MULTIDIMENSIONAL RANDOM PROCESSES WITH NORMAL COVARIANCES

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The definition and basic properties of multidimensional locally stationary and normal covariance functions are given. Necessary and sufficient conditions characterizing these covariance functions are presented and a close connection with normal operators is shown too.

1. INTRODUCTION

Let $\{x(t),\ t \in \mathbb{R}_1\}$ be a second-order random process with vanishing mean and a covariance function $R(\cdot, \cdot)$. Silverman suggested in [8] a generalization of weak stationarity, named by local stationarity, in the following way. A covariance function $R(\cdot, \cdot)$ is called locally stationary if for every pair s, t of reals $(s, t \in \mathbb{R}_1)$

$$R(s, t) = R^{(1)}\left(\frac{s+t}{2}\right) R^{(2)}(s - t)$$

where $R^{(2)}(\cdot)$ is a weakly stationary covariance. Thanks to the facts that $R(s, s) \geqq 0$ for every $s \in \mathbb{R}_1$ and $R^{(2)}(0) \geqq 0$ this definition yields $R^{(1)}(s) \geqq 0$ for every $s \in \mathbb{R}_1$. The definition of local stationarity for random sequences is given in [4]. In this case a covariance function $R(\cdot, \cdot)$, defined on $\mathbb{Z} \times \mathbb{Z}$ (Cartesian product of integers), can be expressed as

$$R(n, m) = R^{(1)}(n + m)\, R^{(2)}(n - m)$$

where $R^{(2)}(\cdot)$ is a stationary covariance. Here, the function $R^{(1)}(\cdot)$ need not be nonnegative.

Under assumption of continuity of $R^{(1)}(\cdot)$, $R^{(2)}(\cdot)$ and nonnegative-definite property of $R^{(1)}(\cdot)$, in the case of a random process, the corresponding locally stationary covariance function can be written in the form

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{sz + t\bar{z}}\, dF_1(\lambda)\, dF_2(\mu)\,, \quad (z = \lambda + i\mu\,, \quad \bar{z} = \lambda - i\mu)\,,$$

as it is shown in [5]. This expression is a special case of a normal covariance function introduced and investigated in [5], [6]. For completeness, we present the definition here.

Definition 1. A covariance function $R(\cdot, \cdot)$ defined on the plane is said to be normal if for every s, t of reals

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{sz + t\bar{z}} \, dd F(\lambda, \mu), \quad z = \lambda + i\mu,$$

where $F(., .)$ is the distribution function corresponding to a bounded nonnegative measure on the Borel sets in the plane.

The definition of a normal covariance function due to a random sequence is given in [4]. The main aim of this paper is to give the definition of multidimensional locally stationary and normal covariance functions together with presenting necessary and sufficient conditions describing these classes. A close connection with groups of normal operators in a Hilbert space is also given.

2. MULTIDIMENSIONAL LOCAL STATIONARITY

Let $\boldsymbol{x}^T(t) = \{(x_1(t), x_2(t), \ldots, x_N(t), \; t \in \mathbb{R}_1\}$ be a multidimensional second order random process with vanishing mean value. Let

$$\boldsymbol{R}(s, t) = \mathsf{E}\{\boldsymbol{x}(s)\, \boldsymbol{x}^T(t)\}$$

be the corresponding covariance function.

Definition 2. We say the process $\boldsymbol{x}^T(\cdot)$ is locally stationary (or its covariance function $\boldsymbol{R}(\cdot, \cdot)$ is locally stationary) if for every N-tuple $\boldsymbol{z}^T = (z_1, z_2, \ldots z_N)$ of complex numbers the random process

$$\xi_z(t) = \sum_{i=1}^{N} z_i \, x_i(t), \quad t \in \mathbb{R}_1$$

has a locally stationary covariance function.

Lemma 1. If an N-dimensional covariance function $\boldsymbol{R}(\cdot, \cdot)$ is locally stationary then for every $u \in \mathbb{R}_1$ the matrix $\boldsymbol{R}(u, u)$ is positive semidefinite and the matrix $\boldsymbol{R}(s - t, t - s)$ is an N-dimensional stationary covariance function.

Proof. If $\boldsymbol{R}(\cdot, \cdot)$ is locally stationary then $\boldsymbol{z}^T \boldsymbol{R}(s, t)\, \boldsymbol{z}$ is for every $\boldsymbol{z}^T = (z_1, \ldots, z_N)$ a one-dimensional local stationary covariance. Then, according to the definition of local stationarity,

$$\boldsymbol{z}^T \boldsymbol{R}(s, t)\, \boldsymbol{z} = R_z^{(1)}\left(\frac{s + t}{2}\right) R_z^{(2)}(s - t).$$

This fact means $\boldsymbol{z}^T \boldsymbol{R}(u, u)\, \boldsymbol{z} = R_z^{(1)}(u)\, R_z^{(2)}(0)$ and

$$\boldsymbol{z}^T \boldsymbol{R}\left(\frac{v}{2}, \frac{-v}{2}\right) \boldsymbol{z} = R_z^{(1)}(0)\, R_z^{(2)}(v).$$

Hence,

$$z^{\mathrm{T}} \boldsymbol{R}(0,0)\, z z^{\mathrm{T}} \boldsymbol{R}(s,t)\, z = z^{\mathrm{T}} \boldsymbol{R}\left(\frac{s+t}{2}, \frac{s+t}{2}\right) z z^{\mathrm{T}} \boldsymbol{R}\left(\frac{s-t}{2}, \frac{t-s}{2}\right) z$$

where $R_z^{(1)}(0)\, R_z^{(2)}(0) = z^{\mathrm{T}} \boldsymbol{R}(0,0)\, z$. As local stationarity demands $R_z^{(1)}(u) \geqq 0$ for every $u \in \mathbb{R}_1$, and $R_z^{(2)}(v)$ must be a stationary covariance, we obtain that for every z

$$z^{\mathrm{T}} \boldsymbol{R}(u,u)\, z \geqq 0 \quad \text{and} \quad \boldsymbol{R}\left(\frac{s-t}{2}, \frac{t-s}{2}\right)$$

is an N-dimensional stationary covariance function. □

Theorem 1. An N-dimensional covariance function $\boldsymbol{R}(\cdot,\cdot)$ is locally stationary if and only if for every $s, t \in \mathbb{R}_1$ and every multiindex $\alpha = (i,j,k,l) \in \{1,2,3,\ldots,N\}^4$

$$R_{ij}(0,0)\, R_{kl}(s,t) + R_{il}(0,0)\, R_{kj}(s,t) + R_{kj}(0,0)\, R_{il}(s,t) + \\ + R_{kl}(0,0)\, R_{ij}(s,t) =$$

$$= R_{ij}\left(\frac{s+t}{2}, \frac{t+s}{2}\right) R_{kl}\left(\frac{s-t}{2}, \frac{t-s}{2}\right) + R_{il}\left(\frac{s+t}{2}, \frac{s+t}{2}\right) R_{kj}\left(\frac{s-t}{2}, \frac{t-s}{2}\right) +$$

$$+ R_{kj}\left(\frac{s+t}{2}, \frac{s+t}{2}\right) R_{il}\left(\frac{s-t}{2}, \frac{t-s}{2}\right) + R_{kl}\left(\frac{s+t}{2}, \frac{t+s}{2}\right) R_{ij}\left(\frac{s-t}{2}, \frac{t-s}{2}\right).$$

where $\boldsymbol{R}(\cdot,\cdot) = \{R_{ij}(\cdot,\cdot)\}_{i,j=1}^{N}$.

Before proving Theorem 1 it is suitable to introduce the following

Lemma 2. Let V_n be an n-dimensional complex vector modul and

$$\Phi\colon V_n \times V_n \times V_n \times V_n \to \mathbb{C}$$

be a mapping having the form

$$\Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} u_i v_j \bar{x}_k \bar{y}_l \Phi_{ijkl}$$

where $\boldsymbol{u} = \sum_1^n u_i \boldsymbol{e}_i$, $\boldsymbol{v} = \sum_1^n v_j \boldsymbol{e}_j$, $\boldsymbol{x} = \sum_{k=1}^n x_k \boldsymbol{e}_k$, $\boldsymbol{y} = \sum_{l=1}^n y_l \boldsymbol{e}_l$, and $\Phi_{ijkl} = \Phi(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k, \boldsymbol{e}_l)$ for a fixed basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of V_n. Then Φ is vanishing on the principal diagonal (i.e. $\Phi(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0$ for every $\boldsymbol{u} \in V_n$) if and only if

$$\Phi_{ijkl} + \Phi_{jikl} + \Phi_{ijlk} + \Phi_{jilk} = 0 \tag{1}$$

for every $i, j, k, l = 1, 2, \ldots, n$.

Proof of Lemma 2. Let the condition (1) hold. Then for every $x \in V_n$, $x = \sum_1^n x_i \boldsymbol{e}_i$,

$$4\Phi(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) = \sum_i \sum_j \sum_k \sum_l x_i x_j \bar{x}_k \bar{x}_l [\Phi_{ijkl} + \Phi_{jikl} + \Phi_{jilk} + \Phi_{ijkl}) = 0\,.$$

Hence, Φ is vanishing on the principal diagonal. Now, assume $\Phi(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) = 0$

for every $x \in V_n$. Then $\Phi(x + t\,\mathrm{e}^{\mathrm{i}\omega}y,\ x + t\,\mathrm{e}^{\mathrm{i}\omega}y, x + t\,\mathrm{e}^{\mathrm{i}\omega}y, x + t\,\mathrm{e}^{\mathrm{i}\omega}y)$ for x, $y \in V_n$ and real t, ω presents a polynomial function of the 4th degree in t having complex coefficients and vanishing everywhere. The coefficient standing by t must satisfy

$$[\Phi(y, x, x, x) + \Phi(x, y, x, x)]\,\mathrm{e}^{\mathrm{i}\omega} + [\Phi(x, x, y, x) + \Phi(x, x, x, y)]\,\mathrm{e}^{-\mathrm{i}\omega} = 0\,.$$

Hence, $\Phi'_y(x) = \Phi(y, x, x, x) + \Phi(x, y, x, x) = 0$ for every x, $y \in V_n$. Now, we shall repeat this consideration twice. First, the coefficient by t in the term $\Phi'_y(x + t\,\mathrm{e}^{\mathrm{i}\omega}y)$ equals

$$[\Phi(y, z, x, x) + \Phi(z, y, x, x)]\,\mathrm{e}^{\mathrm{i}\omega} + [\Phi(y, x, x, z) + \Phi(y, x, z, x) + \\ + \Phi(x, y, x, z) + \Phi(x, y, z, x)]\,\mathrm{e}^{-\mathrm{i}\omega} = 0\,.$$

This fact gives

$$\Phi(y, z, x, x) + \Phi(z, y, x, x) = \Phi''_{y,z}(x) = 0$$

for every x, y, $z \in V_n$. Finally, the expression of $\Phi''_{y,z}(x + t\,\mathrm{e}^{\mathrm{i}\omega}u)$ yields immediately for every x, y, z, $u \in V_n$

$$\Phi(y, z, x, u) + \Phi(y, z, u\ x) + \Phi(z, y, x, u) + \Phi(z, y, u, x) = 0\,.$$

This implies easily condition (1). □

Now, the proof of Theorem 1 is an easy matter.

Proof of Theorem 1. Let an N-dimensional random process $\{x(t), t \in \mathbb{R}_1\}$ be locally stationary. It means that for every $z^{\mathrm{T}} = (z_1, \ldots, z_N)$, an N-couple of complex numbers, and every s, $t \in \mathbb{R}_1$

$$z^{\mathrm{T}} R(0, 0)\, z z^{\mathrm{T}} R(s, t)\, z = z^{\mathrm{T}} R\left(\frac{s+t}{2}, \frac{s+t}{2}\right) z z^{\mathrm{T}} R\left(\frac{s-t}{2}, \frac{t-s}{2}\right) z\,.$$

This equality may be rewritten into the following form

$$0 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} z_i z_j \bar{z}_k \bar{z}_l \Bigg(R_{ij}(0, 0)\, R_{kl}(s, t) - \\ - R_{ij}\left(\frac{s+t}{2}, \frac{s+t}{2}\right) R_{kl}\left(\frac{s-t}{2}, \frac{t-s}{2}\right)\Bigg).$$

At this moment, we can apply Lemma 2 to the function

$$\Phi(u, v, x, y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} u_i v_j \bar{x}_k \bar{y}_l \Bigg(R_{ij}(0, 0)\, R_{kl}(s, t) - \\ - R_{ij}\left(\frac{s+t}{2}, \frac{s+t}{2}\right) R_{kl}\left(\frac{s-t}{2}, \frac{t-s}{2}\right)\Bigg).$$

□

Silverman in [8] proved an assertion dealing with harmonizable locally stationary random processes. This result can be generalized to the multidimensional case.

Theorem 2. Let $\{x(t), t \in \mathbb{R}_1\}$ be an N-dimensional random process with harmonizable (in the strong sense) locally stationary covariance function having a spectral density function. Then, this spectral density function is locally stationary and vice versa.

Proof. Being strongly harmonizable $\{x(t), t \in \mathbb{R}_1\}$ can be expressed in the following form

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$$

where $\{\xi(\lambda), \lambda \in \mathbb{R}_1\}$ is an N-dimensional second-order random process with covariance function

$$F(\lambda, \mu) = \{\mathsf{E}\{\xi_i(\lambda) \, \bar{\xi}_j(\mu)\}\}_{i,j=1}^{N}$$

possessing finite variation $\sum_{i=1}^{N} \sum_k \sum_l |\Delta\Delta F_{ii}(\lambda_k, \mu_l)| \leqq C < \infty$ $(F_{ij}(\lambda, \mu) = \mathsf{E}\{\xi_i(\lambda) \, \bar{\xi}_j(\mu)\})$. Then, the covariance function of $\{x(t), t \in \mathbb{R}_1\}$ can be written as

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} f(\lambda, \mu) \, d\lambda \, d\mu$$

because we assume existence of $\partial^2 F_{ij}(\lambda, \mu)/\partial\lambda \, \partial\mu = f_{ij}(\lambda, \mu)$. As $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary, then by definition, for every $z^T = (z_1, z_2, \ldots, z_N)$

$$z^T R(s, t) \, z = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} z^T f(\lambda, \mu) \, z \, d\lambda \, d\mu$$

must be a locally stationary covariance function. The inverse formula, see [3], gives under local stationarity of $R(\cdot, \cdot)$.

$$z^T f(\lambda, \mu) \, z = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} e^{i(\lambda s - \mu t)} z^T R(s, t) \, z \, ds \, dt =$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp\left(-i\left(\frac{s+t}{2}\right)(\lambda - \mu)\right) \exp\left(-i(s-t)\left(\frac{\lambda+\mu}{2}\right)\right) z^T R(s, t) \, z \, ds \, dt =$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp\left(-i\left(\frac{\lambda+\mu}{2}\right)(s-t)\right) \frac{z^T R\left(\frac{s-t}{2}, \frac{t-s}{2}\right) z}{R_z^{(1)}(0)} \times$$

$$\times \exp\left(-i(\lambda - \mu)\left(\frac{s+t}{2}\right)\right) \frac{z^T R\left(\frac{s+t}{2}, \frac{s+t}{2}\right) z}{R_z^{(2)}(0)} \, ds \, st =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-i\left(\frac{\lambda+\mu}{2}\right) v\right) \frac{z^T R\left(\frac{v}{2}, \frac{v}{2}\right) z}{R_z^{(1)}(0)} \, dv \times \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\lambda-\mu)u} \frac{z^T R(u, u) \, z}{R_z^{(2)}(0)} \, du \, .$$

This means

$$z^T f(\lambda, \mu) \, z = f_z^{(1)}\left(\frac{\lambda+\mu}{2}\right) f_z^{(2)}(\lambda - \mu)$$

where

$$f_z^{(1)}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixv} \frac{z^T R\left(\frac{v}{2}, \frac{v}{2}\right) z}{R_z^{(1)}(0)} dv$$

and

$$f_z^{(2)}(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iyu} \frac{z^T R(u, u)\, z}{R_z^{(2)}(0)} du .$$

We have proved that for every $z^T = (z_1, z_2, \ldots, z_N)$ the covariance function $z^T f(\cdot, \cdot)\, z$ is locally stationary because

$$f_z^{(1)}(x) \geqq 0$$

for every $x \in \mathbb{R}_1$ and $f_z^{(2)}(\cdot)$ is a weakly stationary covariance function. We can summarize that the N-dimensional covariance function $f(\cdot, \cdot)$ is locally stationary. Now, assume $f(\cdot, \cdot)$ to be an N-dimensional locally stationary covariance function. Then, $z^T f(\cdot, \cdot)\, z$ is locally stationary for every $z^T = (z_1, z_2, \ldots, z_N)$, i.e.

$$z^T f(\lambda, \mu)\, z z^T f(0, 0)\, z = z^T f\left(\frac{\lambda + \mu}{2}, \frac{\lambda + \mu}{2}\right) z z^T f\left(\frac{\lambda - \mu}{2}, \frac{\mu - \lambda}{2}\right) z .$$

Hence,

$$z^T R(s, t)\, z = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} z^T f(\lambda, \mu)\, z \,d\lambda\, d\mu =$$

$$= \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \frac{z^T f\left(\frac{\lambda + \mu}{2}, \frac{\lambda + \mu}{2}\right) z}{f_z^{(2)}(0)} \frac{z^T f\left(\frac{\lambda - \mu}{2}, \frac{\mu - \lambda}{2}\right) z}{f_z^{(1)}(0)} d\lambda\, d\mu =$$

$$= \int_{-\infty}^{+\infty} \exp\left(i\left(\frac{s + t}{2}\right) v\right) \frac{z^T f\left(\frac{v}{2}, \frac{v}{2}\right) z}{f_z^{(1)}(0)} dv \int_{-\infty}^{+\infty} e^{i(s-t)u} \frac{z^T f(u, u)\, z\, du}{f_z^{(2)}(0)} =$$

$$= R_z^{(1)}\left(\frac{s + t}{2}\right) R_z^{(2)}(s - t) .$$

It is easy to see that $R_z^{(1)}(\cdot) \geqq 0$ and $R_z^{(2)}(\cdot)$ is a weakly stationary covariance. We proved local stationarity of $z^T R(\cdot, \cdot)\, z$ hence, the process $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary. □

Theorem 2 affirms, roughly speaking, that the Fourier transform of a locally stationary process is a locally stationary one again.

2. MULTIDIMENSIONAL NORMAL COVARIANCES

Let us suppose that an N-dimensional covariance function $R(\cdot, \cdot)$ is locally stationary, i.e. one can write

$$z^T R(0, 0)\, z\, z^T R(s, t)\, z = z^T R\left(\frac{s + t}{2}, \frac{s + t}{2}\right) z\, z^T R\left(\frac{s - t}{2}, \frac{t - s}{2}\right) z$$

for every $z^{\mathrm{T}} = (z_1, z_2, \ldots, z_N)$ of complex numbers and every $s, t \in \mathbb{R}_1$. In general, $R(\frac{1}{2}(s+t), \frac{1}{2}(s+t))$ need not be an N-dimensional covariance function in s, t, it is a positive semidefinite matrix for every fixed s, t as it is proved in Lemma 1. Now, let $R(\frac{1}{2}(s+t), \frac{1}{2}(s+t))$ be a covariance function and $R_{ij}(s, t)$, $i, j = 1, 2, \ldots, N$ be continuous functions on the plane. Then, for every z the function $z^{\mathrm{T}} R(\frac{1}{2}(s+t), \frac{1}{2}(s+t))\, z$ is a covariance with the kernel $(s+t)$, hence,

$$z^{\mathrm{T}} R\left(\frac{s+t}{2}, \frac{s+t}{2}\right) z = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{d}F_z(\lambda)$$

where $F_z(\cdot)$ is a nondecreasing function with finite variation, for detail see [9]. Analogously, by means of Bochner's theorem

$$z^{\mathrm{T}} R\left(\frac{s-t}{2}, \frac{t-s}{2}\right) z = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(s-t)\mu}\, \mathrm{d}G_z(\mu)\,.$$

Let us denote $e^{\mathrm{T}}(j, k) = (0, 0, \ldots, 1, \ldots, 1, \ldots, 0)$ if 1 stands on the jth and kth place $(j < k)$; similarly, $d^{\mathrm{T}}(j, k) = (0, \ldots, 1, \ldots, -\mathrm{i}, \ldots, 0)$. Then,

$$\text{(2)} \qquad e^{\mathrm{T}}(j, k)\, R(s, t)\, e(j, k) = R_{jj}(s, t) + R_{jk}(s, t) + R_{kj}(s, t) + R_{kk}(s, t)\,,$$
$$d^{\mathrm{T}}(j, k)\, R(s, t)\, d(j, k) = R_{jj}(s, t) - \mathrm{i}R_{jk}(s, t) + \mathrm{i}R_{kj}(s, t) + R_{kk}(s, t)\,.$$

The choice of $z_j^{\mathrm{T}} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 stands on the jth place, gives

$$z_j^{\mathrm{T}} R\left(\frac{s+t}{2}, \frac{s+t}{2}\right) z_j = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{d}F_j(\lambda)$$

and

$$z_j^{\mathrm{T}} R\left(\frac{s-t}{2}, \frac{t-s}{2}\right) z_j = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{d}G_j(\mu)\,.$$

This means, of course, that

$$\text{(3)} \qquad R_{jj}(0, 0)\, R_{jj}(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_j(\lambda)\, G_j(\mu)\,.$$

Similarly, for every $z^{\mathrm{T}} = (z_1, z_2, \ldots, z_N)$ local stationarity yields

$$z^{\mathrm{T}} R(0, 0)\, z z^{\mathrm{T}} R(s, t)\, z = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_z(\lambda)\, G_z(\mu)\,.$$

Especially,

$$\text{(4)} \qquad e^{\mathrm{T}}(j, k)\, R(0, 0)\, e(j, k)\, e^{\mathrm{T}}(j, k)\, R(s, t)\, e(j, k) =$$
$$= \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_{e(j,k)}(\lambda)\, G_{e(j,k)}(\mu)\,,$$

$$\text{(5)} \qquad d^{\mathrm{T}}(j, k)\, R(0, 0)\, d(j, k)\, d^{\mathrm{T}}(j, k)\, R(s, t)\, d(j, k) =$$
$$= \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_{d(j,k)}(\lambda)\, G_{d(j,k)}(\mu)\,.$$

Assuming regularity of the matrix $R(0, 0)$ and combining (2), (3), (4), (5) we obtain that

$$\text{(6)} \qquad R_{jk}(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}\left\{\frac{1}{2}\, \frac{F_{e(j,k)}(\lambda)\, G_{e(j,k)}(\mu)}{e^{\mathrm{T}}(j, k)\, R(0, 0)\, e(j, k)} + \right.$$

$$+\frac{\mathrm{i}}{2}\frac{F_{d(j,k)}(\lambda)\,G_{d(j,k)}(\mu)}{d^{\mathrm{T}}(j.k)\,R(0,0)\,d(j,k)}-\frac{1+\mathrm{i}}{2}\left(\frac{F_j(\lambda)\,G_j(\mu)}{R_{jj}(0,0)}+\frac{F_k(\lambda)\,G_k(\mu)}{R_{kk}(0,0)}\right)\Bigg\}\,.$$

We achieved a possibility to express the covariance function $R(\cdot,\cdot)$ in the form

$$R(s,t)=\iint_{-\infty}^{+\infty}\mathrm{e}^{\lambda(s+t)}\,\mathrm{e}^{\mathrm{i}\mu(s-t)}\,\mathrm{dd}F(\lambda,\mu) \tag{7}$$

where $F_{jk}(\lambda,\mu)$ is defined by the formula (6). Thanks to the fact that $z^{\mathrm{T}}R(s,t)\,z$ is a one-dimensional normal covariance function, under our assumptions,

$$z^{\mathrm{T}}R(s,t)\,z=\iint_{-\infty}^{+\infty}\mathrm{e}^{\lambda(s+t)}\,\mathrm{e}^{\mathrm{i}\mu(s-t)}\,\mathrm{dd}\frac{F_z(\lambda)\,G_z(\mu)}{z^{\mathrm{T}}R(0,0)\,z},$$

and thanks to the one-to-one correspondence between a normal covariance function and its spectral measure, see Theorem 3, we can assert

$$z^{\mathrm{T}}F(\lambda,\mu)\,z=\frac{F_z(\lambda)\,G_z(\mu)}{z^{\mathrm{T}}R(0,0)\,z}\,.$$

As for every $z^{\mathrm{T}}=(z_1,z_2,\ldots,z_N)$ of complex numbers

$$\Delta_{h_1}\Delta_{h_2}F_z(\lambda)\,G_z(\mu)\geqq 0$$

this inequality proves that $F(\cdot,\cdot)$ is a matrix spectral measure. $F(\cdot,\cdot)=\{F_{ij}(\cdot,\cdot)\}_{i,j=1}^{N}$ is a matrix spectral measure, see [7], if every component $F_{ij}(\cdot)$ is a complex measure defined on the Borel sets in the plane satisfying

1) $F_{ij}(\cdot)=\bar{F}_{ji}(\cdot)$ for every $i,j=1,2,\ldots,N$
2) $\sum_{i=1}^{N}\sum_{j=1}^{N}c_i\bar{c}_j\,F_{ij}(\Delta)\geqq 0$ for every N-tuple $c_1,c_2,\ldots,c_N$ of complex numbers and every Borel set Δ in the plane $\mathbb{R}_2$.

The spectral decomposition of $R(\cdot,\cdot)$ in the form (7) leads us to the following

Definition 3. An N-dimensional covaraince function $R(\cdot,\cdot)$ will be called normal if it can be expressed in the form

$$R(s,t)=\iint_{-\infty}^{+\infty}\mathrm{e}^{\lambda(s+t)}\,\mathrm{e}^{\mathrm{i}\mu(s-t)}\,\mathrm{dd}F(\lambda,\mu)\quad(\text{for every }(s,t)\in\mathbb{R}_2)$$

where $F(\cdot,\cdot)=\{F_{ij}(\cdot,\cdot)\}_{i,j=1}^{N}$ is a matrix spectral measure.

Properties of Normal Covariances

The existence of $R(s,t)$ for every pair $(s,t)\in\mathbb{R}_2$ implies

$$|R_{ij}(s,t)|\leqq\iint_{-\infty}^{+\infty}\mathrm{e}^{\lambda(s+t)}\,\mathrm{dd}|F_{ij}(\lambda,\mu)|\,,\quad i,j=1,2,\ldots,N\,,$$

where $|F_{ij}(\cdot)|$ is absolute variation of the complex measure $F_{ij}(\cdot)$ because the spectral measure F satisfies the evident relation

$$|F_{ij}(\Delta)|\leqq F_{ii}^{1/2}(\Delta)\,F_{jj}^{1/2}(\Delta) \tag{8}$$

thanks to positive semidefiniteness of $F(\cdot)$. As $F_{ii}(\Delta)\geqq 0$ for every $i=1,2,\ldots,N$

and every Borel set Δ in $\mathbb{R}_2$ we see that every integral

$$\iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, dd F_{ij}(\lambda, \mu), \quad i, j = 1, 2, \ldots, N$$

is absolutely convergent. The above relation (8) gives that every component $F_{ij}(\cdot)$ is of finite absolute variation because

$$\operatorname{Var} F_{ii}(\cdot) = F_{ii}(\mathbb{R}_2) = R_{ii}(0, 0)$$

is finite for every $i = 1, 2, \ldots, N$. Every component $F_{ij}(\cdot)$ can be expressed as the sum $\operatorname{Re} F_{ij}(\cdot) + i \operatorname{Im} F_{ij}(\cdot)$ where both the signed measures are of finite absolute variation. This fact implies that for every $i, j = 1, 2, \ldots, N$

$$(9) \qquad F_{ij}(\Delta) = \operatorname{Re} F_{ij}^{+}(\Delta) - \operatorname{Re} F_{ij}^{-}(\Delta) + i(\operatorname{Im} F_{ij}^{+}(\Delta) - \operatorname{Im} F_{ij}^{-}(\Delta))$$

where all the terms are measures with finite variations. As every one-dimensional normal covariance function is continuous at every point in the plane $\mathbb{R}_2$, see [6], $R_{ij}(\cdot, \cdot)$, which is a sum of normal covariances, cf. (9), must be a continuous function. We can state that every N-dimensional normal covariance function is continuous. Further, every normal covariance can be expressed as

$$\boldsymbol{R}(s, t) = \boldsymbol{S}(s + t, s - t)$$

where $\boldsymbol{S}(\cdot, \cdot) = \{S_{ij}(\cdot, \cdot)\}_{i,j=1}^{N}$ and

$$S_{ij}(u, v) = \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd F_{ij}(\lambda, \mu) .$$

Every function $S_{ij}(\cdot, \cdot)$ is continuous and $\boldsymbol{S}(u, -v) = \overline{\boldsymbol{S}^{T}(u, v)}$ where T means the transposed matrix.

Theorem 3. Every normal covariance function $\boldsymbol{R}(\cdot, \cdot)$ determines unambiguously a matrix spectral measure $\boldsymbol{F}(\cdot, \cdot)$.

Proof. Let the covariance function $\boldsymbol{R}(\cdot, \cdot)$ be normal and let

$$R_{ij}(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, dd F_{ij}(\lambda, \mu), \quad i, j = 1, 2, \ldots, N .$$

The covariance function $\boldsymbol{R}(\cdot, \cdot)$ determines unambiguously the matrix spectral measure $\boldsymbol{F}(\cdot, \cdot)$ if and only if every component $R_{ij}(\cdot, \cdot)$ determines unambigously the corresponding complex measure $F_{ij}(\cdot)$. We begin with the diagonal elements $R_{ii}(\cdot, \cdot)$, $i = 1, 2, \ldots, N$. Then the corresponding spectral measure $F_{ii}(\cdot)$ is non-negative as follows from positive semidefiniteness of $\boldsymbol{F}(\cdot)$. The element $R_{ii}(\cdot, \cdot)$ defines in the unique way $S_{ii}(\cdot, \cdot)$ because

$$S_{ii}(u, v) = R_{ii}\left(\frac{u + v}{2}, \frac{u - v}{2}\right).$$

The integral $S_{ii}(u, v) = \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd F_{ii}(\lambda, \mu)$ is absolutely convergent because

$$\iint_{-\infty}^{+\infty} |e^{\lambda u} e^{i\mu v}| \, dd F_{ii}(\lambda, \mu) = \iint_{-\infty}^{+\infty} e^{\lambda u} \, dd F_{ii}(\lambda, \mu) = S_{ii}(u, 0)$$

exists for every pair $(u, v) \in \mathbb{R}_2$. Now, let us consider a complex number $u = u_1 + i u_2$. Then, the integral

$$\iint_{-\infty}^{+\infty} e^{\lambda u_1} e^{i\mu u_2} e^{i\mu v} \, dd F_{ii}(\lambda, \mu)$$

is also absolutely convergent. In this way we can extend the function $S_{ii}(\cdot,\cdot)$ for every $v \in \mathbb{R}_1$ into the complex plane

$$S_{ii}(u_1 + iu_2, v) = \iint_{-\infty}^{+\infty} e^{\lambda u_1} e^{i\mu u_2} e^{i\mu v} \, ddF_{ii}(\lambda, \mu) .$$

Let us prove that the function $S_{ii}(u, v)$ is for every $v \in \mathbb{R}_1$ a holomorphic function on the complex plane. We introduce, for this purpose, a complex measure $\mathcal{G}_v(\cdot,\cdot)$ defined by the relation

$$\mathcal{G}_v(\lambda, \mu) = \iint_{-\infty}^{\lambda\mu} e^{i\beta v} \, ddF_{ii}(\alpha, \beta) .$$

Surely, $|\mathcal{G}_v(\lambda, \mu)| \leqq F_{ii}(\lambda, \mu)$. Hence, absolute variations of $\{\mathcal{G}_v(\cdot,\cdot), v \in \mathbb{R}_1\}$, are uniformly bounded and

$$(10) \qquad S_{ii}(u, v) = \iint_{-\infty}^{+\infty} e^{\lambda u} \, dd\mathcal{G}_v(\lambda, \mu) = \int_{-\infty}^{+\infty} e^{\lambda u} \, d\mathcal{G}_v^{(1)}(\lambda)$$

where $\mathcal{G}_v^{(1)}(\cdot)$ is the first marginal measure of $\mathcal{G}_v(\cdot,\cdot)$. We see that, by (10), the function $S_{ii}(u, v)$ is for every $v \in \mathbb{R}_1$ the bilateral Laplace transform of $\mathcal{G}_v(\cdot,\cdot)$, and hence, it is a holomorphic function of the variable u. The subset $(-\infty, +\infty) \times \{0\}$ is not isolated in the complex plane. This fact implies that $S_{ii}(u_i + iu_2, v)$ is the unique holomorphic extension that is determined by the values of $S_{ii}(u_1, v)$, $u_1 \in$ $\in (-\infty, +\infty)$. Now, let u_1 be chosen quite arbitrarily. Then,

$$S_{ii}(u_1 + iu_2, v_2) = \iint_{-\infty}^{+\infty} e^{\lambda u_2} e^{i\mu v} \, ddH_{u_1}(\lambda, \mu) ,$$

$$dH_{u_1}(\lambda, \mu) = \iint_{-\infty}^{\lambda\mu} e^{\alpha u_1} \, ddF_{ii}(\alpha, \beta) .$$

We see that for every fixed $u_1 \in (-\infty, +\infty)$ the function $S_{ii}(u_i + iu_2, v)$ is in the variables u_2, v the two-dimensional Fourier transform of $H_{u_1}(\cdot,\cdot)$. Thanks to properties of the Fourier transform the measure $H_{u_1}(\cdot,\cdot)$ is determined unambiguously. As the function $e^{\alpha u_1}$ is the Radon-Nikodym derivative of $H_{u_1}(\cdot,\cdot)$ with respect to $F_{ii}(\cdot,\cdot)$, the measure $F_{ii}(\cdot,\cdot)$ is determined by $H_{u_1}(\cdot,\cdot)$ and $e^{\alpha u_1}$ in the unique way. We have proved a one-to-one correspondence between $R_{ii}(\cdot,\cdot)$ and $F_{ii}(\cdot,\cdot)$.

In the case of a complex measure $F_{ij}(\cdot,\cdot)$ for $i \neq j$ we shall proceed in the following way. Let exist two complex measures such that

$$R_{ij}(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, ddF_{ij}(\lambda, \mu) =$$
$$= \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, ddG_{ij}(\lambda, \mu)$$

for every $s, t \in \mathbb{R}_1$. Then,

$$\iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, dd(F_{ij}(\lambda, \mu) - G_{ij}(\lambda, \mu)) = 0$$

for every $s, t \in \mathbb{R}_1$. This means, we have to prove that the only complex measure satisfying for every $u, v \in \mathbb{R}_1$

$$\iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH(\lambda, \mu) = 0$$

is zero.

Writing $H(\cdot,\cdot) = H_1(\cdot,\cdot) + iH_2(\cdot,\cdot)$ we obtain that

$$\iint_{-\infty}^{+\infty} e^{\lambda u} \cos \mu v \, ddH_1(\lambda, \mu) = \iint_{-\infty}^{+\infty} e^{\lambda u} \sin \mu v \, ddH_2(\lambda, \mu)$$

$$\iint_{-\infty}^{+\infty} e^{\lambda u} \cos \mu v \, ddH_2(\lambda, \mu) = - \iint_{-\infty}^{+\infty} e^{\lambda u} \sin \mu v \, ddH_1(\lambda, \mu) .$$

This fact yields

$$\iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_1(\lambda, \mu) = 0 , \quad \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_2(\lambda, \mu) = 0 .$$

As we consider measures with finite variations we can decompose

$$H_1(\cdot, \cdot) = H_1^+(\cdot, \cdot) - H_1^-(\cdot, \cdot)$$
$$H_2(\cdot, \cdot) = H_2^+(\cdot, \cdot) - H_2^-(\cdot, \cdot)$$

by means of the Jordan decomposition. Then, we have for every $u, v \in \mathbb{R}_1$

$$\iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_1^+(\lambda, \mu) = \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_1^-(\lambda, \mu) ,$$

and similarly

$$\iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_2^+(\lambda, \mu) = \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd H_2^-(\lambda, \mu) .$$

The one-to-one correspondence between one-dimensional normal covariance and spectral measure proved above gives that

$$H_1^+(\cdot) = H_1^-(\cdot) , \quad H_2^+(\cdot) = H_2^-(\cdot) .$$

This fact completes the proof of the theorem. □

Necessary and sufficient conditions given in the following theorem describe the class of multidimensional normal covariances.

Theorem 4. An N-dimensional covariance function $\boldsymbol{R}(\cdot, \cdot)$ defined on the plane $\mathbb{R}_2$ is a normal covariance if and only if there exists a continuous matrix function $\boldsymbol{S}(\cdot, \cdot)$ defined on the plane such that

$$\boldsymbol{R}(s, t) = \boldsymbol{S}(s + t, s - t)$$

and

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k^i \bar{\alpha}_l^j S_{ij}(u_k + u_l, v_k - v_l) \geqq 0$$

for the every $2n$-tuple of real numbers $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and every $n \times N$-matrix of complex numbers $\{\alpha_k^i\}_{k=1,2,\ldots,n,\; i=1,2,\ldots,N}$.

Proof. The proof of this theorem is transformed into the one-dimensional case. Let $\boldsymbol{e}^{\mathrm{T}} = (c_1, c_2, \ldots, c_N)$ be any N-dimensional vector of complex numbers and let us consider the function $R_e(\cdot, \cdot) = \boldsymbol{e}^{\mathrm{T}} \boldsymbol{R}(\cdot, \cdot) \boldsymbol{e}$. We shall prove that $R_e(\cdot, \cdot)$ is a one-dimensional normal covariance function. At the first sight, $R_e(\cdot, \cdot)$ is defined on the plane and is continuous here. Further $\overline{R_e(s, t)} = R_e(t, s)$ because

$$\overline{R_e(s, t)} = \overline{\sum_{i=1}^{N} \sum_{j=1}^{N} c_i \bar{c}_j R_{ij}(s, t)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{c}_i c_j \overline{R_{ij}(s, t)} =$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} c_j \bar{c}_i R_{ji}(t, s) = R_e(t, s) .$$

$R_e(\cdot, \cdot)$ is a covariance function because it is positive semidefinite as follows from the assumptions of the theorem

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\alpha_k\bar{\alpha}_l R_e(s_k, s_l) = \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_k\bar{\alpha}_l c_i\bar{c}_j R_{ij}(s_k, s_l) =$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n}\sum_{l=1}^{n}\alpha_k c_i(\bar{\alpha}_l\bar{c}_j) S_{ij}(s_k + s_l, s_k - s_l) \geqq 0$$

if we put $\alpha_k c_i = \alpha_k^i$ and $s_k = u_k = v_k$.

As we assume that $R_{ij}(s, t) = S_{ij}(s + t, s - t)$ then $R_e(s, t) = e^{\mathrm{T}} S(s + t, s - t)\, e = S(s + t, s - t)$ and the function $R_e(\cdot, \cdot)$ is a function of $s + t$ and $s - t$. There is no problem to prove that $S(\cdot, \cdot)$ is positive semidefinite in the following sense

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\alpha_k\bar{\alpha}_l S(u_k + u_l, v_k - v_l) \geqq 0$$

$$\sum_{\kappa=1}^{n}\sum_{l=1}^{n}\alpha_k\bar{\alpha}_l S(u_k + u_l, v_k - v_l) =$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n}\sum_{l=1}^{n} c_i\alpha_k(\bar{c}_j\bar{\alpha}_l) S_{ij}(u_k + u_l, v_k - v_l) \geqq 0$$

for every matrix $\{c_i\alpha_k\}$ of complex numbers and every $2n$-tuple $u_1, u_2, \ldots, u_n, v_1, v_2 \ldots \ldots, v_n$ of reals. Finally, we have

$$R_e(0, 0) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_i\bar{c}_j R_{ij}(0, 0) \geqq 0$$

and by means of results given in [6] we can assert that the covariance function $R_e(\cdot, \cdot)$ is normal. Hence, there exists a spectral representation of $R_e(\cdot, \cdot)$ in the form

$$R_e(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_e(\lambda, \mu)$$

where $R_e(\cdot, \cdot)$ is a two-dimensional measure with finite variation equal to $R_e(0, 0)$, see [6]. Let us consider now special cases of the vector $\boldsymbol{e}$. Let

$$\boldsymbol{e}_{(k,j)}^{\mathrm{T}} = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where 1 stands on the kth and jth places $(k < j)$; similarly, $\boldsymbol{d}_{(k,j)}^{\mathrm{T}} = (0, \ldots, 0, 1, 0, \ldots \ldots, 0, -\mathrm{i}, 0, \ldots, 0)$ $(k < j)$.

Then,

$$R_{e(k,j)}(\cdot, \cdot) = R_{kk}(\cdot, \cdot) + R_{kj}(\cdot, \cdot) + R_{jk}(\cdot, \cdot) + R_{jj}(\cdot, \cdot)$$
$$R_{d(k,j)} = R_{kk}(\cdot, \cdot) + \mathrm{i}R_{kj}(\cdot, \cdot) - \mathrm{i}R_{jk}(\cdot,) + R_{jj}(\cdot, \cdot)\,;$$

hence,

$$R_{jk} = \tfrac{1}{2}(R_{e(k,j)} - \mathrm{i}R_{d(k,j)} - (1 - \mathrm{i})(R_{kk} - R_{jj}))$$

and thanks to the one-to-one correspondence between R_e and F_e we can state that

$$F_{kj} = \tfrac{1}{2}(F_{e(k,j)} - \mathrm{i}F_{d(k,j)} - (1 - \mathrm{i})(F_{kk} - F_{jj}))\,.$$

We obtain an expression of an off-diagonal component $R_{kj}(\cdot, \cdot)$ in the form

$$R_{kj}(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F_{kj}(\lambda, \mu)\,.$$

We have constructed in this way a matrix complex measure $F = \{F_{kj}\}_{k,j=1}^{N}$. We have to verify that F is a spectral measure. Surely,

$$\bar{F}(\cdot, \cdot) = F^{\mathrm{T}}(\cdot, \cdot)$$

because

$$\bar{R}(\cdot, \cdot) = R^{\mathrm{T}}(\cdot, \cdot) .$$

The function $F_e(\cdot, \cdot)$ defines for every e a measure, hence,

$$\Delta_{h_1} \Delta_{h_2} F_e(\lambda, \mu) \geqq 0$$

for every $(\lambda, \mu) \in \mathbb{R}_2$ and every $h_1 \in \mathbb{R}_1$, $h_2 \in \mathbb{R}_1$. This means, for every vector e of complex numbers

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_i \bar{c}_j \, \Delta_{h_1} \Delta_{h_2} F_{ij}(\lambda, \mu) \geqq 0 .$$

We see, immediately, that the matrix $F(\Delta)$ is positive semidefinite for every Borel subset Δ in the plane $\mathbb{R}_2$. If F is a matrix spectral measure, then, every function

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, dd F(\lambda, \mu)$$

is a normal covariance function, (we assume the existence for every pair $(s, t) \in \mathbb{R}_2$). The function $R(\cdot, \cdot)$ satisfies:

1)
$$\sum_{i=i}^{N} \sum_{j=1}^{N} \alpha_i \bar{\alpha}_j R_{ji}(t, t) = \iint_{-\infty}^{+\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \bar{\alpha}_j e^{2\lambda t} \, dd F_{ij}(\lambda, \mu) =$$
$$= \iint_{-\infty}^{+\infty} e^{2\lambda t} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \bar{\alpha}_j \, dd F_{ij}(\lambda, \mu) \geqq 0$$

because $\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \bar{\alpha}_j F_{ij}(\cdot, \cdot)$ defines a nonnegative measure (F is a matrix spectral measure)

2)
$$\bar{R}(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{\overline{i\mu(s-t)}} \, dd \bar{F}(\lambda, \mu) =$$
$$= \iint_{-\infty}^{+\infty} e^{\lambda(t+s)} e^{i\mu(t-s)} \, dd F^{\mathrm{T}}(\lambda, \mu) = R^{\mathrm{T}}(t, s) .$$

3)
$$|R_{jk}(s, t)| = \left|\iint_{-\infty}^{+\infty} e^{\lambda(t+s)} e^{i\mu(t-s)} \, dd F_{jk}(\lambda, \mu)\right| \leqq$$
$$\leqq \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} \, dd |F_{jk}(\lambda, \mu)| \leqq$$
$$\leqq \left(\iint_{-\infty}^{+\infty} e^{2\lambda s} \, dd F_{jj}(\lambda, \mu)\right)^{1/2} \left(\iint_{-\infty}^{+\infty} e^{2\lambda s} \, dd F_{kk}(\lambda, \mu)\right)^{1/2} .$$

This fact follows from positive definiteness of F because for every complex α the inequality

$$F_{ii}(\Delta) + |\alpha|^2 F_{jj}(\Delta) + \bar{\alpha} F_{ij}(\Delta) + \alpha F_{ij}(\Delta) \geqq 0$$

holds. Then, put $\alpha = F_{ij}(\Delta)/F_{jj}^{1/2}(\Delta)$ if $F_{jj}(\Delta) \neq 0$.

4) Let us consider the function $S(u, v) = R(\frac{1}{2}(u + v), \frac{1}{2}(u - v))$; then,

$$S(u, v) = \iint_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, dd F(\lambda, \mu)$$

and $R(s, t) = S(s + t, s - t)$. Let us prove that this function $S(\cdot, \cdot)$ satisfies the assumption of the theorem.

For this purpose, we need the Karhunen theorem, see [2]. By means of this theorem, wa can express every random process $\{x(t),\ t \in \mathbb{R}_1\}$ having a normal covariance as a stochastic integral understood in the quadratic mean sense

$$x(t) = \iint_{-\infty}^{+\infty} e^{tz}\, dd\xi(z)$$

where $z = \lambda + i\mu$ and $\mathsf{E}\{\xi(z_1)\,\overline{\xi^T(z_2)}\} = F(\min(z_1, z_2))$; $(\min(z_1, z_2) = (\min(\operatorname{Re} z_1,\ \operatorname{Re} z_2),\ \min(\operatorname{Im} z_1,\ \operatorname{Im} z_2)))$. At this moment, let us consider random variables

$$y(u, v) = \iint_{-\infty}^{+\infty} e^{\lambda u_1}\, e^{i\lambda u_2}\, e^{i\mu v}\, dd\xi(z)\,,$$

$u = u_1 + iu_2$, $u_1, u_2 \in \mathbb{R}_1$. These random variables exist because

$$|\mathsf{E}\{y(u, v)\, y^T(x, y)\}| = \left|\iint_{-\infty}^{+\infty} e^{\lambda(u_1+x_1)}\, e^{i\lambda(u_2-x_2)}\, e^{i\mu(v-y)}\, ddF(\lambda, \mu)\right| \leqq$$

$$\leqq \iint_{-\infty}^{+\infty} e^{\lambda(u_1+x_1)}\, dd|F(\lambda, \mu)| < \infty\,.$$

Then,

$$0 \leqq \mathsf{E}\{|\sum_{i=1}^{N}\sum_{p=1}^{n} \alpha_p^i y_i(u_p, v_p)|^2\} =$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p=1}^{n}\sum_{q=1}^{n} \alpha_p^i \bar{\alpha}_q^j\, \mathsf{E}\{y_i(u_p, v_p)\,\overline{y_j(u_q, v_q)}\} =$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p=1}^{n}\sum_{q=1}^{n} \alpha_p^i \bar{\alpha}_q^j \iint_{-\infty}^{+\infty} e^{\lambda u_p}\, e^{\lambda \bar{u}_q}\, e^{i\mu(v_p - v_g)})\, ddF_{ij}(\lambda, \mu)\,.$$

If we put $u_p = \operatorname{Re} u_p$, then, we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p=1}^{n}\sum_{q=1}^{n} \alpha_p^i \bar{\alpha}_q^j\, S_{ij}(u_p + u_q, v_p - v_q) \geqq 0\,.$$

5) Every component $R_{ij}(\cdot, \cdot)$ of $R(\cdot, \cdot)$ is a continuous function because all diagonal elements are one-dimensional normal covariances and off-diagonal elements can be expressed as a linear combinations of one-dimensional normal covariances. This completes the proof of the theorem. □

3. NORMAL COVARIANCES AND NORMAL OPERATORS

In the multidimensional case we can show also a close connection between normal covariances and normal operators. Let a process $x(\cdot) = \{x_i(\cdot)\}_{i=1}^N$ be a random process with a normal covariance function $R(\cdot, \cdot)$, i.e.

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)}\, e^{i\mu(s-t)}\, ddF(\lambda, \mu)\,.$$

As it was mentioned above such a process can be expressed in the form of a stochastic integral

$$x(t) = \iint_{-\infty}^{+\infty} e^{tz}\, dd\xi(z)\,.$$

Let $L(\xi(\cdot))$ be the linear set of all linear combinations

$$\sum_{i=1}^{n} \alpha_i \xi_{j_i}(z_i)$$

and let $H(\xi\cdot)) = \overline{L(\xi(\cdot))}$ be a closure of $L(\xi(\cdot))$ with respect to the convergence in the quadratic mean sense. Let us denote by $H(z)$ the subspace of $H(\xi(\cdot))$ generated by all random variables

$$\sum_{i=1}^{n} \alpha_i \, \xi_{j_i}(z_i) \, , \quad z_i \leqq z \, ;$$

let P_z be the orthogonal projector in $H(\xi(\cdot))$ on the subspace $H(z)$. Thanks to properties of the spectral measure $\boldsymbol{F}$ one can easily prove that the family $\{P_z; z \in \mathbb{C}\}$ forms a complex resolution of the identity in $H(\xi(\cdot))$. We can construct normal operators

$$A_t = \iint_{-\infty}^{+\infty} e^{tz} \, dP_z \, , \quad t \in \mathbb{R}_1$$

with the definition domain

$$\mathscr{D}(A_t) = \{x \in H(\xi(\cdot)) \colon \iint_{-\infty}^{+\infty} e^{2t} \, dd\langle P_z x, x\rangle < \infty\} \, .$$

As $x(0) = \iint_{-\infty}^{+\infty} dd\xi(z) = \underset{z\to\infty}{\text{l.i.m.}} \; \xi(z)$ then $x_i(0) \in H(\xi(\cdot))$ for every $i \in 1, 2, \ldots, N$ and $P_z \, x_i(0) = \xi_i(z)$. Then, we see that

$$x_i(t) = \iint_{-\infty}^{+\infty} e^{tz} \, dP_z \, x_i(0) \, , \quad i = 1, 2, \ldots, N$$

because $dd\langle P_z \, x_i(0), x_i(0)\rangle = dd\langle \xi_i(z), x_i(0)\rangle = dd\langle \xi_i(z), \xi_i(z)\rangle = ddF_{ii}(z)$ and the integral

$$\iint_{-\infty}^{+\infty} e^{2t\lambda} \, ddF_{ii}(\lambda, \mu)$$

exists for every $t \in \mathbb{R}_1$ and every $i = 1, 2, \ldots, N$ as we assume. We obtained that

$$x_i(t) = A_t \, x_i(0) \, , \quad i = 1, 2, \ldots, N \, , \quad t \in \mathbb{R}_1 \, .$$

Corollary to Theorem 4. An N-dimensional covariance function $\boldsymbol{R}(\cdot, \cdot)$ is normal if and only if for every N-tuple $\boldsymbol{z}^{\mathrm{T}} = (z_1, z_2, \ldots, z_N)$ of complex numbers $\boldsymbol{z}^{\mathrm{T}} \boldsymbol{R}(\cdot, \cdot) \, \boldsymbol{z}$ is a one-dimensional normal covariance function.

Another connection between normal covariances and normal operators in a Hilbert space is shown in Theorem 5.

Theorem 5. Let a group $\{T_s, s \in \mathbb{R}_1\}$ of normal, in general unbounded, operators be given in a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Let, for every $x, y \in \mathscr{D} = \bigcap_{s\in R_1} \mathscr{D}(T_s)$, $\langle T_s x, T_t y\rangle$ be a continuous function on the plane. Then for every N-tuple $x_1, x_2, x_3, \ldots, x_N$ of elements in $\mathscr{H}$ belonging to the subset $\mathscr{D}$

$$\boldsymbol{R}(s, t) = \{\langle T_s x_i, T_t x_j\rangle\}_{i,j=1}^{N}$$

is an N-dimensional normal covariance function ($\mathscr{D}(T_s)$ is the definition domain of T_s in $\mathscr{H}$).

Proof. The subset $\mathscr{D}$ is not empty because $0 \in \mathscr{D}$ in every case. Let $x_1, x_2, \ldots, x_N$

belong to $\mathscr{D}$. First, we need to show that the matrix function $\boldsymbol{R}(\cdot, \cdot)$ is a covariance function. Let n be an arbitrary natural number, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an arbitrary n-tuple of complex numbers and $s_1, s_2, \ldots, s_n$ an arbitrary n-tuple of reals. We must prove that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\alpha}_l \langle T_{s_k} x_{i_k}, T_{s_l} x_{i_l} \rangle \geqq 0$$

where $x_{i_k} \in \{x_1, x_2, \ldots, x_N\}$ for every $k = 1, 2, \ldots, n$. This inequality holds evidently because

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\alpha}_l \langle T_{s_k} x_{i_k}, T_{s_l} x_{i_l} \rangle = \left| \sum_{k=1}^{n} \alpha_k T_{s_k} x_{i_k} \right|^2 \geqq 0 \,.$$

For next steps, it is suitable to introduce the function $S_{xy}(u, v)$, $x, y \in \mathscr{D}$, defined by the relation

$$S_{xy}(u, v) = \langle T_{(u+v)/2} x, T_{(u-v)/2}\, y \rangle \,.$$

We immediately see

$$R_{xy}(s, t) = S_{xy}(s + t, s - t) \,;$$

hence, $S_{xy}(\cdot, \cdot)$ is continuous on the plane. Let $\boldsymbol{z}^{\mathrm{T}} = (z_1, z_2, \ldots, z_N)$ be an arbitrary N-tuple of complex numbers and we must prove that

$$\boldsymbol{z}^{\mathrm{T}} \boldsymbol{R}(\cdot, \cdot)\, \boldsymbol{z}$$

is a normal covariance function. To prove this fact we need validity of the equality

$$T_t^* T_s = T_s T_t^*$$

on $\mathscr{D}$. As $\{T_s, s \in \mathbb{R}_1\}$ is a group then $T_{t+s} = T_t T_s = T_s T_t$, i.e. $\mathscr{D}(T_{t+s}) = \mathscr{D}(T_s T_t) = \mathscr{D}(T_t T_s)$ must hold too. Next, it follows $\mathscr{R}(T_t) \subset \mathscr{D}(T_s)$ and simultaneously $\mathscr{R}(T_s) \subset \mathscr{D}(T_t)$ ($\mathscr{R}(T_t)$ is the range of T_t. Let n be an integer. Then,

$$(T_s^*)^n = T_{ns}^*$$

thanks to the group property holding for $\{T_s^*, s \in \mathbb{R}_1\}$ too. Now, let $t = n\,.\,s$. Then

$$T_t^* T_s = T_{n.s}^* T_s = (T_s^*)^n\, T_s = T_s (T_s^*)^n = T_s T_t^*$$

because $T_s^* T_s = T_s T_s^*$. Similarly, in case $t = s\,.\,(p/q)$, where p/q represents a rational number, we can prove

$$T_t^* T_s = T_s T_t^*$$

as

$$T_t^* T_s = T_{s.p/q}^*\, T_{q.s/q} = (T_{s/q}^*)^p\, (T_{s/q})^q = (T_{s/q})^q (\, T_{s/q}^*)^p = T_s T_t^* \,.$$

Finally, let t be quite arbitrary. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ $t_n = s\,.\,p_n/q_n \to t$ where p_n/q_n are rational and continuity of the scalar product in $\mathscr{H}$ proves

$$T_t^* T_s = T_s T_t^*$$

for every pair s, t of reals. If $x \in \mathscr{D}$ then $T_t\, x \in \mathscr{D}$ as well because $T_{t+s} x = T_s(T_t x)$ which implies $T_t x \in \mathscr{D}(T_s)$ for every real s. This proves that $T_t x \in \mathscr{D}$. If $T_s x \in \mathscr{D}$ then $T_t^*(T_s x)$ is well defined as $\mathscr{D}(T_t^*) = \mathscr{D}(T_t)$. In case $s = n\,.\,t$, n is an integer,

$$T_t^* T_s x = T_s T_t^* x$$

as it is proved above and this gives $T_t^* x \in \mathscr{D}(T_{nt})$ for every n, $T_t^* x \in \mathscr{D}$ too. That means both the operators $T_t^* T_s$, $T_s T_t^*$ are well defined on the subset $\mathscr{D}$. Now, we are ready to prove the "nonnegative-definite" property of $z^T \boldsymbol{R}(\cdot, \cdot)\, z$, see [6]. Let n be a natural number, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an n-tuple of complex numbers, let $u_1, u_2, \ldots \ldots, u_n, v_1, v_2, \ldots, v_n$ be a $2n$-tuple of reals. Let us consider the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \sum_{k=1}^{N} \sum_{l=1}^{N} z_k \bar{z}_l \, S_{x_k x_l}(u_i + u_j, v_i - v_j) =$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} z_k \bar{z}_l \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_i \langle T_{(u_i + u_j + v_i - v_j)/2} x_k, \; T_{(u_i + u_j + v_j - v_j)/2} x_l \rangle =$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} z_k \bar{z}_l \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \langle T^*_{(u_i - v_i)/2} T^*_{(u_i + v_i)/2} x_k, \; T^*_{(u_j - v_j)/2} T^*_{(u_j + v_j)/2} x_l \rangle =$$

$$= \Big| \sum_{k=1}^{N} \sum_{i=1}^{n} z_k \alpha_i T^*_{(u_i - v_i)/2} T^*_{(u_i + v_i)/2} x_k \Big|^2 \geqq 0 \, .$$

A necessary and sufficient condition characterizing normal covariances is proved, see [6]. This inequality, together with continuity of $R_{ij}(\cdot, \cdot)$, $i, j = 1, 2, \ldots, N$, show that the matrix covariance function $\boldsymbol{R}(\cdot, \cdot)$ is normal.

4. CONCLUSION

In the literature, we can meet two types of generalization of the notion *weak stationarity*. First generalization, originated by Loève in [3], can be characterized as the nonorthogonal integral representation

$$x(t) = \int_{-\infty}^{+\infty} \varphi(t, \lambda) \, \mathrm{d}\xi(\lambda)$$

in the quadratic mean sense where $\varphi(\cdot, \cdot)$ is a nonrandom complex function and $\xi(\cdot)$ is a second-order random process with covariance function having finite variation on the plane. The second generalization, originated by Karhunen, see [2], can be called the orthogonal integral representation

$$x(t) = \int_{-\infty}^{+\infty} \varphi(t, \lambda) \, \mathrm{d}\eta(\lambda) \tag{11}$$

where $\varphi(\cdot, \cdot)$ is a nonrandom complex function and the process $\eta(\cdot)$ defines an orthogonally scattered random measure on the Borel field in reals. There is no problem to generalize the Karhunen representation in the following way: instead of the Borel sets with the Lebesgue measure we can consider a measure space (Θ, σ, m) and an orthogonally scattered measure $\eta(\cdot)$ satisfying

$$\mathsf{E}(\eta(\Delta_1)\, \bar{\eta}(\Delta_2)\} = m(\Delta_1 \cap \Delta_2)$$

for every $\Delta_1, \Delta_2 \in \sigma$. Then, the corresponding covariance function of the process $\{x(t),\ t \in \mathbb{R}_1\}$ can be expressed as

$$R(s, t) = \int_\Theta \varphi(s, \theta) \, \bar{\varphi}(s, \theta) \, \mathrm{d}m(\theta) \, .$$

Immediately, we see that a process with a normal covariance function belongs into the Karhunen class with $\Theta = \mathbb{R}_2$, σ is the σ-algebra of Borel sets in the plane, $\varphi(s, \theta) = e^{s\lambda + is\mu}$, i.e. $\theta = (\lambda, \mu)$. The measure $m(\cdot)$ defined on the Borel sets is determined by a function $F(\cdot, \cdot)$, see Definition 1. In a similar way, we can handle with the multidimensional case.

As well known, the spectral decomposition of weakly stationary process is connected with groups of unitary shift-operators in the Hilbert space of random process values. Considering normal shift operators we reach, of course, the class of normal covariance functions. In general, if a random process possesses a Karhunen representation (11) then there exists a self-adjoint operator A defined in the mentioned Hilbert space such that

$$x(t) = \varphi(t, A)\, x(0)$$

(see [1]). In case of the nonorthogonal integral representation, mainly in the harmonizable case, the question about the characterization of the corresponding shift operators, has so far been open.

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