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AN ALGEBRAIC APPROACH TO THE SYNTHESIS OF CONTROL FOR LINEAR DISCRETE MEROMORPHIC SYSTEMS

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In the paper, the polynomial approach, successfully used for the synthesis of control of finite-dimensional linear systems, is generalized to the class of meromorphic systems. It is shown that many properties carry over without changes but some have no analogy in the new class, and there are also some new properties emerging.

0. INTRODUCTION

The polynomial approach [1] proved to be a convenient tool for the synthesis of control in linear systems of finite dimension. The basic idea is a formulation of control problems as equations in an appropriate algebraic ring, where divisibility plays a role, first of all in the ring of polynomials. It appeared later that this idea was general and not limited to finite-dimensional systems and polynomials [2]. A possibility is now open to utilize this "algebraic approach" for the synthesis of control in various classes of systems, e.g. those with distributed parameters.

These basic ideas are clear but the algebraic theory of systems and control is still far from fully developed. The control problems are intertwined here with the idea of "system classification": how to define various classes of systems (controlled systems and controllers) with good properties, namely:

- combination of systems according prescribed rules should give results within the class,
- the prescribed control problems should have reasonable solution within the class.

It appears that the properties of systems and control are determined first of all by the corresponding class (divisibility, prime factors, ideals).

The presented paper aims to elaborate the above outlined algebraic approach to the class of linear discrete, time-invariant systems whose transfer functions are meromorphic functions of the delay operator. They are a very simple generalization of systems with rational transfer functions. It appears that the majority of system properties carries from rational to meromorphic without changes. However, there are some properties which have no analogy in the new class, and on the other side, there are some new properties emerging.

The paper is organized as follows. In the mathematical sections 1, 2, 3 the properties of the entire functions and of the meromorphic ones are summarized in a form suitable for the control theory. Most results are either known or easily derivable, that is why the proofs are omitted, only one lemma is proved in the Appendix. Section 4 belongs to the system theory, it introduces the class of meromorphic systems and shows the connections between their state-space models and input-output models. Some simple examples, illustrating the importance of the class of meromorphic systems, are shown in Section 5. The ideas of control synthesis and their connection with algebraic properties are exposed in Section 6.

The symbol $\mathbb{R}$ will stand for the field of real numbers, $\mathbb{C}$ for complex, $\mathbb{P} = \mathbb{C} - \{0\}$.

1. ENTIRE FUNCTIONS

Entire functions are generalization of the polynomials. They are defined as functions $f(q)$ of complex variable q, analytical in $\mathbb{C}$. The polynomials are a special case of them; they have a pole in the point $q = \infty$. The other entire functions have an essential singularity here.

Every entire function can be expressed by a power-series

(1) $$f(q) = \sum_{n=0}^{\infty} f_n q^n$$

with the radius of convergence $R = \infty$. The coefficients f_n fulfil the condition of rapid descent for $n \to \infty$ (faster than exponential):

(2) $$\lim_{n\to\infty} \frac{|f_n|}{a^n} = 0 \quad \text{for} \quad a > 0 .$$

Equivalently,

(3) $$\lim_{n\to\infty} \sqrt[n]{|f_n|} = 0 .$$

The descent of f_n is connected with the ascent of $f(q)$ for $q \to \infty$: this ascent is faster than polynomial (when $f(q)$ is not itself a polynomial):

(4) $$M(r) = \max_{|q|=r} |f(q)| , \quad \lim_{r\to\infty} \frac{r^k}{M(r)} = 0 , \quad k \quad \text{integer} .$$

Like the polynomials, the entire functions form an integrity domain for usual addition and multiplication. On the other side, the degree of a polynomial has no analogy here and Euclidean algorithm cannot be constructed. The entire functions are transcendental, i.e. no algebraical equation of the form

(5) $$\sum_{k=0}^{m} P_k(q) f^k(q) = 0$$

with polynomial P_k is fulfilled.

A polynomial of degree n has always n zero points (respecting the multiplicity). An entire function can have infinity of them. The set of zero points is isolated in $\mathbb{C}$, the only condensation point can be $q = \infty$. However, it can be also only a finite number of them or no zero-point at all. The latter case arises if an only if

$$f(q) = e^{h(q)} \tag{6}$$

with an entire function $h(q)$. Such a function has an inverse $1/f(q)$, which is itself an entire function. In the polynomial domain, such a case (a divisor of unity) is only when $f(q)$ is a constant. Two entire functions are called associate if they differ only by multiplication of a divisor of unity:

$$g(q) = f(q)\, e^{h(q)}\,; \tag{7}$$

it is if and only if they have the same zero-points (respecting the multiplicity).

On the other side, given the sequence of points q_i (not necessarily different) with the condensation point $q = \infty$, an entire function $g(q)$ with zero-points q_i can be constructed:

$$g(q) = \prod_{i=1}^{\infty} (q - q_i)\, e^{h_i(q)} \tag{8}$$

where h_i are appropriate polynomials – for more details, see Weierstrass' approximation theorem [3]: The factors $e^{h_i(q)}$ secure the convergence: the product converges absolutely and uniformly in every bounded finite domain.

Any entire function $f(q)$ can be expressed

$$f(q) = e^{h(q)} \prod_{i=1}^{\infty} (q - q_i)\, e^{h_i(q)} \tag{9}$$

with appropriate $h(q)$, $h_i(q)$. The numbers q_i are uniquely determined but the functions $h(q)$, $h_i(q)$ are not. The factorization (9) is an analogy of a root-factor one in the Gaussian ring of polynomials. For entire functions, the number of root-factors can be infinite, the condition of finite descending chain of divisors is not fulfilled. However, from (9) the existence of a greatest common divisor of two entire function follows as well as its construction: by selecting common factors. In this sense, we can speak about coprime entire functions.

The domain of entire functions is not a principal ideal domain. E.g. it can be easily seen that the set of functions $f(q)$ fulfilling $f(q_i) = 0$, given a sequence q_i with the condensation point $q = \infty$, up to a finite number of points, is an ideal but not a principal one. However, the entire functions form a Bezout domain, as we shall see later.

2. MEROMORPHIC FUNCTIONS

Meromorphic functions are generalizations of rational ones. They are defined as quotients of the entire ones (the quotient field). A meromorphic function $f(q)$ is analytical in $\mathbb{C}$ up to an isolated set of poles. The point $q = \infty$ can be an essential singularity.

A rational function $f(q)$ with a finite limit for $q \to \infty$ is called proper. Any rational function $f(q)$ can be decomposed into a sum of a polynomial $h(q)$ and a proper part. The latter one is determined by a finite set of poles and by a principle part of the Laurent series in every pole:

$$f(q) = h(q) + \sum_{i=1}^{n} \sum_{k=1}^{m_i} \frac{f_{ik}}{(q - q_i)^k} . \tag{10}$$

For meromorphic functions, we cannot define the properness but an equivalence can be introduced: Two functions differ only up to an entire function if and only if they have the same set of poles and the same principal part of Laurent series for every pole.

On the other side, given a sequence of (nonequal) points q_i with condensing point $q = \infty$ and a sequence of functions $f_i(q) = \sum_{k=1}^{m_i} f_{ik}/(q - q_i)^k$, a meromorphic function $f(q)$ can be constructed which has poles q_i and the principal parts of Laurent series in them $f_i(q)$:

$$f(q) = h(q) + \sum_{i=1}^{\infty} \left[\sum_{k=1}^{\infty} \frac{f_{ik}}{(q - q_i)^k} - h_i(q) \right] \tag{11}$$

where $h(q)$ is an entire function and $h_i(q)$ polynomials for securing a convergence (not uniquely determined). The convergence is absolute and almost uniform, for more details, see the Mittag-Leffler theorem [3].

From these results, an existence of the decomposition

$$\frac{c(q)}{a(q)\,b(q)} = \frac{x(q)}{b(q)} + \frac{y(q)}{a(q)} \tag{12}$$

follows. Here $c(q)$, $a(q)$, $b(q)$ are given entire function, $a(q)$, $b(q)$ coprime, $x(q)$, $y(q)$ entire function searched for. Specially, the entire function equation

$$ax + by = 1 \tag{13}$$

with coprime a, b is solvable. When a, b are not coprime but have the greatest common divisor g, the expression

$$ax + by = g \tag{14}$$

is valid. This important property of the entire function domain can be formulated as follows: Every ideal generated by two elements is principal (the Bezout domain).

3. TWO-SIDED ENTIRE AND MEROMORPHIC FUNCTIONS

Besides the polynomials, also "two-sided polynomials"

$$f(q) = \sum_{n=h}^{k} f_n q^n , \quad h, k \text{ integer numbers} , \tag{15}$$

are used in the theory of discrete LQG optimal control. For them, the symmetry

$$f(q) = f(q^{-1}) \tag{16}$$

plays a role. Similarly, the domain of two-sided entire functions can be defined. They are analytical in $\mathbb{P}$; the points 0, ∞ can be essential singularities. These function have an expression

$$f(q) = \sum_{n=-\infty}^{\infty} f_n q^n \tag{17}$$

with f_n rapidly descending for $n \to \pm\infty$. A decomposition

$$f(q) = g(q) + h(q^{-1}) \tag{18}$$

exists with integer g, h, unique up to an additive constant.

Let us call $f(q)$ "with real coefficients" if f_n is real, equivalently $f(\bar{q}) = \overline{f(q)}$. A symmetric two-sided entire function is defined (16), $f_n = f_{-n}$ is valid, its zero points form couples q_i, q_i^{-1}. The unique decomposition

$$f(q) = g(q) + g(q^{-1}) \tag{19}$$

exists with an entire g. For a symmetric $f(q)$ with real coefficients it holds: $f(q)$ is real for $|q| = 1$. Let us call such a function positive if $f(q) > 0$ for $|q| = 1$.

The two-sided entire functions without zero-points (the divisors of unity) are of the form $q^k \, e^{h(q)}$ where k is an integer number and h is a two-sided entire function. For a proof, see the Appendix. For a given sequence of points q_i with two condensation points 0, ∞, a two-sided entire function with zero-points q_i can be constructed:

$$f(q) = q^k \, e^{h(q)} \left[\prod_{i=1}^{\infty} (q - \gamma_i) \, e^{g_i(q)}\right] \left[\prod_{i=1}^{\infty} (q^{-1} - \chi_i) \, e^{h_i(q^{-1})}\right] \tag{20}$$

where γ_i, χ_i are selected subsequences with condensation points ∞, 0, respectively.

From that, the following factorization can be derived. A two-sided entire function $f(q)$ having no zero-points for $|q| = 1$, can be expressed

$$f(q) = g(q) \, h(q^{-1}) \tag{21}$$

with entire $g(q)$, $h(q)$, having zero-points only in $|q| > 1$. The factorization is unique up to multiplication by Kq^k with $K \neq 0$, k integer number. For $f(q)$ symmetric and positive, a factorization

$$f(q) = g(q) \, g(q^{-1}) \tag{22}$$

exists with an entire function g, $g(q) \neq 0$ for $|q| < 1$. The factorization is unique if we require $g(q)$ with real coefficients, $g(1) > 0$.

The quotient field for the domain of two-sided polynomials is nothing else than the field of rational functions. The quotient field for the domain of two-sided entire functions is a new object: two-sided meromorphic functions. They are analytic in $\mathbb{P}$ up to an isolated set of poles, the points 0, ∞ are essential singularities.

Among properties of these functions, the decomposition (18) of the function $f(q)$ with no poles for $|q| = 1$ is interesting. Here $g(q)$, $h(q)$ are meromorphic functions with poles only in $|q| > 1$. The decomposition is unique up to an additive constant. For a symmetric $f(q)$, a unique decomposition (19) exists with meromorphic $g(q)$.

These results lead to the following properties of equations in entire functions. The equation

$$a(q^{-1})\,x(q) + b(q)\,y(q^{-1}) = c(q) \tag{23}$$

with entire $a(q)$, $b(q)$, two-sided entire $c(q)$, where $a(q^{-1})$, $b(q)$ are coprime, is solvable with entire $x(q)$, $y(q)$. The condition $y(0) = 0$ makes the solution unique.

The equation

$$a(q^{-1})\,x(q) + a(q)\,x(q^{-1}) = c(q) \tag{24}$$

with entire a, $a(q) \neq 0$ for $|q| \leqq 1$, with two-sided entire c having no poles for $|q| = 1$, is uniquely solvable.

4. MEROMORPHIC SYSTEMS

Let us consider a linear discrete system

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n\,, \\ y_n &= Cx_n + Du_n\,, \quad n = 0, 1, 2 \ldots \end{aligned} \tag{25}$$

with $u_n \in \mathbb{R}$, $y_n \in \mathbb{R}$, $x_n \in \mathscr{X}$ where $\mathscr{X}$ is a linear space (not necessarily of finite dimension), which is normed and complete (a Banach space). Symbols A, B, C, D stand for linear continuous operators $\mathscr{X} \to \mathscr{X}$, $\mathbb{R} \to \mathscr{X}$, $\mathscr{X} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}$ with norms $\|A\|$, $\|B\|$, $\|C\|$, $\|D\|$.

The system response

$$y_n = CA^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1} B u_k + D u_n \tag{26}$$

can be written by means of the delay operator d:

$$y = G(d)\,x_0 + F(d)\,u \tag{27}$$

where

$$G(d) = C(I - Ad)^{-1}\,, \quad F(d) = Cd(I - Ad)^{-1}\,B + D \tag{28}$$

are the initial state operator and the transfer function. Here I is the identity operator $\mathscr{X} \to \mathscr{X}$. For $|d| < 1/\|A\|$, the operator $(I - Ad)^{-1}$ exists – the resolvent operator of A; it is continuous in $\mathscr{X}$, it is an analytic function of d and can be expressed by a series

$$(I - Ad)^{-1} = \sum_{n=0}^{\infty} A^n d^n\,. \tag{29}$$

Its region of convergence can be broadened: $|d| < 1/r(A)$ where $r(A)$ is the spectral radius of A,

(30) $$r(A) = \lim_{k \to \infty} \|A^k\|^{1/k}, \quad r(A) \leqq \|A\| .$$

Furthermore, suppose the operator A not only continuous but compact (totally continuous): every closed bounded set is mapped to a compact set (containing at least one condensation point). In this case, the resolvent operator is meromorphic [4] and so are $F(d)$, $G(d)$.

From (11), we obtain the spectral decomposition of the transfer function:

(31) $$F(d) = \sum_{i=1}^{\infty} \frac{f_i(d)}{(1 - (d/d_i))^{m_i}} + f_\infty(d)$$

where $f_i(d)$ are polynomials, $f_\infty(d)$ an entire function. The dynamics of the system is described by individual modes (exponential components), defined by numbers d_i. For $|d_i| > 1$, the modes are stable (decreasing in time), for $|d_i| < 1$ unstable. The number of unstable modes is finite (there is no condensation point of poles in $|d_i| < 1$), that of stable ones can be infinite (the point ∞ may be a condensation point). For $i \to \infty$, it is $|d_i| \to \infty$, the descent in time is faster. But it is not the whole story: the mode f_∞ is also present in the dynamics; its descent is faster than exponential. The decomposition (31) is not unique: f_i can be modified, compensated by modifying f_∞. By these modifications, only the beginnings (in time) of the individual modes are influenced, not the ends (time $\to \infty$).

The transfer function can be expressed by means of entire functions

(32) $$F(d) = \frac{b(d)}{a(d)}, \quad a(0) = 1, \quad a, b \quad \text{coprime} .$$

Here $a(d)$, $b(d)$ can have an infinite number of zero-points (9). The expression (32) is not unique: a, b can be multiplied by a divisor of unity (6). All such expressions are of equal validity, no one of them appears to be more "basic" than the other.

A special case of the system arises when $r(A) = 0$. Then $(I - Ad)^{-1}$ is an entire function, in (31) only f_∞ is present, in (32) $a(d) = 1$ can be taken.

If $\mathscr{X}$ is of finite dimension then A is always compact, F is rational, (32) is unique. In (31), there is only a finite number of modes, f_∞ is a polynomial. It is evident that the rapidly descending mode is a generalization of the finite-time mode of rational systems.

The rational systems can be expressed not only by means of the delay operator d but also by means of the advance operator z:

(33) $$F = \frac{b(d)}{a(d)} = \frac{\beta(z)}{\alpha(z)}, \quad \deg \beta \leqq \deg \alpha .$$

In this case, the finite-time mode is expressed by the pole $z = 0$ so that the whole dynamics is described in a unified way by poles or by the characteristic polynomial $\alpha(z)$. In the d-expression, the polynomial $a(d)$ is called pseudo-characteristic.

For meromorphic systems, there is no expression with entire functions of z; the point $z = 0$ is essentially singular.

5. EXAMPLES OF SYSTEMS

Example 1. Let $\mathscr{X}$ be the space of L_2-functions (square integrable) $x(\xi)$ on the finite interval $0 \leqq \xi < X$. It can represent e.g. a distributed electrical charge along a transmission line or a distributed temperature across a wall. Let the operator A be an integral L_2-kernel:

$$A\,x(\xi) = \int_0^X a(\xi, \sigma)\, x(\sigma)\, \mathrm{d}\sigma\,. \tag{34}$$

Then A is always compact and $F(d)$ meromorphic [4]. It is evident that the meromorphic systems cover a very broad class of physical and technical systems.

Example 2. Let the kernel $a(\xi, \sigma)$ in (34) be of Volterra type, i.e. $a(\xi, \sigma) = 0$ for $\xi < \sigma$. It represents e.g. the case where a signal can proceed along the transmission line in one direction only (forwards, not backwards). Then $r(A) = 0$, see [4], $F(d)$ is an entire function.

Example 3. Let a continuous rational system $S(s)$ work in a feedback loop with a transport delay $e^{-T_1 s}$, see Fig. 1. Let the whole feedback loop be controlled by

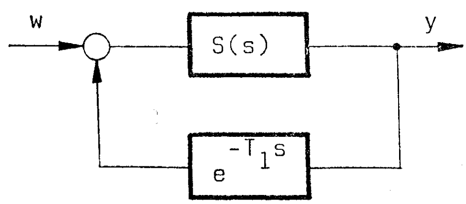

Fig. 1.

a discrete controller with a period T, commensurable with T_1. For its design, we need the discrete transfer function $F(d) = Y(d)/W(d)$ resp. $F(d, \varepsilon) = Y(d, \varepsilon)/W(d)$ where $0 \leqq \varepsilon < 1$ is a relative shift between input and output samples. In [5], these transfer functions are derived by means of state space models. The state has a lumped part (an ordinary differential equation) as well as a distributed part (a partial differential equation of a transport delay). In the simplest case, $S(s) = 1/(T_i s)$, $T = T_1$, $T_1/T_i = k$, we have

$$F(d) = \frac{1 - e^{-kd}}{1 - d\,e^{-kd}}\,, \tag{35}$$

$$F(d, \varepsilon) = \frac{d^{-1}(1 - e^{-k\varepsilon d}) + (e^{-k\varepsilon d} - e^{-kd})}{1 - d\,e^{-kd}}$$

or

$$F(d) = \frac{kd - \frac{k^2}{2} d^2 + \frac{k^3}{6} d^3 - \frac{k^4}{24} d^4 + \ldots}{1 - d + kd^2 - \frac{k^2}{2} d^3 + \frac{k^3}{6} d^4 \ldots},$$

$$F(d, \varepsilon) = \frac{k\varepsilon + \left(-\frac{k^2\varepsilon^2}{2} - k\varepsilon + k\right) d + \left(\frac{k^3\varepsilon^3}{6} + \frac{k^2\varepsilon^2}{2} - \frac{k^2}{2}\right) d^2 + \ldots}{1 - d + kd^2 - \ldots}$$

6. CONTROL PROBLEMS

Problem 1. Consider a system with a meromorphic transfer function $S(d) = B(d)/A(d)$, coprime A, B. It is controlled by a feedback controller $R(d) = Y(d)/X(d)$ of the same class. A well-known method of design is pole assignment: we prescribe poles for the feedback system. For rational systems, we distinguish a complete assignment and an incomplete one. In the former case, we prescribe all modes, i.e. the characteristic polynomial $\alpha(z)$; polynomial equations in z are used. The problem is not always solvable, e.g. it is not possible to obtain a characteristic polynomial of smaller degree than that of the original system. In the incomplete assignment, we use polynomial equations in d, prescribe only the pseudocharacteristic polynomial and let the finite-time mode free. The problem is always solvable, specially $a(d) = 1$ can be obtained (dead-beat). For meromorphic systems, we have only the d-equations. The assignment equation is of form (13); it is always solvable. The rapidly descending mode remains free: for meromorphic systems, the finite-time response cannot be obtained but only a rapid descent. The equation has more than one solution; it is not known (to the author) how to select some "minimal" one. Computational algorithms are not touched in this paper.

Problem 2. (The simplest LQ problem – for simplicity, no loop stabilization is touched here.) Given a stable meromorphic system $S(d) = B(d)/A(d)$, coprime entire A, B, we look for a stable (or on the stability boundary) input signal u_n, $n = 0, 1, 2, \ldots$ which causes the output signal y_n to be as close as possible to the prescribed stable meromorphic function $w = Q(d)/P(d)$, coprime entire P, Q, see Fig. 2. The criterion

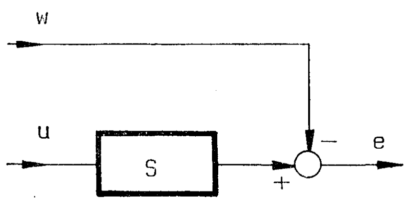

Fig. 2

is minimization of

(37) $$I = \sum_{n=0}^{\infty} e_n^2, \quad e = Su - w.$$

The solution can be found by the Wiener method like for rational systems. We introduce an adjoint system $S^*(d) = S(d^{-1})$ running backward in time, which leads us to the two-sided functions from Section 3. The solution is

$$U = \frac{1}{F_2^+}\left[\frac{F_1}{F_2^-}\right]_+ , \quad F_1 = S^*W , \quad F_2 = S^*S \tag{38}$$

where the symbols G^+, G^- stand for the operation of selecting the stable or the unstable factors, and the symbol $[G]_+$ stands for the operation of selecting (from the two-sided sequence g_n) the terms $n \geqq 0$. With the above stability assumption, the problem is always solvable.

By means of fractions with entire functions the solution process can be written:
1) to find an entire function Φ, $\Phi(d) \neq 0$ for $|d| > 1$ from the equation $\Phi^*\Phi = B^*B$,
2) to find entire X, Y from the equation $\Phi^*X + PY^* = QB^*$ with the condition $Y(0) = 0$,
3) $U = (A/\Phi) . X/P$ with possible cancellation between A, P.
According to Section 3, the equations are always uniquely solvable.

APPENDIX

Lemma. A two-sided entire function $f(q)$ has no zero-points in $\mathbb{P}$ if and only if it can be expressed $f(q) = q^k e^{h(q)}$ where k is an integer number and $h(q)$ is a two-sided entire function.

Proof. Let $f(q) = q^k e^{h(q)}$, it is evident that is has no zero-points in $\mathbb{P}$. Conversely, let $f(q)$ has no zero-points in $\mathbb{P}$. Its derivative $f'(q)$ is also two-sided entire and so is $f'(q)/f(q)$. Let k denote the coefficient for $1/q$ in the Laurent series of $f'(q)/f(q)$ in $\mathbb{P}$. The function $f'(q)/f(q) - k/q$ is two-sided entire and has a primitive function in $\mathbb{P}$, also two-sided entire. It is $h(q) = \ln(q^{-k} f(q))$ as can be easily seen. From that, $f(q) = q^k e^{h(q)}$. We prove that k is an integer number: $q^{-k} = e^{h(q)}/f(q)$ should be two-sided entire, it is possible only for integer k.

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