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ON TRANSFER FUNCTION MATRIX OF LINEAR CASCADE SYSTEMS

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Lower triangular transfer function matrix of cascade linear system is investigated. The paper aims to show and prove that unlike for the other multi-input, multi-output systems general fashions of the coprime transfer function matrix fraction representations can be found in this case.

1. CASCADE SYSTEM TRANSFER FUNCTION MATRIX

Many technological, industrial and other processes are characterized by an one way, one line flow of information and energy ([2]). Such cascade processes represent the special case of multi-input, multi-output structure that may be modelled by the block diagram in Fig. 1 where

P_i denotes the ith stage of the process,
y_i the output of the ith stage (ith controlled variable), and
u_i the input of the ith stage (ith control variables);
$i = 1, \ldots, n$.

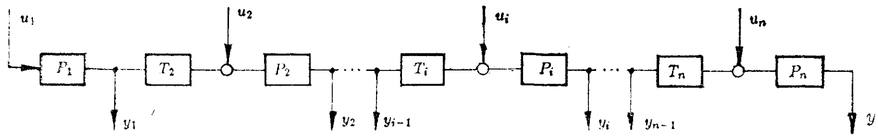

Fig. 1.

Feed properties between parts P_{i-1} and P_i of the process are modelled by elements T_i, $i = 2, \ldots, n$.

The paper deals with the regular n-input, n-output cascade only which is affected by no external disturbances.

Assuming continuous-time, linear dynamics of P_i as well as T_i the following equations can be written (in Laplace transforms):

(1) $$Y_1(s) = P_1(s)\, U_1(s)$$

and

$$Y_i(s) = P_i(s)\left[U_i(s) + T_i(s)\, Y_{i-1}(s)\right], \quad i = 2, \ldots, n .$$

The transfer functions

$$P_i(s) = \frac{b_i(s)}{a_i(s)}, \quad (a_i, b_i) \sim 1 ; \quad i = 1, \ldots, n$$

(2) and

$$T_i(s) = \frac{q_i(s)}{p_i(s)}, \quad (p_i, q_i) \sim 1 ; \quad i = 2, \ldots, n$$

where $a_i(s)$, $b_i(s)$, $p_i(s)$ and $q_i(s)$ are supposed to be polynomials in s. Nevertheless the final results which come in Theorems 1 and 2 of the paper are also applicable if factors $\exp(-\tau_i s)$ corresponding to possible dead times occur in $b_i(s)$ and/or $q_i(s)$.

Note that the standard symbols of polynomial theory ([1]) are used in the paper:

(a, b) for the greatest common divisor (GCD) of a and b,

$a \sim b$ if a and b are associates, i.e., $a = \alpha b$ where $\alpha =$ const. (a polynomial of degree 0) and

$b \mid a$ if b is a divisor of a.

Combining the equations (1) we can write in vector-matrix form

(3) $$\mathbf{Y}(s) = \mathbf{G}(s)\, \mathbf{U}(s)$$

where

(4) $$\mathbf{Y}(s) = [Y_1(s); \ldots; Y_n(s)]^{\mathrm{T}} \quad \text{and} \quad \mathbf{U}(s) = [U_1(s); \ldots; U_n(s)]^{\mathrm{T}}$$

and the system transfer function matrix (TFM)

(5) $$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & & & 0 \\ G_{21}(s) & G_{22}(s) & & \\ \vdots & \vdots & \ddots & \\ G_{n1}(s) & G_{n2}(s) & \ldots & G_{nn}(s) \end{bmatrix}$$

is lower triangular $(n \times n)$ matrix with the elements

$$G_{ij}(s) = \frac{b_i(s)}{a_i(s)} \qquad \text{for} \quad j = i$$

$$G_{ij}(s) = \frac{b_i(s) \ldots b_j(s)\, q_i(s) \ldots q_{j+1}(s)}{a_i(s) \ldots a_j(s)\, p_i(s) \ldots p_{j+1}(s)} \qquad \text{for} \quad j < i$$

$$G_{ij}(s) = 0 \qquad \text{for} \quad j > i , \quad i, j = 1, \ldots, n .$$

The structure given by Fig. 1 includes usual cases which we encounter in practice. Especially, either

– T_i represents the process part which is affected by no external input and has no own measured output or

– two neighbouring $(i-1)$th and ith stages of a process are coupled through a physical transducer T_i to adapt signal y_{i-1} to the actuator of P_i.

Fraction representations of the matrix $\mathbf{G}(s)$ are investigated in the next sections. It is well known that the coprime matrix fraction (CMF) descriptions of a general TFM can be found by numerical ways only (cf. [1]). But numerical algorithms need not be applied in the case of cascade system matrix. It will be shown and proved that $\mathbf{G}$ given by (5) with (6) can be usually transformed into CMF fashion immediately using the elements a_i, b_i, p_i and q_i of single transfer functions in $\mathbf{G}$.

The reader is assumed to be acquainted with the main fundamentals of polynomial and polynomial matrix theory of dynamic systems ([1], [3]).

2. LEFT COPRIME MATRIX FRACTION REPRESENTATION OF G

Any $(n \times r)$ TFM $\mathbf{G}$ of a linear, free of dead times, continuous-time system can be always written in the form (cf. [1])

$$\mathbf{G} = \mathbf{A}_L^{-1}\mathbf{B}_L \tag{7}$$

where $\mathbf{A}_L$ and $\mathbf{B}_L$ are $(n \times n)$ and $(n \times r)$ polynomial matrices in s, respectively. The corresponding input-output equation

$$\mathbf{A}_L\mathbf{Y} = \mathbf{B}_L\mathbf{U} \tag{8}$$

is valid.

The matrices $\mathbf{A}_L$ and $\mathbf{B}_L$ represent a left matrix fraction description of $\mathbf{G}$. Such a representation is left coprime (LCMF) if and only if

$$\mathbf{A}_L = \mathbf{D}\mathbf{F}_L \quad \text{and} \quad \mathbf{B}_L = \mathbf{D}\mathbf{H}_L \tag{9}$$

where the $(n \times n)$ polynomial matrix $\mathbf{D}$ known as the greatest common left divisor (GCLD) of $\mathbf{A}_L$ and $\mathbf{B}_L$ has the property det $\mathbf{D} \sim 1$, i.e., $\mathbf{D}$ is unimodular. The pairs $\mathbf{A}_L$, $\mathbf{F}_L$ and $\mathbf{B}_L$, $\mathbf{H}_L$ are then the pairs of left equivalent polynomial matrices; $\mathbf{A}_L$ and $\mathbf{B}_L$ are called *left coprime matrices.*

Theorem 1. LCMF representation of a cascade system TFM $\mathbf{G}$ standing in (5) with (6) can be written in the form

$$\mathbf{A}_L = \begin{bmatrix} a_1 & & & 0 \\ -b_2q_2 & a_2p_2 & & \\ & \ddots & \ddots & \\ 0 & & -b_nq_n & a_np_n \end{bmatrix} \tag{10}$$

and

$$\mathbf{B}_L = \operatorname{diag}\left[b_1; b_2p_2; \ldots; b_np_n\right] \tag{11}$$

if and only if

$$(p_i, b_ib_{i-1}) \sim 1 \quad \text{for any} \quad i = 2, \ldots, n\,. \tag{12}$$

Proof.

A. At first we must prove that the matrices (10) and (11) represent a left matrix fraction description of $\mathbf{G}$ at all.

Using (10) and (11) in the equation (8) we get

$$a_1 Y_1 = b_1 U_1 \tag{13}$$

and

$$-b_i q_i Y_{i-1} + a_i p_i Y_i = b_i p_i U_i \quad \text{for} \quad i = 2, \ldots, n\,.$$

Hence

$$Y_1 = \frac{b_1}{a_1} U_1 \tag{14}$$

and

$$Y_i = \frac{b_i q_i}{a_i p_i} Y_{i-1} + \frac{b_i}{a_i} U_i \quad \text{for} \quad i = 2, \ldots, n\,.$$

Gradual substitutions Y_i into the $(i + 1)$th equation in (13) $i = 1, \ldots, n - 1$, yield the relation $\mathbf{Y} = \mathbf{G}\mathbf{U}$ with $\mathbf{G}$ standing in (5) with (6). The same result can be obtained if the inverse of $\mathbf{A}_L$ is formed and then $\mathbf{G} = \mathbf{A}_L^{-1}\mathbf{B}_L$ determined. Thus (10) and (11) is a left matrix fraction representation of $\mathbf{G}$.

B. Secondly, it must be proved that the matrices (10) and (11) are left coprime if and only if the conditions (12) are true.

If: Assume a polynomial matrix $\mathbf{D}$ to be GCLD of $\mathbf{A}_L$ and $\mathbf{B}_L$ and denote $d = \det \mathbf{D}$. The expressions (9) can be written as

$$[\mathbf{a}_1; \ldots; \mathbf{a}_n] = \mathbf{D}[\mathbf{f}_1; \ldots; \mathbf{f}_n] \quad \text{and} \quad [\mathbf{b}_1; \ldots; \mathbf{b}_n] = \mathbf{D}[\mathbf{h}_1; \ldots; \mathbf{h}_n] \tag{15}$$

where $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{f}_i$ and $\mathbf{h}_i$ are the columns of $\mathbf{A}_L$, $\mathbf{B}_L$, $\mathbf{F}_L$ and $\mathbf{H}_L$, respectively; $i = 1, \ldots, n$. Obviously $\mathbf{a}_i = \mathbf{D}\mathbf{f}_i$ and $\mathbf{b}_i = \mathbf{D}\mathbf{h}_i$.

Now the set $\{\mathbf{M}_n\}$ of $(n \times n)$ polynomial matrices $\mathbf{M}_n$ can be considered each of them being constructed as a different combination of $\mathbf{a}_i$ and $\mathbf{b}_j$; $i, j = 1, \ldots, n$, i.e., $\mathbf{M}_n$ is formed by n mutually different columns which are taken from $2n$ columns. Combining $\mathbf{f}_i$ and $\mathbf{h}_j$ in similar way the set $\{\mathbf{L}_n\}$ of $(n \times n)$ polynomial matrices $\mathbf{L}_n$ is defined. Note that $\mathbf{A}_L, \mathbf{B}_L \in \{\mathbf{M}_n\}$ and $\mathbf{F}_L, \mathbf{H}_L \in \{\mathbf{L}_n\}$. Then

$$\mathbf{M}_n = \mathbf{D}\mathbf{L}_n \tag{16}$$

and hence

$$\det \mathbf{M}_n = d \det \mathbf{L}_n \tag{17}$$

is true for any corresponding pair of $\mathbf{M}_n \in \{\mathbf{M}_n\}$ and $\mathbf{L}_n \in \{\mathbf{L}_n\}$.

Let all nonzero determinants m_n of all matrices $\mathbf{M}_n$ constitute the set $\{m_n\}$ and the GCD of all m_n be denoted by (m_n).

It follows from (17) that

$$d \mid (m_n)\,. \tag{18}$$

Hence $\mathbf{A}_L$ and $\mathbf{B}_L$ are LCMF of $\mathbf{G}$ if $(m_n) \sim 1$. But it is not easy to survey the finding of $\{m_n\}$ and (m_n) for general n. That is the set $\{\mathbf{M}_n\}$ contains $(2n)!\,(n!)^{-2}$ matrices. To overcome this difficulty we start with $n = 2$.

Here we determine

$$\{\mathbf{M}_2\} = \left\{\begin{bmatrix} a_1 & 0 \\ -b_2q_2 & a_2p_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & a_2p_2 \end{bmatrix}, \begin{bmatrix} a_1 & b_1 \\ -b_2q_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b_2p_2 & a_2p_2 \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ -b_2q_2 & b_2p_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2p_2 \end{bmatrix}\right\},$$

$$\{m_2\} = \{a_1a_2p_2, b_1a_2p_2, b_1b_2q_2, a_1b_2p_2, b_1b_2p_2\} \quad \text{and} \quad (m_2) = (p_2, b_2b_1)$$

Then $(p_2, b_2b_1) \sim 1$ ensures $(m_2) \sim 1$.

For $n = 3$ we obtain

$$\{m_3\} = \{a_1a_2p_2a_3p_3, b_1a_2p_2a_3p_3, a_1a_2p_2b_3p_3, b_1a_2p_2b_3p_3, b_1b_2q_2a_3p_3, b_1b_2q_2b_3p_3, b_1b_2q_2b_3q_3, a_1b_2p_2a_3p_3, a_1b_2p_2b_3p_3, a_1b_2p_2b_3q_3, b_1b_2p_2a_3p_3, b_1b_2p_2b_3q_3, b_1b_2p_2b_3p_3\}$$

and

$$(m_3) = (p_2, b_2b_1)\left(p_3, b_3\frac{b_2}{(b_2, p_2)}(b_2, b_1, p_2)\right).$$

Obviously $(p_2, b_2b_1) \sim 1$ together with $(p_3, b_3b_2) \sim 1$ ensures $(m_3) \sim 1$. Thus far sufficiency of (12) for $n = 2$ and $n = 3$ has been proved.

The structure of determinants m_n for an increasing n must be studied for a general proof.

One can see that

(19) $$m_n = c_1c_2 \dots c_n$$

where either a_1 or b_1 stands at the position c_1 and either a_ip_i or b_ip_i or b_iq_i at the position c_i, $i = 2, \dots, n$. But not all combinations occur in m_n since $c_i = b_iq_i$ can succeed to $c_{i-1} = b_{i-1}p_{i-1}$ or $c_{i-1} = b_{i-1}q_{i-1}$ only.

Then we can decompose

(20) $$\{m_n\} = \{m_{nA}\} \cup \{m_{nB}\} \cup \{m_{nQ}\}$$

where the subset

$\{m_{nA}\}$ contains all m_n ended by $c_n = a_np_n$,
$\{m_{nB}\}$ contains all m_n ended by $c_n = b_np_n$, and
$\{m_{nQ}\}$ contains all m_n ended by $c_n = b_nq_n$.

Let $\lambda\{\cdot\}$ denotes the number of determinants within a set $\{\cdot\}$. Since

(21) $$\lambda\{m_{nA}\} = \lambda\{m_{nB}\} = \lambda\{m_{n-1}\}, \quad \text{and}$$
$$\lambda\{m_{nQ}\} = \lambda\{m_{n-1,B}\} + \lambda\{m_{n-1,Q}\}$$

the recurrent relation

$$\lambda\{m_n\} = 2\lambda\{m_{n-1}\} + \lambda\{m_{n-1,B}\} + \lambda\{m_{n-1,Q}\}$$

is true starting with

$$\lambda\{m_2\} = \lambda\{m_{2A}\} + \lambda\{m_{2B}\} + \lambda\{m_{2Q}\}$$

where

$$\lambda\{m_{2A}\} = \lambda\{m_{2B}\} = 2 \quad \text{and} \quad \lambda\{m_{2Q}\} = 1\,.$$

Suppose now that $(m_j) \sim 1$ is ensured by the conditions

(22) $$(p_i, b_i b_{i-1}) \sim 1 \quad \text{for any} \quad i = 2, \ldots, j < n\,.$$

Considering $i = j + 1$ we need to show that together with (22) the only additional condition

(23) $$(p_{j+1}, b_{j+1} b_j) \sim 1$$

is sufficient to satisfy $(m_{j+1}) \sim 1$.

According to (20) and (21)

$$\{m_{j+1}\} = \{m_{j+1,A}\} \cup \{m_{j+1,B}\} \cup \{m_{j+1,Q}\} =$$

$$= \{m_j a_{j+1} p_{j+1}\} \cup \{m_j b_{j+1} p_{j+1}\} \cup \{m_{jB} b_{j+1} q_{j+1}\} \cup \{m_{jQ} b_{j+1} q_{j+1}\}$$

and then

(24) $$(m_{j+1}) \sim (p_{j+1}, b_{j+1}(m_{jB}, m_{jQ}))\,.$$

Considering (22) we determine

$$(m_{jB}, m_{jQ}) \sim (b_j p_j, b_{j-1} p_{j-1} b_j q_j, b_{j-1} q_{j-1} b_j q_j) \sim b_j (p_j, b_{j-1}) \sim b_j$$

and hence taking into account (24)

(25) $$(m_{j+1}) \sim (p_{j+1}, b_{j+1} b_j)\,.$$

Consequently $(m_{j+1}) \sim 1$ is ensured by (23) for any $j < n$ if (22) are valid. Hence $\mathbf{A}_L$ and $\mathbf{B}_L$ are left coprime if (12) are true.

Only if: The conditions (12) can be decomposed for any i into two separate relations:

(26) $$(p_i, b_i) \sim 1$$

and

(27) $$(p_i, b_{i-1}) \sim 1\,.$$

Assume now that $\mathbf{A}_L$ and $\mathbf{B}_L$ standing in (10) and (11), resp., are left coprime but (12) are not valid.

1. If (12) are broken by $(p_i, b_i) \nsim 1$ for one $i \in [2, n]$ we can denote

(28) $$p_{ii} = \frac{p_i}{(p_i, b_i)} \quad \text{and} \quad b_{ii} = \frac{b_i}{(p_i, b_i)}\,.$$

Then the matrices (10) and (11) can be decomposed into

$$
\text{(29)} \qquad \mathbf{A}_L = \mathbf{D}\begin{bmatrix} a_1 & & & & & & & \\ -b_2q_2 & a_2p_2 & & & & & \mathbf{0} & \\ & \ddots & \ddots & & & & & \\ & & -b_{i-1}q_{i-1} & a_{i-1}p_{i-1} & & & & \\ & & & -b_{ii}q_i & a_ip_{ii} & & & \\ & & & & -b_{i+1}q_{i+1} & a_{i+1}p_{i+1} & & \\ & \mathbf{0} & & & & \ddots & \ddots & \\ & & & & & & -b_nq_n & a_np_n \end{bmatrix} = \mathbf{DA}_{LO}
$$

and

$$
\text{(30)} \qquad \mathbf{B}_L = \mathbf{D}\,\mathrm{diag}\,[b_1;\, b_2p_2, \ldots;\, b_{i-1}p_{i-1};\, b_ip_{ii};\, b_{i+1}p_{i+1};\, \ldots;\, b_np_n] = \mathbf{DB}_{LO}
$$

where

$$
\text{(31)} \qquad \mathbf{D} = \mathrm{diag}\,[1;\, \ldots;1;\, (p_i, b_i);\, 1;\, \ldots;\, 1]
$$

is the GCLD of $\mathbf{A}_L$ and $\mathbf{B}_L$.

As $d = \det \mathbf{D} = (p_i, b_i) \nsim 1$ $\mathbf{A}_L$ and $\mathbf{B}_L$ are not left coprime.

2. If the conditions (12) are broken by $(p_i, b_{i-1}) \nsim 1$ for one $i \in [2, n]$ the denotations

$$
\text{(32)} \qquad p_{i,i-1} = \frac{p_i}{(p_i, b_{i-1})} \quad \text{and} \quad b_{i-1,i} = \frac{b_{i-1}}{(p_i, b_{i-1})}
$$

can be applied.

Then the matrices (10) and (11) may be written as

$$
\text{(33)} \qquad \mathbf{A}_L = \mathbf{D}\begin{bmatrix} a_1 & & & & & & & & \\ -b_2q_2 & a_2p_2 & & & & & & \mathbf{0} & \\ 0 & -b_3q_3 & a_3p_3 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & 0 & -b_{i-1}q_{i-1} & a_{i-1}p_{i-1} & & & & \\ & & & v & y & a_ip_{i,i-1} & & & \\ & \mathbf{0} & & & 0 & -b_{i+1}q_{i+1} & a_{i+1}p_{i+1} & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & 0 & -b_nq_n & a_np_n \end{bmatrix} = \mathbf{DA}_{LO}
$$

and

$$
\text{(34)} \qquad \mathbf{B}_L = \mathbf{D}\begin{bmatrix} b_1 & & & & & & \\ 0 & b_2p_2 & & & & \mathbf{0} & \\ & \ddots & \ddots & & & & \\ & & 0 & b_{i-1}p_{i-1} & & & \\ & & & z & b_ip_{i,i-1} & & \\ & & & & 0 & b_{i+1}p_{i+1} & \\ & \mathbf{0} & & & & \ddots & \ddots \\ & & & & & 0 & b_np_n \end{bmatrix} = \mathbf{DB}_{LO}
$$

with the GCLD

$$
(35) \qquad \mathbf{D} = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & 0 & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & x & (p_i, b_{i-1}) & & & \\ & & & & 0 & 1 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & 0 & 1 \end{bmatrix}
$$

The polynomials x and y satisfy the equation

$$
(36) \qquad a_{i-1}p_{i-1}x + (p_i, b_{i-1})\, y = -b_i q_i
$$

which is always solvable seeing that $(p_{i-1}, b_{i-1}) \sim 1$ as well as $(a_{i-1}, b_{i-1}) \sim 1$ is assumed.

Having x, y the remaining polynomials in (33) and (34) are

$$
(37) \qquad v = \;\; b_{i-1,i}q_{i-1}x\,, \quad \text{and}
$$

$$
z = -b_{i-1,i}p_{i-1}x\,.
$$

For $i = 2$ we put $p_{i-1} = 1$ and $q_{i-1} = 0$ in (36) and (37); the polynomial v in A_{LO} is omitted.

In virtue of (35) we have $d = \det \mathbf{D} = (p_i, b_{i-1}) \nsim 1$ and consequently the matrices (10) and (11) are not left coprime. □

3. RIGHT COPRIME MATRIX FRACTION REPRESENTATION OF $\mathbf{G}$

Any $(n \times r)$ TFM $\mathbf{G}$ of a linear, free of dead times, continuous-time system can be always written also in the form (cf. [1])

$$
(38) \qquad \mathbf{G} = \mathbf{B}_R \mathbf{A}_R^{-1}
$$

where $\mathbf{A}_R$ and $\mathbf{B}_R$ are $(r \times r)$ and $(n \times r)$ polynomial matrices in s, respectively.

The matrices $\mathbf{A}_R$ and $\mathbf{B}_R$ represent a right matrix fraction description of $\mathbf{G}$. Such a representation is right coprime (RCMF) if and only if

$$
(39) \qquad \mathbf{A}_R = \mathbf{F}_R \mathbf{D} \quad \text{and} \quad \mathbf{B}_R = \mathbf{H}_R \mathbf{D}
$$

where the $(r \times r)$ polynomial matrix $\mathbf{D}$ is the greatest common right divisor (GCRD) of $\mathbf{A}_R$ and $\mathbf{B}_R$ and $\det \mathbf{D} \sim 1$, i.e., $\mathbf{D}$ is unimodular. The pairs $\mathbf{A}_R$, $\mathbf{F}_R$ and $\mathbf{B}_R$, $\mathbf{H}_R$ are then the pairs of right equivalent polynomial matrices; $\mathbf{A}_R$ and $\mathbf{B}_R$ are called *right coprime matrices*.

Theorem 2. RCMF representation of a cascade system TFM $\mathbf{G}$ standing in (5) with (6) can be written in the form

$$\mathbf{A}_R = \begin{bmatrix} a_1p_2 & & & \\ -b_1q_2 & a_2p_3 & & 0 \\ & \ddots & \ddots & \\ 0 & & -b_{n-2}q_{n-1} & a_{n-1}p_n \\ & & & -b_{n-1}q_n & a_n \end{bmatrix} \tag{40}$$

and

$$\mathbf{B}_R = \operatorname{diag}\left[b_1p_2; b_2p_3; \ldots; b_{n-1}p_n; b_n\right] \tag{41}$$

if and only if

$$(p_i, b_ib_{i-1}) \sim 1 \quad \text{for any} \quad i = 2, \ldots, n\,. \tag{42}$$

Proof.

A. Considering $r = n$ for a regular cascade system and using an auxiliary $(n \times 1)$ vextor signal $\mathbf{X}$ we can express according to (38)

$$\mathbf{Y} = \mathbf{B}_R\mathbf{A}_R^{-1}\mathbf{U} = \mathbf{B}_R\mathbf{X} \tag{43}$$

and hence

$$\mathbf{U} = \mathbf{A}_R\mathbf{X}\,. \tag{44}$$

If (40) is substituted into (44) the components of $\mathbf{U}$ are

$$U_1 = a_1p_2X_1\,, \tag{45}$$

$$U_i = -b_{i-1}q_iX_{i-1} + a_ip_{i+1}X_i \quad \text{for} \quad i = 2, \ldots, n-1$$

and

$$U_n = -b_{n-1}q_nX_{n-1} + a_nX_n\,.$$

Then by gradual substitutions X_i from ith into $(i+1)$th equation (45), $i = 1, \ldots, n-1$,

$$X_1 = \frac{1}{a_1p_2}U_1\,, \tag{46}$$

$$X_i = \sum_{j=1}^{i-1} \frac{b_{i-1}q_i \ldots b_jq_{j+1}}{a_ip_{i+1} \ldots a_jp_{j+1}} U_j + \frac{1}{a_ip_{i+1}} U_i\,, \quad i = 2, \ldots, n-1,$$

and

$$X_n = \sum_{j=1}^{n-1} \frac{b_{n-1}q_n \ldots b_jq_{j+1}}{a_na_{n-1}p_n \ldots a_jp_{j+1}} U_j + \frac{1}{a_n} U_n\,.$$

Now using $\mathbf{B}_R$ given by (41) in (43) we get

$$Y_i = b_ip_{i+1}X_i \quad \text{for} \quad i = 1, \ldots, n-1\,, \quad \text{and} \tag{47}$$

$$Y_n = b_nX_n$$

If the equations (46) are substituted into (47) the desired form $\mathbf{Y} = \mathbf{G}\mathbf{U}$ with $\mathbf{G}$ given by (5) and (6) is obtained. Thus (40) and (41) is a right matrix fraction description of $\mathbf{G}$.

B. It remains to prove that the matrices (40) and (41) are right coprime if and only if (42) are valid.

If: Assume that a polynomial $(n \times n)$ matrix $\mathbf{D}$ is GCRD of $\mathbf{A}_R$ and $\mathbf{B}_R$ with $d = \det \mathbf{D}$.

Considering (39) we can write

$$\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} \mathbf{D} \quad \text{and} \quad \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_n \end{bmatrix} \mathbf{D} \tag{48}$$

where $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{f}_i$ and $\mathbf{h}_i$ denote now the rows of $\mathbf{A}_R$, $\mathbf{B}_R$, $\mathbf{F}_R$ and $\mathbf{H}_R$, respectively, $i = 1, \ldots, n$. Obviously $\mathbf{a}_i = \mathbf{f}_i\mathbf{D}$ and $\mathbf{b}_i = \mathbf{h}_i\mathbf{D}$.

The approach used in the previous proof of Theorem 1 can be simply transformed here with the rows playing the former role of columns. For this reason the proof of sufficiency of (42) is given very briefly referring for details to the previous section.

Thus the sets $\{\mathbf{M}_n\}$ and $\{\mathbf{L}_n\}$ are considered where $(n \times n)$ polynomial matrices $\mathbf{M}_n$ and $\mathbf{L}_n$ are formed by different combinations of the rows $\mathbf{a}_i$, $\mathbf{b}_j$ and $\mathbf{f}_i$, $\mathbf{h}_j$, respectively; $i, j = 1, \ldots, n$. Then we have

$$\mathbf{M}_n = \mathbf{L}_n\mathbf{D} \tag{49}$$

with

$$\det \mathbf{M}_n = \det \mathbf{L}_n d \tag{50}$$

for any corresponding pair of $\mathbf{M}_n \in \{\mathbf{M}_n\}$ and $\mathbf{L}_n \in \{\mathbf{L}_n\}$.

Although the matrices $\mathbf{M}_n$ and $\mathbf{L}_n$ differ from the ones which considered under the same denotations in the proof of Theorem 1 the set $\{m_n\}$ of all nonzero determinants of $\mathbf{M}_n$ is identical with $\{m_n\}$ which has been considered there.

Since the conditions (42) are also identical with (12) the continuation of the proof can be found in the previous section starting with the equation (18) until (25).

Only if: Suppose that $\mathbf{A}_R$ and $\mathbf{B}_R$ standing in (40) and (41), resp., are right coprime but (42) are not valid.

1. If (42) are broken by $(p_i, b_i) \nsim 1$ for one $i \in [2, n]$ we use the denotations (28). The matrices (40) and (41) can be decomposed into

$$\mathbf{A}_R = \begin{bmatrix} a_1p_2 & & & & & & & & & \\ -b_1q_2 & a_2p_3 & & & & & & & & \\ 0 & -b_2q_3 & \ddots & & & & & 0 & & \\ & 0 & \ddots & a_{i-2}p_{i-1} & & & & & & \\ & & \ddots & -b_{i-2}q_{i-1} & a_{i-1}p_{ii} & & & & & \\ & & & 0 & y & a_ip_{i+1} & & & & \\ & & & & v & -b_iq_{i+1} & \ddots & & & \\ & 0 & & & 0 & \ddots & a_{n-2}p_{n-1} & & & \\ & & & & & \ddots & -b_{n-2}q_{n-1} & a_{n-1}p_n & & \\ & & & & & 0 & -b_{n-1}q_r & a_n & & \end{bmatrix} \mathbf{D} = \mathbf{A}_{RO}\mathbf{D} \tag{51}$$

and

$$B_R = \begin{bmatrix} b_1p_2 & & & & & & & \\ 0 & \ddots & & & & & 0 & \\ & \ddots & b_{i-2}p_{i-1} & & & & & \\ & & 0 & b_{i-1}p_{ii} & & & & \\ & & & z & b_ip_{i+1} & & & \\ & & & & 0 & \ddots & & \\ & 0 & & & & \ddots & b_{n-1}p_n & \\ & & & & & & 0 & b_n \end{bmatrix} D = B_{RO}D$$

with the GCRD

(53)
$$D = \begin{bmatrix} 1 & & & & & & & \\ 0 & \ddots & & & & & 0 & \\ & \ddots & 1 & & & & & \\ & & 0 & (p_i, b_i) & & & & \\ & & & x & 1 & & & \\ & & & & 0 & \ddots & & \\ & 0 & & & & \ddots & 1 & \\ & & & & & & 0 & 1 \end{bmatrix}$$

The polynomials x, y represent a solution of the equation

(54)
$$a_ip_{i+1}x + (p_i, b_i)\,y = -b_{i-1}q_i$$

which is always solvable since $(p_{i+1}, b_i) \sim 1$ as well as $(a_i, b_i) \sim 1$ is assumed.

Then

(55)
$$v = b_{ii}q_{i+1}x\,, \quad \text{and}$$
$$z = -b_{ii}p_{i+1}x\,.$$

For $i = n$ we must put $p_{i+1} = 1$ and $q_{i+1} = 0$ in (54) and (55). The polynomial v in A_{RO} is omitted.

According to (53) $d = \det D = (p_i, b_i) \nsim 1$ and hence the matrices (40) and (41) are not right coprime.

2. If (42) are broken by $(p_i, b_{i-1}) \nsim 1$ for one $i \in [2, n]$ then using the denotations (32) the matrices (40) and (41) can be written in the form

$$A_R = \begin{bmatrix} a_1p_2 & & & & & & \\ -b_1q_2 & \ddots & & & & 0 & \\ & \ddots & a_{i-2}p_{i-1} & & & & \\ & & -b_{i-2}q_{i-1} & a_{i-1}p_{i,i-1} & & & \\ & & & -b_{i-1,i}q_i & a_ip_{i+1} & & \\ & & & & -b_iq_{i+1} & \ddots & \\ & 0 & & & & \ddots & a_{n-1}p_n \\ & & & & & -b_{n-1}q_n & a_n \end{bmatrix} D = A_{RO}D$$

and

(57) $$\mathbf{B}_R = \text{diag}\,[b_1 p_2; \ldots; b_{i-2} p_{i-1}; b_{i-1} p_{i,i-1}; b_i p_{i+1}; \ldots; b_{n-1} p_n; b_n]\,\mathbf{D} = \\ = \mathbf{B}_{RO}\mathbf{D}$$

where

(58) $$\mathbf{D} = \text{diag}\,[1; \ldots; 1; (p_i, b_{i-1}); 1; \ldots; 1]$$

Since $d = \det \mathbf{D} = (p_i, b_{i-1}) \nsim 1$ the matrices (40) and (41) are not right coprime. □

Note. Polynomial fraction fashions of $P_i(s)$ as well as $T_i(s)$ have been assumed so far. If dead times τ_i are present in a cascade system factors $\exp(-\tau_i s)$ occur in the numerators $b_i(s)$ and/or $q_i(s)$.

But the results of both Theorems 1 and 2 are valid for dead-time systems too, of course, with $\mathbf{A}_L$, $\mathbf{B}_L$ and $\mathbf{A}_R$, $\mathbf{B}_R$ being not necessarily polynomial matrices. The algebraic structure containing the elements $t_i(s) \exp(-\tau_i s)$ instead of polynomials can be considered where $t_i(s)$ is a polynomial and $\tau_i \geqq 0$. Then going through the steps of presented proofs no special pressumptions and operations in this structure are required. It can be appreciated from the physical system viewpoint that $\exp(-\tau_i s)$ and an arbitrary polynomial $c(s)$ have only the common factor equivalent to unit, i.e., $(\exp(-\tau_i s), c) \sim 1$.

4. EXAMPLE

Let the three-stage cascade process (according to Fig. 1) be described by

$$P_1(s) = \frac{b_1(s)}{a_1(s)} = \frac{10}{s(s+2)}, \quad P_2(s) = \frac{b_2(s)}{a_1(s)} = \frac{1}{s+2}, \quad P_3(s) = \frac{b_3(s)}{a_3(s)} = \frac{s+2}{s(s+1)},$$

$$T_2(s) = \frac{q_2(s)}{p_2(s)} = \frac{\exp(-s)}{s+0{\cdot}1}, \quad T_3(s) = \frac{q_3(s)}{p_3(s)} = \frac{1}{s+1}$$

and hence its TFM

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{10}{s(s+2)} & 0 & 0 \\ \dfrac{10\exp(-s)}{s(s+2)^2(s+0{\cdot}1)} & \dfrac{1}{s+2} & 0 \\ \dfrac{10\exp(-s)}{s^2(s+2)(s+0{\cdot}1)(s+1)^2} & \dfrac{1}{s(s+1)^2} & \dfrac{s+2}{s(s+1)} \end{bmatrix}$$

Observing that $(p_2, b_2 b_1) \sim 1$ as well as $(p_3, b_3 b_2) \sim 1$ the conditions (12) = (42) are fulfilled.

Using (10) and (11) the LCMF representation of $\boldsymbol{G}$ is given by

$$\boldsymbol{A}_L = \begin{bmatrix} s(s+2) & 0 & 0 \\ -\exp(-s) & (s+2)(s+0{\cdot}1) & 0 \\ 0 & -(s+2) & s(s+1)^2 \end{bmatrix},$$

$$\boldsymbol{B}_L = \begin{bmatrix} 10 & 0 & 0 \\ 0 & s+0{\cdot}1 & 0 \\ 0 & 0 & (s+2)(s+1) \end{bmatrix}$$

and according to (40) and (41) the RCMF description of $\boldsymbol{G}$ is formed by

$$\boldsymbol{A}_R = \begin{bmatrix} s(s+2)(s+0{\cdot}1) & 0 & 0 \\ -10\exp(-s) & (s+2)(s+1) & 0 \\ 0 & -1 & s(s+1) \end{bmatrix},$$

$$\boldsymbol{B}_R = \begin{bmatrix} 10(s+0{\cdot}1) & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+2 \end{bmatrix}.$$

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REFERENCES

[1] V. Kučera: Discrete Linear Control — The Polynomial Equation Approach. J. Wiley, Chichester 1979.

[2] M. G. Singh: Dynamical Hierarchical Control. North Holland Publ. Co., Amsterdam 1977.

[3] L. N. Volgin: Optimalnoje diskretnoje upravlenije dinamičeskimi sistemami. Nauka, Moskva 1986.

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