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ASSOCIATED SPECTRA OF SOME NON-STATIONARY PROCESSES

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In this paper asymptotic stationarity of some classes of nonstantionary random processes is investigated.

The notion of weak stationarity was generalized gradually in several directions. The first extension, suggested by Loève, see [1], is called strong harmonizability in contemporary times. Weak harmonizability was studied first by Rozanov, [2], another approach based on Fourier analysis methods was proposed by Bochner under the name V-boundedness [3].

Further extensions, considered by Kampé de Fériet and Frankiel in [4], are the notions of associated spectrum and of asymptotic stationarity. This was also considered under the same name "asymptotic stationarity" by Parzen [5] and by Rozanov [2] without a name. Rozanov established that every strongly harmonizable process is asymptotically stationary and hence it possesses an associated spectral function. A generalization of this result for a weakly harmonizable process one can find in [6]. The goal of this work is to prove asymptotic stationarity for some classes of non-stationary processes.

Let $\{x(t), t \in \mathbb{R}_1\}$ be a random process, complex in general, with finite second moments. Let us assume, for simplicity, its expected value is vanishing everywhere. We shall say that the process $\{x(t), t \in \mathbb{R}_1\}$ is asymptotically stationary if there exists a finite function $r(\cdot): \mathbb{R}_1 \to \mathbb{C}$ such that for every $h \in \mathbb{R}_1$

$$r(h) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathsf{E}\{x(s + h)\overline{x(s)}\} \, \mathrm{d}s \, .$$

As far as the function $r(\cdot)$ exists then it is positively semidefinite as was shown in [2] e.g. and hence, there exists a unique bounded Borel measure $\mu(\cdot)$ such that

$$r(h) = \int_{-\infty}^{+\infty} \mathrm{e}^{ihu} \, \mathrm{d}\mu(u)$$

a.e. with respect to the Lebesgue measure. The measure $\mu(\cdot)$ represents the associated spectrum of the process $\{x(t), t \in \mathbb{R}_1\}$.

Following Silverman [7], a random process $\{x(t), t \in \mathbb{R}_1\}$, is locally stationary if its covariance function $R(s, t) = \mathsf{E}\{x(s)\,\overline{x(t)}\}$ can be written in the form

$$R(s, t) = R_1\left(\frac{s+t}{2}\right) R_2(s-t)$$

where $R_1(\cdot) \geqq 0$ and $R_2(\cdot)$ is a stationary covariance function. In this case we shall call covariance function $R(\cdot, \cdot)$ locally stationary, too.

Let us assume that the process $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary. In order to consider

$$\int_0^t R(s+h, s)\,\mathrm{d}s$$

we must require measurability of the function $R_2(\cdot)$ and the existence of

$$\int_{t_1}^{t_2} R_1(u)\,\mathrm{d}u$$

for every pair $t_1 < t_2$ of real numbers. Under these assumptions the integral

$$\int_0^t R(s+h, s)\,\mathrm{d}s$$

exists for every $h \in \mathbb{R}_1$ and every $t > 0$ because

$$\int_0^t R(s+h, s)\,\mathrm{d}s = \int_0^t R_1(s+h/2)\,R_2(h)\,\mathrm{d}s = R_2(h)\int_0^t R_1(s+h/2)\,\mathrm{d}s =$$
$$= R_2(h)\int_{h/2}^{t+h/2} R_1(u)\,\mathrm{d}u\,.$$

This fact immediately implies that the locally stationary process $\{x(t), t \in \mathbb{R}_1\}$ is asymptotically stationary if and only if there exists a finite limit

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R_1(s+h)\,\mathrm{d}s$$

for every $h \in \mathbb{R}_1$. It is easy to see that this limit, if exists, is not depending on h. It holds

$$\frac{1}{t}\int_0^t R_1(s+h)\,\mathrm{d}s = \frac{1}{t}\int_h^{t+h} R_1(u)\,\mathrm{d}u =$$
$$= \frac{1}{t}\int_0^t R_1(u)\,\mathrm{d}u + \frac{1}{t}\int_t^{t+h} R_1(u)\,\mathrm{d}u - \frac{1}{t}\int_0^h R_1(u)\,\mathrm{d}u\,.$$

As follows from the last equality

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R_1(s+h)\,\mathrm{d}s$$

exists if and only if

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R_1(s)\,\mathrm{d}s$$

exists and then both the limits are equal. Thus, we can state if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R_1(s)\, ds = A$$

exists then the process $\{x(t), t \in \mathbb{R}_1\}$ is asymptotically stationary and

$$r(h) = A\, R_2(h)\,.$$

The corresponding associated spectrum given by the function $r(\cdot)$, in this case, is determined by the limit value A and by the spectral measure of the stationary covariance $R_2(\cdot)$. This analysis yields that asymptotic stationarity of a locally stationary process depends fully on the behaviour of its covariance function along the principal diagonal in the plane because

$$R(s + h, s) = R_1(s + h/2)\, R_2(h) = R(s + h/2, s + h/2)\, R_2(h)$$

thanks the relation

$$R(s, t) = R\left(\frac{s + t}{2}, \frac{s + t}{2}\right) R\left(\frac{s - t}{2}, \frac{t - s}{2}\right)$$

holding for locally stationary covariances. The obtained results can be summarized in the following assertion.

Theorem 1. A locally stationary process with a covariance function $R(\cdot, \cdot)$ is asymptotically stationary if and only if there exists a finite

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(u, u)\, du\,.$$

One sees immediately that in case $A = 0$ the associated spectrum is vanishing. This case occurs e.g. if the covariance function $R(\cdot, \cdot)$ is integrable along the principal diagonal.

The existence of an associated spectrum can be used for estimation of the asymptotic mean square error of prediction. Let $\tau > 0$ and let us consider a prediction $\tilde{x}(t)$ of $x(t)$ where $\tilde{x}(t)$ belongs to the subspace $H_{t-\tau}$ generated by all random variables of the type

$$y(t) = \sum_{j=1}^{n} \alpha_{\tau_j} x(t - \tau_j)\,, \quad \tau_j \leqq \tau\,, \quad \alpha_{\tau_j} \text{ complex}\,.$$

Under the assumption that the process $\{x(t), t \in \mathbb{R}_1\}$ is asymptotically stationary there exists for every $y(t)$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathsf{E}\{|x(s) - y(s)|^2\}\, ds = \int_{-\infty}^{+\infty} \left|1 - \sum_{j=1}^{n} \alpha_{\tau_j} e^{-i\tau_j u}\right|^2 d\mu(u) = \sigma_\tau(y(\cdot))\,.$$

The value

$$\sigma_\tau = \inf_{y(t)\in H_{t-\tau}} \sigma_\tau(y(\cdot)) = \inf_{\phi(\cdot)} \int_{-\infty}^{+\infty} |1 - \phi(u)|^2\, d\mu(u)\ \left(\phi(u) = \sum_{j=1}^{n} \alpha_{\tau_j} e^{-i\tau_j u}\right)$$

is said to be the asymptotic mean square error of prediction. It is clear that in case of locally stationarity the error σ_τ is proportional to the mean square error derived from the stationary covariance $R_2(\cdot)$. If $A = 0$ we reach, of course, a singular case.

At this moment, let us assume that the locally stationary process $\{x(t),\ t \in \mathbb{R}_1\}$ is strongly harmonizable, too. Then its covariance function can be expressed as

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)}\, dd\gamma(\lambda, \mu)$$

where $\gamma(\cdot, \cdot)$ is a covariance function with finite variation in the plane. Hence, there exists

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(s + h, s)\, ds = r(h)$$

as proved in [2]. Fisrt, we assume $dd\gamma(\lambda, \mu) = f(\lambda, \mu)\, d\lambda\, d\mu$. The Rozanov result yields that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(s + h, s)\, ds = \iint_\Delta e^{ih\lambda} f(\lambda, \mu)\, d\lambda\, d\mu$$

where $\Delta = \{(\lambda, \mu): \lambda = \mu\}$. This limit can be expressed as follows

$$r(h) = \lim_{t\to\infty} \frac{1}{t} \int_0^t R(s + h, s)\, ds = \int_{-\infty}^{+\infty} e^{ihu}\, d\mu(u)$$

where $\mu(B) = \iint_{B\times B\cap\Delta} dd\gamma(\lambda, \mu)$, B is a Borel set in $\mathbb{R}_1$. In case a density $f(\cdot, \cdot)$ exists we obtain $\mu(B) = 0$ for every Borel set $B \subset \mathbb{R}_1$ and we get a singular case. In a general case, the function $r(\cdot)$ can be written in the form

$$r(h) = \iint_\Delta e^{ih\lambda}\, dd\gamma(\lambda, \mu)\,.$$

Let us denote $\Gamma(u, v) = \iint_{-\infty}^{uv} \psi_\Delta(\lambda, \mu)\, dd\gamma(\lambda, \mu)$, $\psi_\Delta(\cdot, \cdot)$ is the indicator of the principal diagonal Δ. Then, evidently,

$$r(h) = \int_{-\infty}^{+\infty} e^{ih\lambda}\, d\Gamma_1(\lambda)$$

where

$$\Gamma_1(\lambda) = \Gamma(\lambda, +\infty) = \iint_{-\infty}^{+\infty} \psi_\Delta(\alpha, \beta)\, dd\gamma(\alpha, \beta) = \iint_{-\infty}^{\lambda\lambda} \psi_\Delta(\alpha, \beta)\, dd\gamma(\alpha, \beta)\,.$$

Now, we can use the results obtained in [8]. Making use of the transformation

$$T: \mathbb{R}_2 \to \mathbb{R}_2\ T(\lambda, \mu) = \left(\frac{\lambda + \mu}{2}, \lambda - \mu\right)$$

the complex measure $dd\gamma(\lambda, \mu)$ can be transformed into a product complex measure $dF_1(u)\, dF_2(v)$ where $F_1(\cdot)$ is non-negative even. The relation between these measures and locally stationary decomposition $R(s, t) = R_1((s + t)/2)\ R_2(s - t)$ is given by

$$R_1(x) = \iint_{-\infty}^{+\infty} e^{ixv}\, dF_2(v)\,, \quad R_2(y) = \int_{-\infty}^{+\infty} e^{iyu}\, dF_1(u)\,.$$

Under the transformation T the principal diagonal Δ of the plane (λ, μ) becomes the axis u in the plane (u, v), hence

$$\Gamma_1(\lambda) = \iint_{-\infty}^{\lambda\lambda} \psi_\Delta(\alpha, \beta) \, \mathrm{d}\mathrm{d}\gamma(\alpha, \beta) = \iint_{u+v/2 \leq \lambda} \psi_{\langle u,0 \rangle}(u, v) \, \mathrm{d}F_1(u) \, \mathrm{d}F_2(v) = \\ = F_1(\lambda) \, \Delta F_2(0)$$

where $\psi_{\langle u,0 \rangle}$ is the indicator of the axis u and $\Delta F_2(0)$ is the jump of the function $F_2(\cdot)$ at 0. It is easy to see that a strongly harmonizable locally stationary process has a non-vanishing associated spectrum if and only if the stationary component $F_2(\cdot)$ of its spectrum has a jump at 0. This fact implies, further, a close connection between asymptotic stationarity and the law of large numbers.

Theorem 2. If a strongly harmonizable locally stationary process has a vanishing associated spectrum then fulfiles the law of large numbers in the quadratic mean sense.

Proof. It follows immediately from the results of [9] and the previous conclusions. □

The contrary of Theorem 2 does not hold because the condition

$$\Delta F_1(0) \, \Delta F_2(0) = 0$$

is necessary and sufficient for fulfilling of the law of large numbers in case of locally stationary processes. We can end this part by the statement that for every strongly harmonizable locally stationary process there exists an associated spectrum and

$$r(h) = \Delta F_2(0) \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}hu} \, \mathrm{d}F_1(u) \, .$$

Some locally stationary covariances belong into the class of normal covariances, which were studied first by the author in [9]. Let us now study asymptotic stationary of normal covariances. A covariance $R(\cdot, \cdot)$ is normal if there exists a finite two-dimensional non-negative measure $F(\cdot, \cdot)$ such that for every $s, t \in \mathbb{R}_1$

$$R(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)} \, \mathrm{e}^{\mathrm{i}\mu(s-t)} \, \mathrm{d}\mathrm{d}F(\lambda, \mu) \, .$$

We know, see [11], that every normal covariance is everywhere continuous, and hence we can consider

$$\frac{1}{t} \int_0^t R(s + h, s) \, \mathrm{d}s \quad \text{for every} \quad t > 0 \, .$$

With respect to unboundedness of the function $\mathrm{e}^{\lambda s}$ we are obliged to restrict ourselves to a suitable subclass of normal covariances.

Theorem 3. Let $R(\cdot, \cdot)$ be a normal covariance, let the corresponding measure $F(, \cdot)$ has a support in the left half-plane. Then the covariance $R(\cdot, \cdot)$ possesses an associated spectrum.

Proof. As the triple integral

$$\int_0^t \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda h} \, \mathrm{e}^{2\lambda s} \, \mathrm{e}^{\mathrm{i}\mu h} \, \mathrm{d}s \, \mathrm{d}\mathrm{d}F(\lambda, \mu)$$

exists, we can write

$$\frac{1}{t}\int_0^t R(s+h,s)\,\mathrm{d}s = \iint_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu h}\,\mathrm{e}^{\lambda h}\left\{\frac{1}{t}\int_0^t \mathrm{e}^{2\lambda s}\,\mathrm{d}s\right\}\mathrm{dd}F(\lambda,\mu)\,.$$

Surely

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{e}^{2\lambda s}\,\mathrm{d}s = \lim_{t\to\infty}\frac{\mathrm{e}^{4\lambda t}-1}{4\lambda t} = \begin{cases}1 & \text{for } \lambda = 0\,.\\ 0 & \text{for } \lambda < 0\\ \infty & \text{for } \lambda > 0\,.\end{cases}$$

As the function $(\mathrm{e}^{4\lambda t}-1)/(4^\lambda t)$ is a.e. $[\mathrm{F}(\cdot,\cdot)]$ bounded the Lebesgue theorem on the dominated convergence proves the existence of an associated spectrum

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(s+h,s)\,\mathrm{d}s = \iint_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\mu h}\,\psi_{\langle 0\rangle}(\lambda)\,\mathrm{dd}F(\lambda,\mu) = \int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\mu h}\,\mathrm{d}G(\mu)$$

where $G(u) = \int_{-\infty}^{+\infty}\int_{-\infty}^{u}\psi_{\langle 0\rangle}(\alpha)\,\mathrm{dd}F(\alpha,\beta) = \int_{\langle 0\rangle}\int_{-\infty}^{u}\mathrm{dd}F(\alpha,\beta)$. We see, in this case, that the associated spectrum is given by the spectral measure

$$\mu(B) = \int_B \mathrm{d}G(u) = \int_{\langle 0\rangle}\int_B \mathrm{dd}F(\alpha,\beta)\,.$$

If there exists a density $\mathrm{dd}F(\lambda,\mu) = f(\lambda,\mu)\,\mathrm{d}\lambda\,\mathrm{d}\mu$ then the corresponding associated spectrum is vanishing. A very important class of nonstationary processes are those of the Karhunen type. Their covariance functions can be expressed as

$$R(s,t) = \int_{-\infty}^{+\infty} f(s,\lambda)\,\overline{f(t,\lambda)}\,\mathrm{d}F(\lambda)$$

where $F(\cdot)$ is, in general, a σ-finite non-negative measure. Assuming measurability of $f(\cdot,\cdot)$ in the plane we can consider

$$\int_{t_1}^{t_2}\int_{-\infty}^{+\infty} f(s,\lambda)\,\overline{f(t,\lambda)}\,\mathrm{d}F(\lambda)\,\mathrm{d}s\,,\quad t_1 < t_2\,,$$

which exists under the condition

$$\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|f(s,\lambda)|^2\,\mathrm{d}F(\lambda)\,\mathrm{d}s < \infty$$

for every pair $t_1 < t_2$ of real numbers. This assumption ensures the existence of

$$\int_0^t R(s+h,s)\,\mathrm{d}s$$

for every $h\in\mathbb{R}_1$. Surely, then

$$\frac{1}{t}\int_0^t R(s+h,s)\,\mathrm{d}s = \int_{-\infty}^{+\infty}\left\{\frac{1}{t}\int_0^t f(s+h,\lambda)\,\overline{f(s,\lambda)}\,\mathrm{d}s\right\}\mathrm{d}F(\lambda)\,.$$

At this moment let us suppose that for every $h \geqq 0$ and for every $\lambda\in\mathbb{R}_1$ there exists (this assumption is not so restrictive)

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(s+h,\lambda)\,\overline{f(s,\lambda)}\,\mathrm{d}s = \phi(h,\lambda)\,.$$

There is no problem to prove the existence of $\phi(h,\lambda)$ for $h < 0$, too. We shall prove that the function $\phi(\cdot,\cdot)$ is positively semidefinite in h. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be arbitrary complex numbers, let $h_1, h_2, \ldots, h_n$ be arbitrary real numbers. Then

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\phi(h_j - h_k, \lambda) = \lim_{t\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\bar{\alpha}_k \frac{1}{t}\int_0^t f(s + h_j - h_k, \lambda)\overline{f(s, \lambda)}\,\mathrm{d}s =$$

$$= \lim_{t\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\bar{\alpha}_k \frac{1}{t}\int_{-h_k}^{t-h_k} f(s + h_j, \lambda)\overline{f(s + h_k, \lambda)}\,\mathrm{d}s =$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\bar{\alpha}_k f(s + h_j, \lambda)\overline{f(s + h_k, \lambda)}\,\mathrm{d}s =$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t \Big|\sum_{j=1}^{n}\alpha_j f(s + h_j, \lambda)\Big|^2\,\mathrm{d}s \geqq 0\,.$$

This means that $\phi(\cdot, \lambda)$ is a.e. with respect to the Lebesgue measure a characteristic function, i.e.

$$\phi(h, \lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{ihu}\,\mathrm{d}g(u, \lambda) \quad \text{a.e.}\,.$$

Let us suppose that $\phi(\cdot, \lambda)$ is continuous for every $\lambda \in \mathbb{R}_1$. Further, let us suppose the existence of an integrable majorant function $\Phi(\cdot)$ such that for every $t > t_0$

$$\frac{1}{t}\int_0^t |f(s, \lambda)|^2\,\mathrm{d}s \leqq \Phi(\lambda) \quad \text{a.e.} \quad [F],\ \Phi(\cdot) \in \mathscr{L}_1(F)\,.$$

The Lebesgue theorem on the dominated convergence ensures then the existence of

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(s + h, s)\,\mathrm{d}s$$

for every $h \in \mathbb{R}_1$ and the equality

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(s + h, s)\,\mathrm{d}s = \int_{-\infty}^{+\infty}\Big\{\int_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{d}g(u, \lambda)\Big\}\,\mathrm{d}F(\lambda)$$

holds for every $h \in \mathbb{R}_1$.

The function $g(\cdot, \cdot)$ is everywhere defined in the plane and thus

$$G(u, v) = \int_{-\infty}^{v}\Big\{\int_{-\infty}^{u}\mathrm{d}g(\cdot, \lambda)\Big\}\,\mathrm{d}F(\lambda) = \int_{-\infty}^{v} g(u, \lambda)\,\mathrm{d}F(\lambda)$$

defines a two-dimensional non-negative finite measure. As the function e^{ihu} is bounded

$$\iint_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{dd}G(u, v)$$

exists for every $h \in \mathbb{R}_1$. By means of the double integral properties

$$\iint_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{dd}G(u, v) = \int_{-\infty}^{+\infty}\mathrm{e}^{ihu}\Big[\int_{-\infty}^{+\infty}\mathrm{dd}G(u, v)\Big] = \int_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{d}G_1(u)$$

where $G_1(u) = \int_{-\infty}^{+\infty}\mathrm{d}G(u, v)$. On the other hand,

$$\iint_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{dd}G(u, v) = \int_{-\infty}^{+\infty}\Big\{\int_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{dd}G(u, v)\Big\} = \int_{-\infty}^{+\infty}\Big\{\int_{-\infty}^{+\infty}\mathrm{e}^{ihu}\,\mathrm{d}g(u, \lambda)\Big\}\,\mathrm{dd}F(\lambda)\,.$$

This equality proves that the associated spectral measure is equal to the marginal measure $G_1(\cdot)$ derived from $G(\cdot, \cdot)$. The obtained result will be introduced as the following theorem.

Theorem 4. Let the covariance function $R(\cdot, \cdot)$ of a random process $\{x(t), t \in \mathbb{R}_1\}$ have the Karhunen form

$$R(s, t) = \int_{-\infty}^{+\infty} f(s, \lambda) \overline{f(t, \lambda)} \, dF(\lambda),$$

$s, t \in \mathbb{R}_1$. Let for every pair $t_1 < t_2$ of real numbers there exist

$$\int_{t_2}^{t_1} \int_{-\infty}^{+\infty} |f(s, \lambda)|^2 \, dF(\lambda) \, ds .$$

Let for every $h \geqq 0$ and for every $\lambda \in \mathbb{R}_1$ there exist

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(s + h, \lambda) \overline{f(s, \lambda)} \, ds = \phi(h, \lambda)$$

provided that $\phi(\cdot, \lambda)$ is continuous and for every $t > t_0$

$$\frac{1}{t} \int_0^t |f(s, \lambda)|^2 \, ds \leqq \Phi(\lambda) \quad \text{a.e.}$$

with respect to $F(\cdot)$ and let $\Phi(\cdot) \in \mathscr{L}_1(F(\cdot))$, Then the process $\{x(t), t \in \mathbb{R}_1\}$ is asymptotically stationary and

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t R(s + h, s) \, ds = \int \left[\int_{-\infty}^{+\infty} e^{ihu} \, dg(u, \lambda)\right] dF(\lambda)$$

where

$$\phi(h, \lambda) = \int_{-\infty}^{+\infty} e^{ihu} \, dg(u, \lambda) .$$

The associated spectral measure equals

$$\int_{-\infty}^{+\infty} g(u, \lambda) \, dF(\lambda) .$$

A similar result under analogical assumption can be derived for a wider class of stochastic processes than those of Karhunen, namely, for the class introduced by Cramér in [12]. Covariances of these processes can be expressed in the form

$$(*) \qquad R(s, t) = \iint_{-\infty}^{+\infty} f(s, \lambda) \overline{f(t, \mu)} \, dd F(\lambda, \mu)$$

where $F(\cdot, \cdot)$ is a covariance function with locally finite total variation in the plane. The integral (*) is understood in the following sense

$$R(s, t) = \lim_{\substack{a, a' \to -\infty \\ b, b' \to +\infty}} \int_a^b \int_{a'}^{b'} f(s, \lambda) \overline{f(t, \mu)} \, dd F(\lambda, \mu) .$$

Cramér constructed in [12] the corresponding stochastic integral, too.

Theorem 5. Let a covariance function $R(\cdot, \cdot)$ belong to the Cramér class. Let there exist

$$\iint_{-\infty}^{+\infty} |f(s, \lambda) \overline{f(t, \mu)}| \, |dd| F(\lambda, \mu)| < \infty$$

bounded in s, t on every compact rectangle in the plane. Let for every $h \geqq 0$ and for every pair (λ, μ) of real numbers there exist

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(s + h, \lambda) \overline{f(s, \mu)} \, ds = \phi(h, \lambda, \mu)$$

continuous in h and let for every $T > T_0$

$$\frac{1}{T} \int_0^T |f(s, \lambda)|^2 \,\mathrm{d}s \leqq \Phi(\lambda)$$

a.e. with respect to the total variation $|F(\cdot, \cdot)|$ with $\iint_{-\infty}^{+\infty} (\Phi(\lambda, \lambda)\, \Phi(\mu, \mu))^{1/2}$. $\mathrm{dd}|F(\lambda, \mu)| < \infty$. Then the covariance function $R(\cdot, \cdot)$ possesses an associated spectrum.

Proof. The assumptions of the theorem ensure the existence of integrals

$$\int_0^T \{\iint_{-\infty}^{+\infty} f(s + h, \lambda) \overline{f(s, \mu)} \,\mathrm{dd}F(\lambda, \mu)\} \,\mathrm{d}s$$

for every $h \in \mathbb{R}_1$ and the possibility to change the order of integration, i.e.

$$\int_0^T \{\iint_{-\infty}^{+\infty} f(s + h, \lambda) \overline{f(s, \mu)} \,\mathrm{dd}F(\lambda, \mu)\} \,\mathrm{d}s =$$
$$= \iint_{-\infty}^{+\infty} \{\int_0^T f(s + h, \lambda) \overline{f(s, \mu)} \,\mathrm{d}s\} \,\mathrm{dd}F(\lambda, \mu) .$$

By means of the Lebesgue theorem on the dominated convergence we can assert that there exists a limit function $r(\cdot)$ such that

$$r(h) = \lim_{T \to \infty} \frac{1}{T} \int_0^T R(s + h, s) \,\mathrm{d}s = \iint_{-\infty}^{+\infty} \phi(h, \lambda. \mu) \,\mathrm{dd}F(\lambda, \mu) .$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be arbitrary real numbers. Let us prove that the matrix function

$$\{\phi(\cdot, \lambda_k, \lambda_l)\}_{k,l=1}^n$$

is positively semidefinite. We have for arbitrary complex $\alpha_1, \alpha_2, \ldots, \alpha_N$ and real $h_1, h_2, \ldots, h_N$

$$\sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \,\phi(h_i - h_j, \lambda_{h_i}, \lambda_{h_j}) =$$
$$= \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s + h_i - h_j, \lambda_{k_j}) f(s, \lambda_{k_j}) \,\mathrm{d}s =$$
$$= \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j \lim_{T \to \infty} \frac{1}{T} \int_{h_j}^{T+h_j} f(s + h_i, \lambda_{k_i}) \overline{f(s + h_j, \lambda_{h_j})} \,\mathrm{d}s =$$
$$= \lim_{T \to \infty} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j \frac{1}{T} \int_{h_j}^{T+h_j} f(s + h_i, \lambda_{h_i}) \overline{f(s + h_j, \lambda_{k_j})} \,\mathrm{d}s =$$
$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j f(s + h_i, \lambda_{k_j}) \overline{f(s + h_j, \lambda_{k_j})} \,\mathrm{d}s \geqq 0$$

This fact implies the existence of a matrix spectral measure

$$\{g(\cdot, \lambda_i, \lambda_j)\}_{i,j=1}^n$$

such that for every pair i, j

$$\phi(h, \lambda_i, \lambda_j) = \int_{-\infty}^{+\infty} \mathrm{e}^{ihu} \,\mathrm{d}g(u, \lambda_i, \lambda_j) \quad \text{a.e.}$$

with respect to the Lebesgue measure in h. Since $\phi(\cdot, \lambda_i, \lambda_j)$ is assumed to be continuous this equality holds for every $h \in \mathbb{R}_1$. Then, evidently

$$r(h) = \iint_{-\infty}^{+\infty} \{\int_{-\infty}^{+\infty} e^{ihu} \, dg(u, \lambda, \mu)\} \, ddF(\lambda, \mu) .$$

Let us consider the complex measure G defined by the relation

$$\Delta\Delta\Delta G(u, \lambda, \mu) = \Delta g(u, \lambda, \mu) \, \Delta\Delta H(\lambda, \mu) .$$

As $g(\cdot, \lambda, \mu)$ is a term of a matrix spectral measure

$$|\Delta g(u, \lambda, \mu)| = [\Delta g(u, \lambda, \lambda)]^{1/2} \, [\Delta g(u, \mu, \mu)]^{1/2}$$

and further,

$$\Delta g(u, \lambda, \lambda) \leqq g(+\infty, \lambda, \lambda) = \int_{-\infty}^{+\infty} e^{i0u} \, dg(u, \lambda, \lambda) =$$

$$= \phi(0, \lambda, \lambda) = \lim_{T \to \infty} \frac{1}{T} \int_0^T |f(s, \lambda)|^2 \, ds \leqq \Phi(\lambda, \lambda) .$$

Thus,

$$|\Delta\Delta\Delta G(u, \lambda, \mu)| \leqq \Phi(\lambda, \lambda)^{1/2} \, \Phi^{1/2}(\mu, \mu) \, |\Delta\Delta F(\lambda, \mu)| .$$

We have proved that the complex measure $G(\cdot, \cdot, \cdot)$ is of finite total variation and hence

$$\iiint_{-\infty}^{+\infty} e^{ihu} \, dddG(u, \lambda, \mu)$$

exists. Then, of course,

$$\iiint_{-\infty}^{+\infty} e^{ihu} \, dddG(u, \lambda, \mu) = \int_{-\infty}^{+\infty} e^{ihu} \{\iint_{-\infty}^{+\infty} dddG(u, \lambda, \mu)\}$$

where

$$\iint_{-\infty}^{+\infty} ddG(u, \lambda, \mu) = \iint_{-\infty}^{+\infty} G(u, \lambda, \mu) \, ddF(\lambda, \mu) .$$

On the other hand, there exists the following integral

$$\iint_{-\infty}^{+\infty} \{\int_{-\infty}^{+\infty} e^{ihu} \, dddG(u, \lambda, \mu)\} = \iint_{-\infty}^{+\infty} \{\int_{-\infty}^{+\infty} e^{ihu} \, dG(u, \lambda, \mu)\} \, ddF(\lambda, \mu)$$

and both the integrals are equal. The proof of the theorem is finished by the assertion that the corresponding associated spectral measure is

$$m(h) = \int_{-\infty}^{+\infty} e^{ihu} \{\iint_{-\infty}^{+\infty} dG(u, \lambda, \mu)\} \, ddF(\lambda, \mu) . \qquad \square$$

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