^c(R''(z), S''(z))$. This has already been done in step c) of the proof of Theorem 2.2.

Completeness of ϕ^c

Let $(R'(z), S'(z))$ and $(R''(z), S''(z))$ be two elements of S_c such that $\phi^c(R'(z), S'(z)) = \phi^c(R''(z), S''(z)) = (\nu_i^c, \alpha_{ijk}^c, \gamma_{ijk}^c)$. It must be proved that $(R'(z), S'(z))$ $E\,(R''(z), S''(z))$. Since ϕ^c: $S_c \to K_c$, the pairs $(R'(z), S'(z))$ and $(R''(z), S''(z))$ have the same canonical form. Because of steps c) and d) in the proof of Theorem 2.2 it follows that $(R'(z), S'(z))$ and $(R''(z), S''(z))$ belong to the same equivalence class of S_c.

Independence of ϕ^c

Let $(\nu_1^c, \ldots, \nu_r^c)$ be an arbitrary element of N^r with $\nu_i^c \neq 0$ and $n = \nu_1^c + \ldots + \nu_r^c$, (α_{ijk}^c) an arbitrary element of $\mathscr{F}^n$ and (γ_{ijk}^c) an arbitrary element of $\mathscr{F}^{(n \times m)}$. It must be proved that there exists an element of S_c, $(R(z), S(z))$, such that $\phi^c(R(z), S(z)) = (\nu_i^c, \alpha_{ijk}^c, \gamma_{ijk}^c)$ i.e. that ϕ^c is surjective with respect to $N^r \times \mathscr{F}^n \times \mathscr{F}^{(n \times m)}$. This will ensure the independence of the considered set of functions. By means of relations

(2.3.3), (2.3.4), (2.3.5) and (2.3.6) an element $(\tilde{R}(z), \tilde{S}(z))$ of K_c can be obtained such that $\phi^c(\tilde{R}(z), \tilde{S}(z)) = (v_i^c, \alpha_{ijk}^c, \gamma_{ijk}^c)$ and since K_c is a subset of S_c this completes the proof. □

Remark 2.6. Theorem 2.3 can be obtained as a corollary of Theorems 2.2 and 1.1. Similarly, Theorem 2.2 could be considered as a corollary of Theorems 2.3 and 1.1. The way this material has been presented allows either of these two ways to be selected.

Example 2.3.1.

A numerical example regarding the application of Algorithm 2.1 to an element of S_c in order to obtain the equivalent canonical form is now proposed. Let us consider the pair $(R(z), S(z))$ given by

$$R(z) = \begin{bmatrix} -2z^3 + 6z - 2 & -2z^2 + 2z \\ 2z^3 + 8z^2 + 5z - 6 & z^2 + 3z - 2 \end{bmatrix}$$

$$S(z) = \begin{bmatrix} 5z + 5{\cdot}5 & -0{\cdot}5z + 0{\cdot}5 \\ 2z^2 + 8z + 9 & z + 3 \end{bmatrix}$$

The application of Algorithm 2.1 to the previous pair will be performed step by step.

STEP 1. $R(z)$ is already column-proper since the real matrix, whose columns are obtained from the coefficients of the terms in the columns of $R(z)$, whose degree equals the row degree, is the nonsingular matrix

$$K = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix}.$$

STEP 2. Column condition (2.3.1a) is already satisfied.

STEP 3. The only element to be tested is $r_{21}(z)$. Since $\deg\{r_{21}(z)\} = \deg\{r_{11}(z)\} = 3$ and $\deg\{r_{21}(z)\} > \deg\{r_{22}(z)\} = 2$ the degree of $r_{21}(z)$ is lowered by subtracting from the first column of $R(z)$ the second one multiplied by $2z$. The same operation is performed on the columns of $S(z)$. The matrices obtained are

$$R_1(z) = R(z)\,M_1(z) = \begin{bmatrix} 2z^3 - 4z^2 + 6z - 2 & -2z^2 + 2z \\ 2z^2 + 9z - 6 & z^2 + 3z - 2 \end{bmatrix}$$

$$S_1(z) = S(z)\,M_1(z) = \begin{bmatrix} z^2 + 5{\cdot}5 & -0{\cdot}5z + 0{\cdot}5 \\ 2z + 9 & z + 3 \end{bmatrix}$$

where

$$M_1(z) = \begin{bmatrix} 1 & 0 \\ -2z & 1 \end{bmatrix}.$$

STEP 4. The only element to be tested is again $r_{21}(z)$. Since $\deg\{r_{21}(z)\} = \deg\{r_{22}(z)\}$ the degree of $r_{21}(z)$ is lowered by subtracting from the first column of $R(z)$ the second one multiplied by 2. The matrices obtained are

$$R_2(z) = R_1(z)\, M_2(z) = \begin{bmatrix} 2z^3 + 2z - 2 & -2z^2 + 2z \\ 3z - 2 & z^2 + 3z - 2 \end{bmatrix}$$

$$S_2(z) = S_1(z)\, M_2(z) = \begin{bmatrix} z^2 + z + 0{\cdot}5 & -0{\cdot}5z + 0{\cdot}5 \\ 3 & z + 3 \end{bmatrix}$$

where

$$M_2(z) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Since condition (2.3.1c) in the left lower triangular part of $R(z)$ is already achieved the next step can be considered.

STEP 5. The only element to be tested is $r_{12}(z)$. Since $\deg \{r_{12}(z)\} < \deg \{r_{11}(z)\}$ no operations must be performed.

STEP 6. The first column of $R(z)$ must be divided by 2 in order to make $r_{11}(z)$ monic while $r_{22}(z)$ is already monic. The matrices obtained are

$$\tilde{R}(z) = R_2(z)\, M_3(z) = \begin{bmatrix} z^3 + z - 1 & -2z^2 + 2z \\ 1{\cdot}5z - 1 & z^2 + 3z - 2 \end{bmatrix} \tag{2.3.10}$$

$$\tilde{S}(z) = S_2(z)\, M_3(z) = \begin{bmatrix} 0{\cdot}5z^2 + 0{\cdot}5z + 0{\cdot}25 & -0{\cdot}5z + 0{\cdot}5 \\ 1{\cdot}5 & z + 3 \end{bmatrix} \tag{2.3.11}$$

where

$$M_3(z) = \begin{bmatrix} 0{\cdot}5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Remark 2.7. Note that the transformation to the canonical form of the given pair $(R(z), S(z))$ has been performed by postmultiplying $R(z)$ and $S(z)$ for the non-singular unimodular matrix

$$M(z) = M_1(z)\, M_2(z)\, M_3(z) = \begin{bmatrix} 0{\cdot}5 & 0 \\ -z - 1 & 1 \end{bmatrix}.$$

Remark 2.8. The image $\phi^c(R(z), S(z))$ is given by

$$\begin{array}{lll}
\nu_1^c = 3 & & \nu_2^c = 2 \\
\alpha_{113}^c = 0 & & \\
\alpha_{112}^c = -1 & & \alpha_{212}^c = -1{\cdot}5 \\
\alpha_{111}^c = 1 & & \alpha_{211}^c = 1 \\
\alpha_{123}^c = 2 & & \\
\alpha_{122}^c = -2 & & \alpha_{222}^c = -3 \\
\alpha_{121}^c = 0 & & \alpha_{221}^c = 2 \\
\gamma_{113}^c = 0{\cdot}25 & & \gamma_{213}^c = 1{\cdot}5 \\
\gamma_{112}^c = 0{\cdot}5 & & \gamma_{212}^c = 0 \\
\gamma_{111}^c = 0{\cdot}5 & & \gamma_{211}^c = 0 \\
\gamma_{122}^c = 0{\cdot}5 & & \gamma_{222}^c = 3 \\
\gamma_{121}^c = -0{\cdot}5 & & \gamma_{221}^c = 1
\end{array}$$

It is now possible to express the state vector as a function of the partial state. In fact, in the considered state space canonical form the system is decomposed into r interconnected subsystems whose states are the components $x_1(t), \ldots$ $\ldots, x_{\nu_1^c}(t); \ldots; x_{\nu_1^c+\ldots+\nu_{r-1}^c+1}(t), \ldots, x_{\nu_1^c+\ldots+\nu_r^c}(t)$. If the jth subsystem is now considered it is possible to write the following relations between its state and the partial state because of the simple structure of $\tilde{F}$.

(2.4.2)
$$
\begin{aligned}
x_{\nu_1^c+\ldots+\nu_{j-1}^c+1}(t) = & -\alpha^c_{j1\nu_{j1}^c} z^{(\nu_{j1}^c-2)}\, w_1(t) - \ldots - \alpha^c_{j12}\, w_1(t) + \\
& \ldots \\
& + z^{(\nu_j^c-1)} w_j(t) - \alpha^c_{jj\nu_j^c} z^{(\nu_j^c-2)} w_j(t) - \ldots - \alpha^c_{jj2}\, w_j(t) - \\
& \ldots \\
& - \alpha^c_{jr\nu_{jr}^c} z^{(\nu_{jr}^c-2)}\, w_r(t) - \ldots - \alpha^c_{jr2}\, w_r(t) \\
& \vdots \\
x_{\nu_1^c+\ldots+\nu_j^c-1}(t) = & -\alpha^c_{j1\nu_{j1}^c}\, w_1(t) - \alpha^c_{j2\nu_{j2}^c}\, w_2(t) - \ldots \\
& \ldots \\
& + z\, w_j(t) - \alpha^c_{jj\nu_j^c}\, w_j(t) - \alpha^c_{j(j+1)\nu_{j(j+1)}^c}\, w_{(j+1)}(t) - \\
& \ldots \\
& - \ldots - \alpha^c_{jr\nu_{jr}^c}\, w_r(t) \\
x_{\nu_1^c+\ldots+\nu_j^c}(t) = & \; w_j(t)
\end{aligned}
$$

If relations (2.4.2) are considered for every subsystem, it is possible to obtain the relation between the system state $x(t)$ and the components of the partial state $w(t)$ in the following way

(2.4.3)
$$x(t) = M\, V(z)\, w(t)$$

where

(2.4.4a)
$$M = [M_{ij}] \qquad (i, j = 1, \ldots, r)$$

(2.4.4b)
$$
\underset{(\nu_i^c \times \nu_i^c)}{M_{ii}} = \begin{bmatrix}
1 & -\alpha^c_{ii\nu_i^c} & \ldots & -\alpha^c_{ii3} & -\alpha^c_{ii2} \\
 & 1 & \ldots & -\alpha^c_{ii4} & -\alpha^c_{ii3} \\
 & & & & \vdots \\
 & & & 1 & -\alpha^c_{ii\nu_i^c} \\
 & & & & 1
\end{bmatrix}
$$

(2.4.4c)
$$
\underset{(\nu_i^c \times \nu_j^c)}{M_{ij}} = \begin{bmatrix}
0 & \ldots & 0 & -\alpha^c_{ij\nu_{ij}^c} & \ldots & -\alpha^c_{ij2} \\
0 & \ldots & 0 & 0 & \ldots & -\alpha^c_{ij3} \\
\vdots & & & & & \vdots \\
0 & \ldots & 0 & 0 & \ldots & -\alpha^c_{ij(\nu_{ij}^c-1)} \\
0 & \ldots & 0 & 0 & \ldots & -\alpha^c_{ij\nu_{ij}^c}
\end{bmatrix}
$$

$$
V(z) = \underset{(n\times r)}{\begin{bmatrix} z^{(\nu_1^c - 1)} & \dots & 0 \\ \vdots & & \vdots \\ z & & 0 \\ 1 & & 0 \\ \vdots & & \vdots \\ 0 & & z^{(\nu_r^c - 1)} \\ \vdots & & \vdots \\ 0 & & z \\ 0 & \dots & 1 \end{bmatrix}} \tag{2.4.5}
$$

The substitution of the state vector expression (2.4.3) in equation (1.2.1a) leads to the following equation in $w(t)$

$$
(zI - \tilde{F})\, M\, V(z)\, w(t) = \tilde{G}\, u(t)\,. \tag{2.4.6}
$$

Computation of the left side of expression (2.4.6) shows that only the first, $(\nu_1^c + 1)$th, ..., $(\nu_1^c + \dots + \nu_{r-1}^c + 1)$th relations are significant while the remaining ones are simple identities. The r significant relations can be written in the following more compact form

$$
\tilde{R}(z)\, w(t) = u(t) \tag{2.4.7}
$$

where

$$
\tilde{R}(z) = \begin{bmatrix} \tilde{r}_{11}(z) & \dots & \tilde{r}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{r}_{r1}(z) & \dots & \tilde{r}_{rr}(z) \end{bmatrix} \tag{2.4.8}
$$

$$
\tilde{r}_{ii}(z) = z^{\nu_i^c} - \alpha_{ii\nu_i^c}^c z^{\nu_i^c - 1} - \dots - \alpha_{ii2}^c z - \alpha_{ii1}^c \tag{2.4.9}
$$

$$
\tilde{r}_{ij}(z) = -\alpha_{ij\nu_{ij}^c}^c z^{\nu_{ij}^c - 1} - \dots - \alpha_{ij2}^c z - \alpha_{ij1}^c\,. \tag{2.4.10}
$$

Expression (2.4.3) of the system state can now be substituted in the output equation (1.2.1b). The system output can thus be related to the partial state by the following equation

$$
y(t) = \tilde{H} M\, V(z)\, w(t)\,. \tag{2.4.11}
$$

Let us now define the matrix $\bar{H}$ given by

$$
\bar{H} = \tilde{H} M \tag{2.4.12}
$$

$$
\bar{H} = [\bar{H}_1 \dots \bar{H}_r] \qquad \underset{(m\times \nu_i^c)}{\bar{H}_i} = \begin{bmatrix} \gamma_{1i1}^c & \cdots & \gamma_{1i\nu_i^c}^c \\ \vdots & & \vdots \\ \gamma_{mi1}^c & \cdots & \gamma_{mi\nu_i^c}^c \end{bmatrix}. \tag{2.4.13}
$$

Relation (2.4.13) can also be written in the equivalent, more simple form

$$
y(t) = \tilde{S}(z)\, w(t) \tag{2.4.14}
$$

$$
\tilde{S}(z) = \begin{bmatrix} \tilde{s}_{11}(z) & \dots & \tilde{s}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{s}_{m1}(z) & \dots & \tilde{s}_{mr}(z) \end{bmatrix} \tag{2.4.15}
$$

$$
\tilde{s}_{ij}(z) = \gamma_{ij1} z^{\nu_i^c - 1} + \dots + \gamma_{ij(\nu_i^c - 1)}^c z + \gamma_{ij\nu_i^c}^c \tag{2.4.16}
$$

Matrix M (2.4.4) is structurally nonsingular for every set (α_{ijk}^{c}) since in every case $\det\{M\} = 1$.

From relations (2.2.5) it follows that the degrees of the polynomials in $\tilde{R}(z)$ and $\tilde{S}(z)$ satisfy the following conditions

$$\deg\{\tilde{r}_{jj}(z)\} \geqq \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad j > i \tag{2.4.17a}$$

$$\deg\{\tilde{r}_{jj}(z)\} > \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad j < i \tag{2.4.17b}$$

$$\deg\{\tilde{r}_{ii}(z)\} > \deg\{\tilde{r}_{ij}(z)\} \quad \text{if} \quad i \neq j \tag{2.4.17c}$$

$$\deg\{\tilde{r}_{jj}(z)\} > \deg\{\tilde{s}_{ij}(z)\} \tag{2.4.18}$$

The n initial conditions on the partial state requested by the definition of strict equivalence between state-space and input – partial state – output models are given by relation (2.4.3) written for $t = t_0$. It can be noted that the conditions requested on the first component of the partial state vector are v_1^c, those on the second component $v_2^c, \ldots$, and those on the rth component v_r^c.

Relation (2.4.8) shows that the on-diagonal elements of $\tilde{R}(z)$ are monic and since the obtained conditions (2.4.17a), (2.4.17b), (2.4.17c) and (2.4.18) are coincident with conditions (2.3.1a), (2.3.1b), (2.3.1c) and (2.3.2) it follows that the obtained pair $(\tilde{R}(z), \tilde{S}(z))$ is canonical. This completes the proof of the theorem. □

Corollary 2.2. For every element of Σ_c/E there exists a strictly equivalent element of K_c.

Example 2.4.1.

Let us consider the canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ of Σ_c given by (2.2.27)–(2.2.29) and the following initial state

$$x(0) = [0\ 0\ 1\ 0\ 0]^{\mathrm{T}}. \tag{2.4.19}$$

A strictly equivalent pair $(\tilde{R}(z), \tilde{S}(z))$ of K_c as well as the associated initial conditions on its partial state will be determined.

Matrix $\tilde{R}(z)$ can be written by direct inspection of $\tilde{F}$. In fact, from (2.4.8) and (2.4.9) it follows that

$$\tilde{R}(z) = \begin{bmatrix} z^3 + z - 1 & -2z^2 + 2z \\ 1{\cdot}5z - 1 & z^2 + 3z - 2 \end{bmatrix}. \tag{2.4.20}$$

Determination of $\tilde{S}(z)$ requires the previous construction of the matrix M (2.4.4). This matrix, too, can be written by direct inspection of $\tilde{F}$ on the basis of (2.4.4).

$$M = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1{\cdot}5 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{2.4.21}$$

Matrix $\bar{H}$ (2.4.12) is therefore given by

$$(2.4.22) \qquad \bar{H} = \tilde{H}M = \left[\begin{array}{ccc|cc} 0.5 & 0.5 & 0.25 & -0.5 & 0.5 \\ 0 & 0 & 1.5 & 1 & 3 \end{array}\right].$$

From (2.4.13) the scalars γ^c_{ijk} are given by

$$\begin{aligned} \gamma^c_{111} &= 0.5 \qquad & \gamma^c_{121} &= -0.5 \\ \gamma^c_{112} &= 0.5 & \gamma^c_{122} &= 0.5 \\ \gamma^c_{113} &= 0.25 \\ \gamma^c_{211} &= 0 & \gamma^c_{221} &= 1 \\ \gamma^c_{212} &= 0 & \gamma^c_{222} &= 3 \\ \gamma^c_{213} &= 1.5 \end{aligned}$$

The matrix $\tilde{S}(z)$ (2.4.15) is thus given by

$$(2.4.23) \qquad \tilde{S}(z) = \begin{bmatrix} 0.5z^2 + 0.5z + 0.25 & -0.5z + 0.5 \\ 1.5 & z + 3 \end{bmatrix}$$

The initial conditions on the partial state components are given by $w_1(0)$, $w_1(1)$, $w_1(2)$, $w_2(0)$ and $w_2(1)$. With the initial state (2.4.19) it follows that

$$\begin{aligned} w_1(0) &= 1 \qquad & w_2(0) &= 0 \\ w_1(1) &= 0 & w_2(1) &= -1.5 \\ w_1(2) &= -4 \end{aligned}$$

It can be noted from comparison of (2.4.20) and (2.4.23) with (2.3.10) and (2.3.11), respectively, that the obtained canonical pair $(\tilde{R}(z), \tilde{S}(z))$ is the same as that considered in Example 2.3.1.

Invariance Properties of the Transformations to the Canonical Forms on Σ_c and on S_c

Parametrization of the elements of C_c has been performed by means of the image $(v^c_i, \alpha^c_{ijk}, c^c_{ijk})$ of a complete set of independent invariants, f_c, for E on Σ_c. Similarly, parametrization of the elements of K_c has been performed by means of the image $(v^c_i, \alpha^c_{ijk}, \gamma^c_{ijk})$ of a complete set of independent invariants, ϕ_c, for E on S_c.

The map g_c: $\mathscr{F}^{(n\times m)} \to \mathscr{F}^{(n\times m)}$ described by relation (2.4.12), which transforms the set of scalars (c^c_{ijk}) into the set (γ^c_{ijk}) is, because of the structural nonsingularity of matrix M (2.4.4), one to one. Also the function c_c: $N^r \times \mathscr{F}^n \times \mathscr{F}^{(n\times m)} \to N^r \times \mathscr{F}^n \times \mathscr{F}^{(n\times m)}$ defined by $c_c(v^c_i, \alpha^c_{ijk}, c^c_{ijk}) = (v^c_i, \alpha^c_{ijk}, \gamma^c_{ijk})$ is, therefore, a bijection.

Because of Property 1.1 it follows therefore that the function δ_c: $\Sigma_c \to N^r \times \mathscr{F}^n \times \mathscr{F}^{(n\times m)}$ given by $\delta_c{:}c_c \,.\, f_c$ constitutes a complete set of independent invariants for E on Σ_c. Similarly the function d_c: $S_c \to N^r \times \mathscr{F}^n \times \mathscr{F}^{(n\times m)}$ given by $d_c = c_c^{-1} \cdot \phi_c$ constitutes a complete set of independent invariants for E on S_c.

The following theorems have thus been proved.

Theorem 2.5. Every canonical form $(\tilde{F}, \tilde{G}, \tilde{H})$ of C_c is parametrized by the image in d_c of any strictly equivalent element, $(R(z), S(z))$, of S_c.

Theorem 2.6. Every canonical form $(\tilde{R}(z), \tilde{S}(z))$ of K_c is parametrized by the image in δ_c of any strictly equivalent element, (F, G, H), of Σ_c.

Remark 2.11. In Sections 2 and 3 all the algorithms for the construction of functions f_c, ϕ_c, d_c and δ_c have been described. This allows the performance of every transformation between state-space controllable and input – partial state – output models.

The considered transformations between state-space controllable and input – partial state – output canonical forms are summarized by the commutative diagram of Figure 2.1 where Π_c and Π'_c are sets whose elements are all the sets of scalars $(v_i^c, \alpha_{ijk}^c, c_{ijk}^c)$ and $(v_i^c, \alpha_{ijk}^c, \gamma_{ijk}^c)$ respectively.

$$\begin{tikzcd}
& \Pi_c \arrow[rr, shift left, "c_c"] \arrow[dr, shift left, "\gamma_c"] & & \Pi'_c \arrow[ll, shift left, "c_c^{-1}"] \arrow[dr, shift left, "\eta_c"] & & S_c \arrow[ll, "\phi_c"'] \arrow[dl, "\omega_c"] \\
\Sigma_c \arrow[ur, "f_c"] \arrow[rr, "\rho_c"'] & & C_c \arrow[ul, shift left, "\gamma_c^{-1}"] \arrow[rr, shift left, "\psi_c"] & & K_c \arrow[ll, shift left, "\psi_c^{-1}"] \arrow[ul, shift left, "\eta_c^{-1}"] &
\end{tikzcd}$$

Fig. 2.1.

Let us now denote with C_{cm} the subset of C_c, whose elements are the canonical forms of the equivalence classes of Σ_m, and with K_{cm} the subset of K_c, whose elements are the canonical forms of the equivalence classes on S_{co}. The following theorem, analogous to Theorem 2.4, can be stated.

Theorem 2.7. For every canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ of C_{cm} there exists a strictly equivalent canonical pair $(\tilde{R}(z), \tilde{S}(z))$ of K_{cm}.

The proof follows directly from the properties of the elements of S_{co} [6] and from Theorem 2.4.

5. MINIMAL SYSTEMS AND CONCLUDING DISCUSSION

When systems that are both completely reachable and completely observable, i.e. minimal, are considered it is possible, according to strict equivalence Definitions 2.1 and 2.2, to represent their dynamical behavior by means of elements of Σ_m, S_{oc} and S_{co}. All the results that have been previously deduced for the elements of Σ_o, S_o, Σ_c and S_c are now simultaneously valid. This can be summarized by the diagram of Fig. 2.2.

In Fig. 2.2 τ denotes the transformation from canonical state-space representations

(2.2.20)–(2.2.22) to canonical state-space representations (1.4.20)–(1.4.22) and τ^{-1} the inverse transformation.

As has been pointed out by R. E. Kalman, Definition 1.7, even if general, does not associate any advantage with respect to all remaining elements of the same equi-

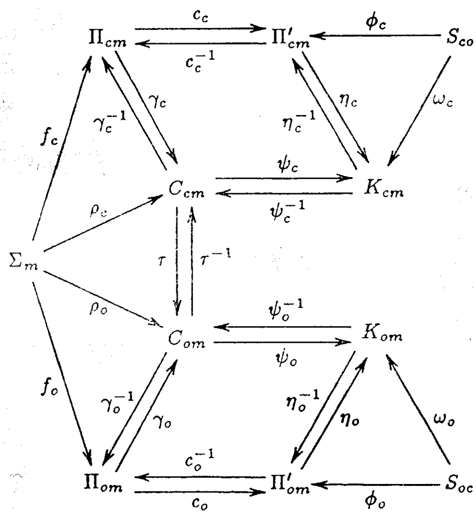

Fig. 2.2.

valence class to the elements of a canonical set for a given equivalence relation. This is due to the fact that Definition 1.7 considers as canonical with respect to a given equivalence relation on a set X, every subset C of X if and only if there is a one-to-one correspondence between the elements of C and the equivalence classes in X. This does not imply any particular selection among the elements of the equivalence classes to obtain the elements of C. The canonical forms considered in this work correspond, on the contrary, to a well-defined selection procedure and share nice structural and parametric properties. A restriction of Definition 1.7 that could be considered in connection with the problem of defining nice canonical forms for dynamical systems could be the following.

Definition 2.3. Let X be a set of models for dynamical systems and E an equivalence relation on X. A subset C of X will be called a set of nice canonical forms for E if and only if: 1) Every element of X is equivalent, under E, to one and only one element of C; 2) Every element of C is parametrized by the image in a complete set of independent invariants of every equivalent element of X; 3) The elements of C exhibit, in their structure, the Kronecker indices associated to the pair (F, G) or $(F^{\mathrm{T}}, H^{\mathrm{T}})$ of every strictly equivalent state-space model.

It must however be noted that condition 3) is somehow restrictive since some useful (and nice) canonical forms would be excluded (e.g. the Jordan form).

6. CONCLUDING REMARKS

The models considered in this paper refer to purely dynamical systems; the extension of the given results to systems where an algebraic input-output link is present is very simple and can be performed according to the lines followed, for instance, in [8].

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