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TABLES FOR AR(1) PROCESSES WITH EXPONENTIAL WHITE NOISE

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A new method was recently proposed for estimating the parameter of the AR(1) process with non-negative values. The exact distribution of this estimator was derived for the case that the white noise has an exponential distribution. Here we present tables containing the expectation and standard deviation of the new estimator.

1. INTRODUCTION

Let X_1 be a non-negative random variable such that $\mathsf{E}X_1^2 < \infty$. Let $Y_2, Y_3, \ldots, Y_n$ be i.i.d. non-negative random variables with a distribution function F having a finite second moment. Let $Y_2, \ldots, Y_n$ be independent of X_1. Consider the AR(1) process $\{X_t, 1 \leqq t \leqq n\}$ given by

(1.1) $$X_t = bX_{t-1} + Y_t \quad (2 \leqq t \leqq n)$$

where $b \in [0, 1)$. Bell and Smith [2] proposed this model for investigating non-negative time series. The parameter b can be estimated by

$$b^* = \min_{2 \leqq t \leqq n} (X_t/X_{t-1}).$$

Theorem 1.1. The estimator b^* has a positive bias. As $n \to \infty$, b^* is consistent if and only if there exist no numbers c, d such that $0 < c < d < \infty$, $F(d) - F(c) = 1$.

Proof. See [2].

If the condition introduced in Theorem 1.1 is satisfied, then b^* is even strongly consistent.

It is clear that the effect of X_1 on b^* is diminished as time increases.

The most important case is when Y_t has an exponential distribution $Ex(a)$ with the density

$$f(y) = a^{-1} \mathrm{e}^{-y/a}, \quad y > 0 .$$

Anděl [1] proposed to consider the model, in which $X_1 \sim Ex[a/(1 - b)]$, because in this case $\mathsf{E}X_1$ is the same as the expectation of the stationary distribution. He derived some explicit results.

Theorem 1.2. Let $X_1 \sim Ex[a/(1-b)]$, $Y_t \sim Ex(a)$. Then the distribution of b^* is given by $\mathsf{P}(b^* < v) = 1 - G(v)$, where

$$G(v) = (1-b)\{[v + (1-b)][v^2 + (1-b)(1+v)] \ldots$$
$$\ldots [v^{n-2} + (1-b)(1 + v + \ldots + v^{n-3})] \,.$$
$$.\, [v^{n-1} + (1-b)(1 + v + \ldots + v^{n-2}) - b]\}^{-1}$$

for $v \geqq b$, and $G(v) = 1$ for $v < b$.

Proof. See [1].

Critical values of this distribution are introduced in [1]. It was proved in the same paper that

$$b + n^{-1}(1-b)^2 \leqq \mathsf{E}b^* \leqq b + (n-2)^{-1}\,,$$
$$\operatorname{var} b^* \leqq 2b[(n-2)^{-1} - n^{-1}(1-b)^2] + 2(n-2)^{-1}(n-3)^{-1} - n^{-2}(1-b)^4\,.$$

Unfortunately, for $b \neq 0$ these inequalities give only very rough bounds for $\mathsf{E}b^*$ and $\operatorname{var} b^*$. On the other hand, simulations show that the estimator b^* has considerably smaller standard deviation in comparison with the classical least squares estimator. If the bias of b^* were known exactly, b^* could serve even much better. However, no explicit formulas are known for the integral $\mathsf{E}b^* = -\int vG'(v)\,\mathrm{d}v$.

Table 1 contains $\mathsf{E}b^*$, Table 2 $(\operatorname{var} b^*)^{1/2}$ for $b = 0(0{\cdot}1)\,0{\cdot}9, 0{\cdot}95\,(0{\cdot}01)\,0{\cdot}99$ and $n = 10(5)\,50(50)150$. $\mathsf{E}b^*$ and $\operatorname{var} b^*$ were computed using formulas

$$\mathsf{E}b^* = b + \int_b^\infty G(v)\,\mathrm{d}v\,,$$
$$\operatorname{var} b^* = 2\int_b^\infty v\,G(v)\,\mathrm{d}v - 2b\int_b^\infty G(v)\,\mathrm{d}v - \left[\int_b^\infty G(v)\,\mathrm{d}v\right]^2$$

and the integrals

$$\int_b^\infty G(v)\,\mathrm{d}v \quad \text{and} \quad \int_b^\infty v\,G(v)\,\mathrm{d}v$$

were calculated numerically. In each case, the interval (b, ∞) was written in the form $(b, \infty) = (b, B] \cup (B, \infty)$. The constant B $(B \geqq b)$ was chosen so that the integral over (B, ∞) was smaller than 10^{-6}, and the integral over $(b, B]$ was then calculated using the Gauss method.

2. AN APPROXIMATION

Since

$$b^* = b + \min_{2 \leqq t \leqq n} \frac{Y_t}{X_{t-1}}$$

it suffices to consider the distribution of

$$\xi = \min_{2 \leqq t \leqq n} \frac{Y_t}{X_{t-1}}\,. \tag{2.1}$$

Table 1.

b	n										
	10	15	20	25	30	35	40	45	50	100	150
0·00	0·1023	0·0673	0·0503	0·0401	0·0334	0·0286	0·0250	0·0222	0·0200	0·0100	0·0067
0·10	0·1923	0·1606	0·1453	0·1361	0·1301	0·1258	0·1225	0·1200	0·1180	0·1090	0·1060
0·20	0·2824	0·2540	0·2403	0·2321	0·2268	0·2229	0·2200	0·2178	0·2160	0·2080	0·2053
0·30	0·3726	0·3474	0·3353	0·3281	0·3234	0·3201	0·3175	0·3156	0·3140	0·3070	0·3047
0·40	0·4628	0·4407	0·4303	0·4242	0·4201	0·4172	0·4150	0·4134	0·4120	0·4060	0·4040
0·50	0·5531	0·5342	0·5253	0·5202	0·5168	0·5144	0·5125	0·5111	0·5100	0·5050	0·5033
0·60	0·6435	0·6276	0·6204	0·6162	0·6135	0·6115	0·6100	0·6089	0·6080	0·6040	0·6027
0·70	0·7340	0·7212	0·7155	0·7122	0·7101	0·7087	0·7076	0·7067	0·7060	0·7030	0·7020
0·80	0·8246	0·8148	0·8106	0·8083	0·8069	0·8058	0·8051	0·8045	0·8040	0·8020	0·8013
0·90	0·9149	0·9085	0·9059	0·9045	0·9036	0·9030	0·9026	0·9023	0·9021	0·9010	0·9007
0·95	0·9593	0·9552	0·9535	0·9526	0·9521	0·9517	0·9514	0·9513	0·9511	0·9505	0·9503
0·96	0·9680	0·9645	0·9630	0·9622	0·9618	0·9614	0·9612	0·9611	0·9609	0·9604	0·9603
0·97	0·9766	0·9737	0·9725	0·9718	0·9714	0·9712	0·9710	0·9708	0·9707	0·9703	0·9702
0·98	0·9850	0·9828	0·9819	0·9814	0·9811	0·9809	0·9807	0·9806	0·9805	0·9802	0·9801
0·99	0·9931	0·9917	0·9912	0·9909	0·9907	0·9905	0·9905	0·9904	0·9903	0·9901	0·9901

Table 2.

b	n = 10	15	20	25	30	35	40	45	50	100	150
0·00	0·0944	0·0635	0·0480	0·0387	0·0324	0·0279	0·0244	0·0218	0·0196	0·0099	0·0066
0·10	0·0855	0·0573	0·0433	0·0348	0·0292	0·0251	0·0220	0·0196	0·0177	0·0089	0·0060
0·20	0·0767	0·0511	0·0385	0·0310	0·0259	0·0223	0·0196	0·0174	0·0157	0·0079	0·0053
0·30	0·0680	0·0449	0·0338	0·0272	0·0227	0·0195	0·0171	0·0153	0·0138	0·0069	0·0046
0·40	0·0593	0·0388	0·0291	0·0234	0·0195	0·0168	0·0147	0·0131	0·0118	0·0059	0·0040
0·50	0·0508	0·0327	0·0244	0·0195	0·0163	0·0140	0·0123	0·0109	0·0098	0·0050	0·0033
0·60	0·0425	0·0267	0·0198	0·0157	0·0131	0·0112	0·0098	0·0088	0·0079	0·0040	0·0027
0·70	0·0344	0·0208	0·0151	0·0120	0·0099	0·0085	0·0074	0·0066	0·0059	0·0030	0·0020
0·80	0·0263	0·0151	0·0107	0·0083	0·0068	0·0058	0·0050	0·0045	0·0040	0·0020	0·0013
0·90	0·0181	0·0096	0·0064	0·0047	0·0038	0·0031	0·0027	0·0024	0·0021	0·0010	0·0007
0·95	0·0131	0·0066	0·0042	0·0030	0·0023	0·0019	0·0016	0·0014	0·0012	0·0005	0·0003
0·96	0·0119	0·0060	0·0038	0·0027	0·0020	0·0016	0·0014	0·0012	0·0010	0·0004	0·0003
0·97	0·0105	0·0053	0·0033	0·0023	0·0017	0·0014	0·0011	0·0010	0·0008	0·0003	0·0002
0·98	0·0089	0·0044	0·0027	0·0019	0·0014	0·0011	0·0009	0·0008	0·0007	0·0003	0·0002
0·99	0·0066	0·0033	0·0020	0·0014	0·0010	0·0008	0·0007	0·0005	0·0005	0·0002	0·0001

Without loss of generality we can assume that $a = 1$, because the distribution of ξ does not depend on a (see Theorem 1.2). Denote $m = \mathsf{E}X_t$. Since $X_1, \ldots, X_n$ can be considered from practical point of view as stationary, we have from (1.1)

$$m = bm + 1 \,,$$

i.e.

$$m = 1/(1 - b) \,.$$

If we substitute m for X_{t-1} in (2.1), we have for ξ an approximation

$$\xi_{\mathrm{appr}} = \frac{1}{m} \min_{2 \leqq t \leqq n} Y_t \,.$$

Since $Y_t \sim Ex(1)$ we have $\min_{2 \leqq t \leqq n} Y_t \sim Ex(1/(n - 1))$. Thus

$$\mathsf{E}\xi_{\mathrm{appr}} = \frac{1 - b}{n - 1} \,,$$

$$\operatorname{var} \xi_{\mathrm{appr}} = \frac{(1 - b)^2}{(n - 1)^2} \,.$$

The quality of this approximation can be judged using Table 3. The exact values are taken from Table 1 and Table 2.

Table 3.

b	n	Exact values $\mathsf{E}b^*$	Exact values $(\operatorname{var} b^*)^{1/2}$	Approximate values $\mathsf{E}b^*$	Approximate values $(\operatorname{var} b^*)^{1/2}$
0·2	10	0·2824	0·0767	0·2889	0·0889
0·2	100	0·2080	0·0079	0·2081	0·0081
0·9	10	0·9149	0·0181	0·9111	0·0111
0·9	100	0·9010	0·0010	0·9010	0·0010

For the practical purposes our approximation can be used in the form

$$\mathsf{E}b^* \doteq b + \frac{1 - b}{n - 1} \,,$$

$$(\operatorname{var} b^*)^{1/2} \doteq \frac{1 - b}{n - 1} \,.$$

3. AN APPLICATION

It was mentioned that $\mathsf{E}b^* > b$. Using Table 1, we can reduce the bias of the estimator b^*. We can proceed in the following way:

1. Calculate b^*.

2. Find b such that $\mathsf{E}b = b^*$; denote this b by b_0.

3. Use b_0 as a new estimator.

To illustrate this approach, we produced a small simulation study. For each value of b introduced in Table 4, 100 simulations of the stationary AR(1) process $X_1, \ldots, X_{50}$ with $Y_t \sim Ex(1)$ were produced, In the column b^* the averages of the corresponding estimates are given. The next column s.d. b^* contains empirical standard deviations. In the column b_0 the new estimator is presented, which is calculated from values placed the column b^*. It was obtained by interpolation in Table 1. To compare these results with classical estimators, we introduce also the average of the least squares estimates b^0 and the empirical standard deviation s.d. b^0.

Table 4.

b	b^*	s. d. b^*	b_0	b^0	s. d. b^0
0	0·021	0·019	0·001	−0·008	0·142
0·5	0·511	0·011	0·501	0·492	0·121
0·9	0·902	0·002	0·900	0·828	0·103

Table 4 shows that b_0 is more concentrated around b than b^*. Further, s.d. b^* is much smaller than s.d. b^0. Thus in the AR(1) processes with exponential white noise the new method gives considerably better estimators than the classical least squares method.

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