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A VANISHING DISCOUNT LIMIT THEOREM FOR CONTROLLED MARKOV CHAINS

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Finite controlled Markov chains with discounted cost criterion are considered. It is proved that the average cost optimal control yields the stochastically smallest distribution of the discounted cost asymptotically as the discount rate tends to zero.

1. INTRODUCTION

Papers dealing with limit inequalities for the probability distributions of the total costs in controlled Markov processes as time tends to infinity were published recently (see [3], [5], [7]). In the present paper analogous inequalities are derived for the discounted cost criterion with low discount rate in finite controlled Markov chains. Martingale methods are used together with the Skorokhod representation of random variables by means of stopping of the Wiener process (see [4], [1]). Under additional hypotheses (see e.g. [6]) these methods can be applied also to chains with countable state space.

The total discounted cost is proved to have stochastically smallest probability distribution in the asymptotic sense for the average cost optimal stationary control. If we interpret the discounting as the means to relate a future payment to the present time, the paper gives asymptotic solution of the following problem. What is the smallest security A needed for the defrayment of random cost in the future with a given probability α. For low discount rate, i.e. for discount factor β near to 1, the solution is written in the form

$$A \sim \frac{\theta}{1-\beta} + \mathrm{u}_\alpha \sqrt{\frac{\Delta}{2(1-\beta)}}\,.$$

θ (resp. Δ) is the minimal average cost of the considered chain (resp. of an auxiliary chain) obtained by solving a quasi-linear (resp. linear) system of equations and u_α is the α-quantile of the standardized normal distribution.

2. PROBLEM FORMULATION

Consider a controlled Markov chain with finite set of states I, the evolution of which is defined by means of transition probabilities

$$p(i, k, z)\,, \quad z \in Z(i)\,, \quad i, k \in I\,.$$

z denotes the control parameter. Its range in state i, $Z(i)$, is assumed to be finite for $i \in I$. Further, let X_n be the state of the chain and Z_n be the control parameter value at time n. In general

$$Z_n = \mathbf{z}_n(X_0, \ldots, X_n)\,.$$

In the stationary case, Z_n is a function of X_n only,

(1) $$Z_n = \mathbf{z}(X_n)\,.$$

(1) will be called briefly stationary control $\mathbf{z}$.

To evaluate the trajectory $\{X_n\}$ and the control $\{Z_n\}$ introduce the discounted cost

$$C_\infty = \sum_{n=0}^{\infty} \beta^n\, c(X_n, X_{n+1}, Z_n)\,.$$

The function

$$c(i, k, z)\,, \quad z \in Z(i)\,, \quad i, k \in I\,,$$

gives the cost from transition $i \to k$ under parameter value z, β is the discount factor, $0 < \beta < 1$.

The well known Abel type arguments yield the connection between

$$(1 - \beta)\, C_\infty\,, \quad \beta \to 1\,,$$

and

(2) $$\frac{1}{N} \sum_{n=0}^{N} c(X_n, X_{n+1}, Z_n)\,, \quad N \to \infty\,.$$

We therefore base our considerations on the optimal stationary control with respect to the average cost criterion (2). We make the following hypothesis.

Assumption 1. For each stationary control $\mathbf{z}$ the matrix

$$\left\| p(i, k, \mathbf{z}(i)) \right\|_{i,k \in I}$$

is indecomposable.

Assumption 1 implies that $\{X_n\}$ is an ergodic Markov chain under any stationary control $\mathbf{z}$.

Let $\pi_i(\mathbf{z})$, $i \in I$, denote the stationary distribution of the chain. The corresponding average cost per transition equals

$$\theta(\mathbf{z}) = \sum_i \sum_j \pi_i(\mathbf{z})\, p(i, j, \mathbf{z}(i))\, c(i, j, \mathbf{z}(i))\,.$$

Set

$$\theta = \min_{\mathbf{z}} \theta(\mathbf{z})\,.$$

$\mathbf{z}$ is average cost optimal if $\theta = \theta(\mathbf{z})$. θ and $\mathbf{z}$ are obtained from the following optimality equation (see [2]). θ is the unique number such that constants w_i, $i \in I$, can be found so that

$$\min_{z \in Z(i)} \left[\sum_j p(i, j, z)\left(c(i, j, z) + w_j\right) - w_i - \theta\right] = 0\,, \quad i \in I\,. \tag{3}$$

Denote by $\varphi(i, z)$ the expression in the square brackets in (3). Under Assumption 1 $\mathbf{z}$ is average cost optimal if and only if

$$\varphi(i, \mathbf{z}(i)) = 0\,, \quad i \in I\,. \tag{4}$$

Assumption 2. The stationary control $\mathbf{z}$ satisfying (4) is unique.

In the following text $\mathbf{z}$ denotes the unique average cost optimal control.

3. THE SKOROKHOD REPRESENTATION OF A DISCRETE RANDOM VARIABLE

Let ξ be a random variable taking on $m + m'$ values, $\mathsf{E}\xi = 0$,

$$\mathsf{P}[\xi = y_i] = q_i\,, \quad i = 1, 2, \ldots, m\,,$$
$$\mathsf{P}[\xi = x_i] = p_i\,, \quad i = 1, 2, \ldots, m'\,,$$
$$x_{m'} \leqq \ldots \leqq x_1 < 0 \leqq y_1 \leqq \ldots \leqq y_m\,.$$

Proposition 1. Let $W(t)$ be a Wiener process independent of ξ. There exists a stopping time τ such that $W(\tau)$ has the same distribution as ξ. Further, it holds

$$\mathsf{E}\tau = \operatorname{var} \xi\,, \tag{5}$$

$$\mathsf{E}\tau^2 \leqq \text{const.}\ \mathsf{E}\xi^4\,. \tag{6}$$

Method of proof. Proposition 1 can be proved by applying Skorokhod's construction to the random variable

$$\xi_\varepsilon = \xi + \varepsilon\eta\,,$$

where η has uniform distribution, and by letting $\varepsilon \to 0+$.

Let us present some details regarding the limiting stopping time

$$\tau = \lim_{\varepsilon \to 0+} \tau_\varepsilon\,.$$

Set

$$e(y_i) = \sum_{j=1}^{i} q_j y_j\,, \qquad i = 1, \ldots, m\,,$$

$$h(x_i) = -\sum_{j=1}^{i} p_j x_j\,, \quad i = 1, \ldots, m'\,.$$

τ is the minimal root of the equation

$$(W(t) - \xi)(W(t) - \bar{\xi}) = 0\,,$$

where $\bar{\xi}$ is constructed from ξ by randomization.

Let, e.g., $\xi = y_i$ and

$$h(x_{j-1}) < e(y_{i-1}) < h(x_j) < h(x_{j+1}) < \dots < h(x_l) < \dots$$
$$\dots < h(x_{k-1}) < e(y_i) < h(x_k).$$

Then the conditional distribution of $\bar{\xi}$ is given by

$$\mathsf{P}[\bar{\xi} = x_j \mid \xi = y_i] = \frac{h(x_j) - e(y_{i-1})}{e(y_i) - e(y_{i-1})} = \frac{h(x_j) - e(y_{i-1})}{q_i y_i},$$

$$\mathsf{P}[\bar{\xi} = x_l \mid \xi = y_i] = \frac{h(x_l) - h(x_{l-1})}{e(y_i) - e(y_{i-1})} = -\frac{p_l x_l}{q_i y_i}, \quad l = j+1, \dots, k-1,$$

$$\mathsf{P}[\bar{\xi} = x_k \mid \xi = y_i] = \frac{e(y_i) - h(x_{k-1})}{e(y_i) - e(y_{i-1})} = \frac{e(y_i) - h(x_{k-1})}{q_i y_i}.$$

Taking into account that the definition of τ is analogous in other cases, we shall calculate $\mathsf{P}[W(\tau) = y_i]$. The event $W(\tau) = y_i$ is possible if ξ takes on one of the following values:

$$y_i, x_k, x_{k-1}, \dots, x_l, \dots, x_j.$$

Moreover, the probabilities of reaching first ξ or $\bar{\xi}$ by the Wiener process are inverse proportional to their moduluses. Consequently,

$$\begin{aligned}
\mathsf{P}[W(\tau) = y_i] &= q_i \frac{h(x_j) - e(y_{i-1})}{q_i y_i} \frac{-x_j}{y_i - x_j} + \\
&+ \sum_{l=j+1}^{k-1} q_i \frac{-p_l x_l}{q_i y_i} \frac{-x_l}{y_i - x_l} + q_i \frac{e(y_i) - h(x_{k-1})}{q_i y_i} \frac{-x_k}{y_i - x_k} + \\
&+ p_k \frac{h(x_{k-1}) - e(y_i)}{-p_k x_k} \frac{-x_k}{y_i - x_k} + \sum_{l=j+1}^{k-1} p_l \frac{-x_l}{y_i - x_l} + \\
&+ p_j \frac{h(x_j) - e(y_{i-1})}{-p_j x_j} \frac{-x_j}{y_i - x_j} = \\
&= \frac{h(x_j) - e(y_{i-1})}{y_i} \left(\frac{-x_j}{y_i - x_j} + \frac{y_i}{y_i - x_j} \right) + \\
&+ \sum_{l=j+1}^{k-1} \frac{-p_l x_l}{y_i} \left(\frac{-x_l}{y_i - x_l} + \frac{y_i}{y_i - x_l} \right) + \\
&+ \frac{e(y_i) - h(x_{k-1})}{y_i} \left(\frac{-x_k}{y_i - x_k} + \frac{y_i}{y_i - x_k} \right) = \\
&= \frac{h(x_j) - e(y_{i-1}) + h(x_{k-1}) - h(x_j) + e(y_i) - h(x_{k-1})}{y_i} = \frac{q_i y_i}{y_i} = q_i. \quad \square
\end{aligned}$$

4. AUXILIARY MARTINGALES

Consider a general control $\{Z_n\}$. The investigation of the asymptotic behaviour of the cost C_∞ is performed using two martingales. Introduce the discounted cost up to time N,

$$C_N = \sum_{n=0}^{N-1} \beta^n c(X_n, X_{n+1}. Z_n) .$$

Let $\mathscr{F}_n$ be the Borel field of random events defined in terms of $X_0, X_1, \ldots, X_n$. The first martingale with respect to $\{\mathscr{F}_N\}$ is

$$ {}^1M_N = \sqrt{(1-\beta)}\left[C_N - \theta\frac{\beta^N - 1}{\beta - 1} + w_{X_N}\beta^{N-1} - w_{X_0} + \right.$$

$$\left. + (1-\beta)\sum_{n=1}^{N-1} w_{X_n}\beta^{n-1} - \sum_{n=0}^{N-1}\varphi(X_n, Z_n)\,\beta^n\right], \quad N = 1, 2, \ldots, ,$$

where $\varphi(i, z)$, w_i, $i \in I$, θ were introduced in Section 2. It holds

$$ {}^1M_N = \sum_{n=0}^{N-1} {}^1Y_n ,$$

$$ {}^1Y_n = \sqrt{(1-\beta)}\,[\beta^n c(X_n, X_{n+1}, Z_n) - \theta\beta^n + w_{X_{n+1}}\beta^n -$$
$$ - w_{X_n}\beta^n - \varphi(X_n, Z_n)\,\beta^n] =$$
$$ = \sqrt{(1-\beta)}\,[\beta^n c(X_n, X_{n+1}, Z_n) + w_{X_{n+1}}\beta^n -$$
$$ - \beta^n \sum_k p(X_n, k, Z_n)\,(c(X_n, k, Z_n) + w_k)] .$$

From here it is seen that

$$\mathsf{E}({}^1Y_n \mid \mathscr{F}_n) = 0 .$$

Further, it is computed that

$$\mathsf{E}({}^1Y_n^2 \mid \mathscr{F}_n) = \beta^{2n} c_2(X_n, Z_n)\,(1-\beta) , \quad n = 0, 1, \ldots,$$

where

$$c_2(i, z) = \sum_k p(i, k, z)\,(c(i, k, z) + w_k)^2 -$$
$$ - [\sum_k p(i, k, z)\,(c(i, k, z) + w_k)]^2 , \quad i \in I , \quad z \in Z(i) .$$

Denote

$$\bar{C}_N = \sum_{n=0}^{N-1} \beta^{2n} c_2(X_n, Z_n) .$$

We shall associate to $\{\bar{C}_N\}$ an analogous martingale $\{\bar{M}_N\}$ as in the case of $\{C_N\}$. . $\{{}^2M_N\}$ will be a sum of $\{\bar{M}_N\}$ and of another martingale $\{M_N^*\}$.

Let the constants Δ, v_i, $i \in I$, fulfil

$$\Delta = \sum_i \sum_j \pi_i(\mathbf{z})\, p(i, j, \mathbf{z}(i))\, c_2(i, j, \mathbf{z}(i)) ,$$

$$\sum_j p(i, j, \mathbf{z}(i))\,(c_2(i, j, \mathbf{z}(i)) + v_j) - v_i - \Delta = 0 , \quad i \in I .$$

Set

$$\psi(i, z) = \sum_j p(i, j, z) (c_2(i, j, z) + v_j) - v_i - \Delta, \quad z \in Z(i), \quad i \in I.$$

We define

$$\bar{M}_N = \sum_{n=0}^{N-1} \bar{Y}_n = (1 - \beta) \Big[\bar{C}_N - \Delta \frac{\beta^{2N} - 1}{\beta^2 - 1} +$$

$$+ v_{X_N} \beta^{2(N-1)} - v_{X_0} + (1 - \beta^2) \sum_{n=0}^{N-1} v_{X_n} \beta^{2(n-1)} - \sum_{n=0}^{N-1} \psi(X_n, Z_n) \beta^{2n} \Big],$$

$$N = 1, 2, \ldots.$$

Letting $N \to \infty$ it follows

(7) $$^1M_\infty = \sum_{n=0}^{\infty} {}^1Y_n = \sqrt{(1 - \beta)} \left(C_\infty - \frac{\theta}{1 - \beta} \right) +$$

$$+ \sqrt{(1 - \beta)} \left[-w_{X_0} + (1 - \beta) \sum_{n=1}^{\infty} w_{X_n} \beta^{n-1} - \sum_{n=0}^{\infty} \varphi(X_n, Z_n) \beta^n \right],$$

(8) $$\bar{M}_\infty = \sum_{n=0}^{\infty} \bar{Y}_n = (1 - \beta) \left(\bar{C}_\infty - \frac{\Delta}{1 - \beta^2} \right) +$$

$$+ (1 - \beta) \left[-v_{X_0} + (1 - \beta^2) \sum_{n=1}^{\infty} v_{X_n} \beta^{2(n-1)} - \sum_{n=0}^{\infty} \psi(X_n, Z_n) \beta^{2n} \right].$$

Applying successively the Skorokhod representation to the martingale differences of $\{{}^1M_N\}$ we obtain

(9) $$^1M_\infty = \sum_{n=0}^{\infty} {}^1Y_n = W\Big(\sum_{n=0}^{\infty} \tau_n\Big),$$

where τ_n is the stopping time corresponding to the martingale difference ${}^1Y_n$, and it holds according to (5)

$$\mathsf{E}(\tau_n \mid \mathscr{F}_n) = \mathsf{E}({}^1Y_n^2 \mid \mathscr{F}_n) = \beta^{2n}(1 - \beta)\, c_2(X_n, Z_n).$$

Further, let

$$M_N^* = \sum_{n=0}^{N-1} [\tau_n - \mathsf{E}(\tau_n \mid \mathscr{F}_n)] = \sum_{n=0}^{N-1} \tau_n - \sum_{n=0}^{N-1} \beta^{2n}(1 - \beta)\, c_2(X_n, Z_n) =$$

$$= \sum_{n=0}^{N-1} \tau_n - (1 - \beta) \bar{C}_N, \quad N = 1, 2, \ldots.$$

Using (5), (6) it can be verified that

(10) $$\mathsf{E}(M_\infty^*)^2 = \mathsf{E}\Big[\sum_{n=0}^{\infty} (\tau_n - \mathsf{E}(Y_n^2 \mid \mathscr{F}_n))^2\Big] \leqq (1 - \beta)^2 \text{ const.} \sum_{n=0}^{\infty} \beta^{4n}.$$

From (8) it follows

(11) $$\mathsf{E}(\bar{M}_\infty)^2 \leqq (1 - \beta)^2 \text{ const.} \sum_{n=0}^{\infty} \beta^{4n}.$$

By Assumption 2 $\varphi(i, z) > 0$ for $z \neq \mathbf{z}(i)$. Consequently,

$$|\psi(i, z)| \leqq \text{const.}\, \varphi(i, z)\,, \quad z \in Z(i)\,, \quad i \in I\,. \tag{12}$$

Finally, define

$$^2M_\infty = M_\infty^* + \overline{M}_\infty = \sum_{n=0}^{\infty} \tau_n - \frac{\Delta}{1+\beta} + \tag{13}$$

$$+ (1-\beta)\left(-v_{X_0} + (1-\beta^2)\sum_{n=1}^{\infty} v_{X_n}\beta^{2(n-1)}\right) - (1-\beta)\sum_{n=0}^{\infty}\psi(X_n, Z_n)\,\beta^{2n}\,.$$

5. STATEMENT OF RESULTS

Φ will denote the distribution function of the standardized normal distribution.

Proposition 2. Let Assumptions 1,2 hold. Under arbitrary control $\{Z_n\}$

$$\limsup_{\beta\to 1} \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \leqq \Phi\left(\frac{y}{\sqrt{(\Delta/2)}}\right), \quad y \in (-\infty, \infty)\,. \tag{14}$$

Proof. According to (7) and (9) it holds for $\delta > 0$

$$\mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \leqq \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y + \delta\right] + \tag{15}$$

$$+ \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) - \sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) > \delta\right] =$$

$$= \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y + \delta\right] + \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) - W\left(\sum_{n=0}^{\infty}\tau_n\right) > \delta + \right.$$

$$\left. + \sqrt{(1-\beta)}\left(w_{X_0} - (1-\beta)\sum_{n=1}^{\infty} w_{X_n}\beta^{n-1}\right) + \sqrt{(1-\beta)}\sum_{n=0}^{\infty}\varphi(X_n, Z_n)\,\beta^n\right].$$

Further, as $\beta \to 1$,

$$\mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y + \delta\right] \to \mathsf{P}\left[W\left(\frac{\Delta}{2}\right) \leqq y + \delta\right]. \tag{16}$$

Now, we prove the neglibility of the second probability on the right-hand side of (15). For β close to 1 this probability is majorized using (13) by

$$\varepsilon + \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) - W\left(\sum_{n=0}^{\infty}\tau_n\right) > \frac{\delta}{2} + \sqrt{(1-\beta)}\sum_{n=0}^{\infty}\varphi\beta^n\,;\right. \tag{17}$$

$$\left.\left|\frac{\Delta}{1+\beta} - \sum_{n=0}^{\infty}\tau_n\right| \leqq |^2M_\infty| + \varepsilon + \left|(1-\beta)\sum_{n=0}^{\infty}\psi\beta^{2n}\right|\right], \quad \varepsilon > 0\,.$$

(10), (11) imply

$$\lim_{\beta\to 1} \mathsf{E}(^2M_\infty)^2 = 0\,. \tag{18}$$

From here and from (12) it follows that (17) can be further estimated by

$$(19)\quad 2\varepsilon + \mathsf{P}\Big[\sup_{|\frac{1}{2}\Delta - t| \leqq 2\varepsilon + \text{const.}(1-\beta)\Sigma\varphi\beta^{2n}} (W(\tfrac{1}{2}\Delta) - W(t)) > \tfrac{1}{2}\delta + \sqrt{(1-\beta)}\sum \varphi\beta^n\Big] \leqq$$

$$2\varepsilon + \sum_{j=0}^{\infty} \mathsf{P}\Big[\sup_{|\Delta\frac{1}{2} - t| \leqq 2\varepsilon + \text{const.}(j+1)\sqrt{(1-\beta)}} (W(\tfrac{1}{2}\Delta) - W(t)) > \tfrac{1}{2}\delta + j\Big] \leqq$$

$$2\varepsilon + \sum_{j=0}^{\infty} 4\left[1 - \Phi\left(\frac{\delta/2 + j}{(2\varepsilon + \text{const.}\,(j+1)\sqrt{(1-\beta)})^{1/2}}\right)\right].$$

In the last step we used the well known relation for the Wiener process (see [1] § 1.3)

$$\mathsf{P}\big[\sup_{0 \leqq t \leqq b} W(t) > a\big] = 2\left(1 - \Phi\left(\frac{a}{\sqrt{b}}\right)\right).$$

The last term in (19) converges to zero as $\beta \to 1$, $\varepsilon \to 0$.

From (16) we conclude that

$$\limsup_{\beta \to 1} \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \leqq \mathsf{P}\big[W(\tfrac{1}{2}\Delta) \leqq y + \delta\big]$$

and letting $\delta \to 0$ we get (14). □

Proposition 3. Let Assumptions 1,2 hold and let

$$(20)\qquad \lim_{\beta \to 1} \sqrt{(1-\beta)} \sum_{n=0}^{\infty} \varphi(X_n, Z_n)\, \beta^n = 0 \quad \text{in prob.}$$

Then

$$\lim_{\beta \to 1} \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] = \Phi\left(\frac{y}{\sqrt{(\Delta/2)}}\right), \quad y \in (-\infty, \infty).$$

Proof. With regard to Proposition 2 we have to verify

$$\liminf_{\beta \to 1} \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \geqq \Phi\left(\frac{y}{\sqrt{(\Delta/2)}}\right).$$

From (7) and from (9) it follows for $\delta > 0$

$$(21)\quad \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \geqq \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y - \delta\right] -$$

$$- \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) - W\left(\frac{\Delta}{1+\beta}\right) > \delta\right] =$$

$$= \mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y - \delta\right] - \mathsf{P}\left[W\left(\sum_{n=0}^{\infty} \tau_n\right) - W\left(\frac{\Delta}{1+\beta}\right) \geqq\right.$$

$$\geqq \delta - w_{X_0}\sqrt{(1-\beta)} + \sqrt{(1-\beta)}\,(1-\beta)\sum_{n=1}^{\infty} w_{X_n}\beta^{n-1} -$$

$$\left. - \sqrt{(1-\beta)} \sum_{n=0}^{\infty} \varphi(X_n, Z_n)\, \beta^n\right].$$

As $\beta \to 1$,

$$\mathsf{P}\left[W\left(\frac{\Delta}{1+\beta}\right) \leqq y - \delta\right] \to \mathsf{P}[W(\tfrac{1}{2}\Delta) \leqq y - \delta] .$$

The second probability on the right-hand side of (21) will be proved to be negligible. With regard to assumption (20) it holds for $\varepsilon > 0$

$$\mathsf{P}[\sqrt{(1-\beta)} \sum_{n=0}^{\infty} \varphi(X_n, Z_n) \geqq \tfrac{1}{2}\delta] < \varepsilon .$$

Moreover, using (13) and neglecting small terms it is seen that the last probability in (21) does not exceed for β close to 1

$$\varepsilon + \mathsf{P}\left[W(\textstyle\sum\tau_n) - W\left(\dfrac{\Delta}{1+\beta}\right) \geqq \tfrac{1}{2}\delta ; \quad \left|\textstyle\sum\tau_n - \dfrac{\Delta}{1+\beta}\right| \leqq \right.$$
$$\left. |^2M_\infty| + \varepsilon + |(1-\beta) \textstyle\sum\psi\beta^{2n}|\right] . \tag{22}$$

Finally, (12), (20), and (18) are used to majorize (22) by

$$2\varepsilon + \mathsf{P}[\sup_{|t-\frac{1}{2}\Delta| \leqq 2\varepsilon} (W(t) - W(\tfrac{1}{2}\Delta)) \geqq \tfrac{1}{2}\delta] \leqq 2\varepsilon + 4[1 - \Phi(\delta/(4\varepsilon))] .$$

As $\varepsilon \to 0$, $2\varepsilon + 4[1 - \Phi(\delta/(4\varepsilon))]$ converges to zero.

Letting $\delta \to 0$ we infer that

$$\liminf_{\beta \to 1} \mathsf{P}\left[\sqrt{(1-\beta)}\left(C_\infty - \frac{\theta}{1-\beta}\right) \leqq y\right] \geqq \mathsf{P}[W(\tfrac{1}{2}\Delta) \leqq y] = \Phi\left(\frac{y}{\sqrt{(\Delta/2)}}\right) . \quad \square$$

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